Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.1, No.3, 2011

New Multi-Step Runge –Kutta Method For Solving Fuzzy Differential Equations

Nirmala. V[1*] Chenthur Pandian.S[2]

1. Department of Mathematics, University College of Engineering Tindivanam
(Anna University of Technology Chennai), Tamil Nadu, India, 604001.
2. Department of Electrical and Electronics Engineering,Dr Mahalingam College of Engineering and Technology,(Anna University of Technology Coimbatore)Pollachi,Tamil Nadu,India

*E-mail of the corresponding auther: nirmala_2005@mail.com

Abstract

This paper presents solution for the first order fuzzy differential equation by Runge –Kutta method of order two with new parameters and Harmonic mean of k_i's which are used in the main formula in order to increase the order of accuracy of the solution. This method is discussed in detail followed by a complete error analysis. The accuracy and efficiency of the proposed method is illustrated by solving a fuzzy initial value problem.

Keywords: Fuzzy differential equations, multi-step Runge-Kutta method, higher order derivative approximations, harmonic mean.

1. Introduction

Fuzzy Differential Equation (FDE) models have wide range of applications in many branches of engineering and in the field of medicine. The concept of fuzzy derivative was first introduced by S.L.Chang and L.A.Zadeh [7].D.Dubois and Prade [8] discussed differentiation with fuzzy features.M.L.puri and D.A.Ralesec [18] and R.Goetschel and W.Voxman [10] contributed towards the differential of fuzzy functions. The fuzzy differential equation and initial value problems were extensively studied by O.Kaleva [11,12] and by S.Seikkala [19].Recently many research papers are focused on numerical solution of fuzzy initial value problems (FIVPS).Numerical Solution of fuzzy differential equations has been introduced by M.Ma, M. Friedman, A. Kandel [14] through Euler method and by S.Abbasbandy and T.Allahviranloo [2] by Taylor method.Runge – Kutta methods have also been studied by authors [3,17]. Numerical Solution of fuzzy differential equations by Runge –Kutta method of order two with new parameters has been by V.Nirmala, Saveetha, N and S.Chenthur Pandian[15] and by Runge –Kutta method of order four with new parameters by V.Nirmala and S.Chenthur Pandian [16].

This paper is organised as follows: In section 2, some basic results on fuzzy numbers and definition of fuzzy derivative are given. Section 3 contains fuzzy Cauchy problem whose numerical solution is the main interest of this paper. Second order multi -step Runge –Kutta method with new parameters based on harmonic mean is discussed in section 4.The proposed method is illustrated by a solved numerical example in section 5and the result is compared with Euler's approximation and with the approximation by Runge-Kutta method proposed in [15] and conclusion is in section 6.

2. Preliminaries

Consider the initial value problem

$$\begin{cases} y'(t) = f\big(t, y(t)\big); t_0 \le t \le b \\ y(t_0) = y_0 \end{cases} \tag{2.1}$$

We assume that

1.$f\big(t, y(t)\big)$ is defined and continuous in the strip $t_0 \le t \le b, -\infty < y < \infty$ with t_0 and b finite.

2. There exists a constant L such that for any 't' in $[t_0, b]$ and any two numbers y and y^*, $|f(t,y) - f(t,y^*)| \le L|y - y^*|$

These conditions are sufficient to prove that there exists on $[t_0, b]$, a unique continuous, differentiable function $y(t)$ satisfying (2.1).

The basis of all Runge-Kutta methods is to express the difference between the value of y at t_{n+1} and t_n as $y_{n+1} - y_n = \sum_{i=0}^{m} w_i k_i$ (2.2) Where w_i's are constant for all i and $k_i = hf(t_n + a_i h, y_n + \sum_{j=1}^{i-1} c_{ij} k_j)$ (2.3) Most efforts to increase the order of accuracy of the Runge-Kutta methods have been accomplished by increasing the number of Taylor's series terms used and thus the number of functional evaluations required [6].The method proposed by Goeken .D and Johnson .O[9] introduces new terms involving higher order derivatives of 'f' in the Runge-Kutta k_i terms $(i > 1)$ to obtain a higher order of accuracy without a corresponding increase in evaluations of'f', but with the addition of evaluations of f'.

The second order Runge-Kutta method for autonomous systems proposed by Goeken.D andJohnson.O [9]:

Consider $y_{n+1} = y_n + b_1 k_1 + b_2 k_2$ (2.4)

Where$k_1 = hf(y_n)$ (2.5)

$k_2 = hf(y_n + a_{21} k_1 + h a_{22} f_y(y_n) k_1)$ (2.6)

New Multi-step Runge- Kutta method with harmonic mean was discussed and used approximate f' in the stage k_2 by Ababneh,O.Y,Ahmad,R.,and Ismail,E.S [1]. In this paper, we use exact f'in stagek_2.Then, the new multi-step Runge –Kutta method of order two is given by

$$y_{n+1} = y_n + \frac{2k_1 k_2}{k_1 + k_2} \quad (2.7)$$

Where $k_1 = hf(y_n)$ (2.8)

$k_2 = hf(y_n + a_{21} k_1 + h a_{22} f_y(y_n) k_1)$ (2.9)

Utilizing the Taylor's series expansion techniques in (2.7),we get the parameter $a_{21} = 1$ and there is no any order condition to choose a_{22} and hence we take $a_{22} = 9/10$

Runge-Kutta method of order two is given by:

$$y(t_{n+1}) = y(t_n) + \frac{2k_1 k_2}{k_1 + k_2} \quad (2.10)$$

Where$k_1 = hf(y(t_n))$ (2.11)

$k_2 = hf(y(t_n) + k_1 + h\frac{9}{10}(f_y(y(t_n))k_1))$ (2.12)

Theorem 2.1. Let $f(t, y)$belong to $C^2[a, b]$ and its partial derivatives be bounded and let us assume that there exist positive constants L, M, such that $|f(t, y)| < M$, $\left|\frac{\partial^{i+j} f}{\partial t^i \partial y^j}\right| < \frac{L^{i+j}}{M^{j-1}}$, $i + j \leq m$, then in the Runge –Kutta method of order two ,we have([13]),

$$y(t_{i+1}) - y_{i+1} \cong (63/30) M L^2 h^3 + O(h^4).$$

Definition 2.1. A fuzzy number u is a fuzzy subset of R (ie) $u: R \to [0,1]$ satisfying the following conditions:

1.u is normal (ie) $\exists\, x_0 \in R$ with $u(x_0) = 1$.

2.u is convex fuzzy set

(ie) $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}, \forall t \in [0,1], x, y \in R$.

3.u is upper semi continuous on R. 4.$\overline{\{x \in R, u(x) > 0\}}$ is compact.

Let E be the class of all fuzzy subsets of R.Then E is called the space of fuzzy numbers [11].

Clearly, $R \subset E$ and $R \subset E$ is understood as $R = \{\aleph_x : \aleph \text{ is usual real number}\}$.

An arbitrary fuzzy number is represented by an ordered pair of functions

$(\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$ that satisfies the following requirements.

1.$\underline{u}(r)$is a bounded left continous non-decreasing function over [0,1],with respect to any 'r'.

2. $\overline{u}(r)$ is a bounded right continuous non-increasing function over [0,1] with respect to any 'r'.

3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1.$

Then the r-level set is $[u]_r = \{x \setminus u(x) \geq r\}, 0 < r \leq 1$ is a closed and bounded interval, denoted by $[u]_r = [\underline{u}(r), \overline{u}(r)]$. And clearly, $[u]_0 = \{x \setminus u(x) > 0\}$ is compact.

Definition 2.2. A triangular fuzzy number u is a fuzzy set in E that is characterized by an ordered triple $(u_l, u_c, u_r) \in R^3$ with $u_l \leq u_c \leq u_r$ such that $[u]_0 = [u_l; u_r]$ and $[u]_1 = \{u_c\}$.

The membership function of the triangular fuzzy number u is given by

$$u(x) = \begin{cases} \frac{x-u_l}{u_c-u_l}, & u_l \leq x \leq u_c \\ 1, & x = u_c \\ \frac{u_r-x}{u_r-u_c}, & u_c \leq x \leq u_r \end{cases}.$$

We will write (i) $u > 0, if\ u_l > 0.$ (ii) $u \geq 0, if\ u_l \geq 0.$ (iii) $u < 0, if\ u_c < 0.$

(iv) $u \leq 0, if\ u_c \leq 0.$

Let I be a real interval. A mapping $y: I \to E$ is called a fuzzy process, and its $\alpha - level$ set is denoted by $[y(t)]_\alpha = \left[\underline{y}(t;\alpha), \overline{y}(t;\alpha)\right], t \in I, 0 < \alpha \leq I$. The seikkala derivative $y'(t)$ of a fuzzy process is defined by $[y'(t)]_\alpha = \left[\underline{y}'(t;\alpha), \overline{y}'(t;\alpha)\right], t \in I, 0 < \alpha \leq I$, provided that this equation defines a fuzzy number, as in [19].

Lemma2.1. Let $u, v \in E$ and s scalar, then for $r \in (0,1]$

$$[u+v]_r = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)]$$

$$[u-v]_r = [\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)]$$

$$[u,v]_r = [\min\{\underline{u}(r).\underline{v}(r), \underline{u}(r).\overline{v}(r), \overline{u}(r).\underline{v}(r), \overline{u}(r).\overline{v}(r)\},$$
$$\max\{\underline{u}(r).\underline{v}(r), \underline{u}(r).\overline{v}(r), \overline{u}(r).\underline{v}(r), \overline{u}(r).\overline{v}(r)\}],$$

$$[su]_r = s\,[u]_r.$$

3. A Fuzzy Cauchy Problem

Consider the fuzzy initial value problem

$$\begin{cases} y'(t) = f\big(t, y(t)\big), t \in I = [0,T] \\ y(0) = y_0. \end{cases} \quad (3.1)$$

Where f is a continuous mapping from $R_+ \, X \, R$ into R and $y_0 \in E$ with r-level sets

$$[y_0]_r = \left[\underline{y}(0,r), \overline{y}(0,r)\right], r \in (0,1].$$

The extension principle of Zadeh leads to the following definition of $f(t,y)$ when $y = y(t)$ is a fuzzy number, $f(t,y)(s) = \sup\{y(\tau) \setminus s = f(t,\tau)\}, s \in R.$

It follows that, $[f(t,y)]_r = \left[\underline{f}(t,y;r), \overline{f}(t,y;r)\right], r \in (0,1]$,

Where $\underline{f}(t,y;r) = \min\left\{f(t,u) \setminus u \in \left[\underline{y}(r), \overline{y}(r)\right]\right\}$, $\overline{f}(t,y;r) = \max\left\{\overline{f}(t,u) \setminus u \in \left[\underline{y}(r), \overline{y}(r)\right]\right\}$.

Theorem3.1. Let f satisfy $\left|f(t,v) - f(t,\overline{v})\right| \leq g(t, |v - \overline{v}|), t \geq 0, v, \overline{v} \in R,$ (3.2)

where $g: R_+ X\, R_+ \to R_+$ is a continuous mapping such that $r \to g(t,r)$ is non decreasing, the initial value problem $u'(t) = g\big(t, u(t)\big), u(0) = u_0,$ (3.3)

has a solution on R_+ for $u_0 > 0$ and that $u(t) \equiv 0$ is the only solution of (3.3) for $u_0 = 0$. Then the fuzzy initial value problem (3.1) has a unique fuzzy solution.

Proof: see [19].

In this paper we suppose (3.1) satisfies the hypothesis of theorem3.1, also.

4 .The Second Order Runge –Kutta Method with Harmonic Mean

Let the exact solution $[Y(t)]_r = [\underline{Y}(t;r), \overline{Y}(t;r)]$ is approximated by some $[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)]$.From (2.7) to (2.9), we define

$$\underline{y}(t_{n+1};r) - \underline{y}(t_n;r) = 2\frac{\underline{k_1}(t_n,y(t_n;r))\ \underline{k_2}(t_n,y(t_n;r))}{\underline{k_1}(t_n,y(t_n;r)) + \underline{k_2}(t_n,y(t_n;r))} \qquad (4.1)$$

$$\overline{y}(t_{n+1};r) - \overline{y}(t_n;r) = 2\frac{\overline{k_1}(t_n,y(t_n;r))\ \overline{k_2}(t_n,y(t_n;r))}{\overline{k_1}(t_n,y(t_n;r)) + \overline{k_2}(t_n,y(t_n;r))} \qquad (4.2)$$

$$\underline{k_1}(t,y(t;r)) = \min\{h.f(t,u) \setminus u \in [\underline{y}(t,r), \overline{y}(t,r)]\} \qquad (4.3)$$

$$\overline{k_1}(t,y(t,r)) = \max\{h.f(t,u) \setminus u \in [\underline{y}(t,r), \overline{y}(t,r)]\} \qquad (4.4)$$

$$\underline{k_2}(t,y(t,r)) = \min\{h.f(t,u) \setminus u \in [\underline{z_1}(t,y(t,r)), \overline{z_1}(t,y(t,r))]\}. \qquad (4.5)$$

$$\overline{k_2}(t,y(t,r)) = \max\{h.f(t,u) \setminus u \in [\underline{z_1}(t,y(t,r)), \overline{z_1}(t,y(t,r))]\} \qquad (4.6)$$

Where

$$\underline{z_1}(t,y(t,r)) = \underline{y}(t,r) + \underline{k_1}(t,y(t,r)) + \frac{9}{10}\underline{a}$$

$$\overline{z_1}(t,y(t,r)) = \overline{y}(t,r) + \overline{k_1}(t,y(t,r)) + \frac{9}{10}\overline{a}$$

$$\underline{a} = \min\{h.f_y(t,u).v \setminus u \in [\underline{y}(t,r), \overline{y}(t,r)]\ \&v \in [\underline{k_1}(t,y(t;r)), \overline{k_1}(t,y(t;r))]\}$$

$$\overline{a} = \max\{h.f_y(t,u).v \setminus u \in [\underline{y}(t,r), \overline{y}(t,r)]\ \&v \in [\underline{k_1}(t,y(t;r)), \overline{k_1}(t,y(t;r))]\}$$

Define,

$$F[t,y(t;r)] = 2\frac{\underline{k_1}(t,y(t;r))\underline{k_2}(t,y(t;r))}{\underline{k_1}(t,y(t;r)) + \underline{k_2}(t,y(t;r))} \qquad (4.7)$$

$$G[t,y(t;r)] = 2\frac{\overline{k_1}(t,y(t;r))\ \overline{k_2}(t,y(t;r))}{\overline{k_1}(t,y(t;r)) + \overline{k_2}(t,y(t;r))} \qquad (4.8)$$

The exact and approximate solutions at $t_n, 0 \le n \le N$, are denoted by $[Y(t_n)]_r = [\underline{Y}(t_n;r), \overline{Y}(t_n;r)]$ and $[y(t_n)]_r = [\underline{y}(t_n;r), \overline{y}(t_n;r)]$,respectively

The solution is calculated by grid points at $a = t_0 \le t_1 \le t_2 \le \cdots \ldots \ldots \ldots \ldots \ldots \le t_N = b$ and $h = \frac{(b-a)}{N} = t_{n+1} - t_n$.

Therefore, we have

$$\underline{Y}(t_{n+1};r) = \underline{Y}(t_n;r) + F[t_n, Y(t_n,r)] \qquad (4.9)$$

$$\overline{Y}(t_{n+1};r) = \overline{Y}(t_n;r) + G[t_n, Y(t_n,r)] \qquad (4.10)$$

And $\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + F[t_n, y(t_n,r)]$ (4.11)

$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + G[t_n, y(t_n,r)] \qquad (4.12)$$

The following lemmas will be applied to show the convergences of theses approximates.
i.e., $\lim_{h\to 0} \underline{y}(t,r) = \underline{Y}(t,r)$ and $\lim_{h\to 0} \overline{y}(t,r) = \overline{y}(t,r)$.

Lemma: 4.1 Let a sequence of numbers $\{W_n\}_{n=0}^{N}$ satisfy $|W_{n+1}| \le A|W_n| + B,\ 0 \le n \le N-1$, for some given positive constants A and B, then $|W_n| \le A^n|W_0| + B\frac{A^n-1}{A-1}, 0 \le n \le N-1.$

Proof: see [14]

Lemma: 4.2 Let the sequence of numbers $\{W_n\}_{n=0}^{N}$, $\{V_n\}_{n=0}^{N}$ satisfy $|W_{n+1}| \leq |W_n| + Amax\{|W_n|, |V_n|\} + B,\ |V_{n+1}| \leq |V_n| + Amax\{|W_n|, |V_n|\} + B,$
for some given positive constants A and B, and denote $U_n = |W_n| + |V_n|, 0 \leq n \leq N.$

Then $U_n \leq \overline{A}^n U_0 + \overline{B}\frac{\overline{A}^n - 1}{\overline{A} - 1}, 0 \leq n \leq N$, where $\overline{A} = 1 + 2A$ and $\overline{B} = 2B$

Proof: see [14].

Let $F(t, u, v)$ and $G(t, u, v)$ be obtained by substituting $[y(t)]_r = [u, v]$ in (4.7&4.8),

$$F[t, u, v] = 2\frac{\underline{K_1}[t,u,v]\underline{K_2}\,[t,u,v]}{\underline{K_1}[t,u,v] + \underline{K_2}\,[t,u,v]},\quad G[t, u, v] = 2\frac{\overline{K_1}[t,u,v]\overline{K_2}[t,u,v]}{\overline{K_1}[t,u,v] + \overline{K_2}[t,u,v]}.$$

The domain where F and G are defined is therefore

$$K = \{(t, u, v)\backslash 0 \leq t \leq T, -\infty < v < \infty, -\infty < u \leq v\}.$$

Theorem 4.1: let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^2\,(K)$ and let the partial derivatives of F and G be bounded over K .Then, for arbitrary fixed $r, 0 \leq r \leq 1$, the approximate solutions (4.11&4.12) converge to the exact solutions $\underline{Y}(t; r)$ and $\overline{Y}(t; r)$ uniformly in t.

Proof: See [14].

5 .Numerical Example

Example 1. Consider the fuzzy initial value problem,

$$\begin{cases} y'(t) = y(t), t \in [0,1], \\ y(0) = (0.75 + 0.25r, 1.125 - 0.125r), 0 < r \leq 1. \end{cases}$$

The exact solution is given by $\underline{Y}(t; r) = \underline{y}(t; r)e^t, \overline{Y}(t; r) = \overline{y}(t; r)e^t$, which at $t = 1$,

$$Y(1; r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e], 0 < r \leq 1.$$

The exact and approximate solutions obtained by the Euler method and by the Runge-Kutta method of order two with new parameters [New RK] [15] and by the proposed New Multi-Step Runge-Kutta method [New MSRK] with ‘$h = 0.1$’ are given in Table: 1

6. Conclusion

In this work, we have used the proposed second-order Runge-Kutta method to find a numerical solution of fuzzy differential equations. Taking into account the convergence order of the Euler method is $O(h)$, a higher order of convergence $O(h^3)$ is obtained by the proposed method and by the method proposed in [15]. Comparison of the solutions of example 5.1 shows that the proposed method gives a better solution than the Euler method and by the Runge-Kutta method of order 2 proposed in [15].

References

[1]Ababneh, O.Y, Ahmad, R., and Ismail, E.S (2009), “New Multi-Step Runge-Kutta Method”, Applied Mathematical Sciences, **3**(45) pp.2255-2262.

[2] Abbasbandy, S., Allah Viranloo, T.(2002), “Numerical Solution of fuzzy differential equations by Taylor method”, *Journal of Computational Methods in Applied Mathematics* **2**(2),pp.113-124.

[3]Abbasbandy, S., Allah Viranloo, T. (2004), “Numerical solution of fuzzy differential equations by Runge-Kutta method”, *Nonlinear studies* .**11**(1),pp. 117-129.

[4]Buckley, J.J., Eslami, E., and Feuring, T.(2002),“Fuzzy Mathematics in Economics and Engineering”, Heidelberg, Germany: Physica-verlag

[5] Buckley, J.J, Feuring, T (2000), “Fuzzy differential equations”, *Fuzzy sets and Systems* 110, pp.43-54.

[6] Butcher, J.C (1987), “The Numerical Analysis of Ordinary Differential equations Runge-Kutta and General Linear Methods”, New York: Wiley

[7] Chang, S.L.,.Zadeh, L.A. (1972), "On fuzzy mapping and control", *IEEETransactions on Systems Man cybernetics*.**2**(1), pp.30-34.

[8] Dubois, D., Prade, H. (1982), "Towards fuzzy differential calculus: Part3, Differentiation", *Fuzzy sets and systems* 8, pp.2 25-233.

[9]Goeken, D., Johnson.(2000), "Runge–Kutta with higher order derivative Approximations" *Applied. Numerical Mathematics* 34, pp.207-218.

[10] Goetschel, R.., Voxman, W. (1986), "Elementary Fuzzy Calculus", *Fuzzy Sets and Systems* 18pp.31-43.

[11] Kaleva, O. (1987), "Fuzzy differential equations", *Fuzzy sets and systems* 24pp.301-317.

[12] Kaleva, O. (1990), "The Cauchy problem for Fuzzy differential equations", *Fuzzy sets and systems* 35, pp.389-396.

[13] Lambert, J.D (1990), "Numerical methods for Ordinary differential systems", New York: Wiley

[14] Ma, M., Friedman, M., Kandel, A (1999), "Numerical solutions of fuzzy differential equations", *Fuzzy sets and System* 105, pp. 133-138.

[15] Nirmala, V., Saveetha, N., Chenthurpandian, S. (2010), "Numerical Solution of Fuzzy Differential Equation by Runge-Kutta Method with Higher Order Derivative Approximations", *Proceedins of the International Conference on Emerging Trends in Mathematics and Computer Applications*, India: MEPCO Schlenk Engineering College, Sivakasi Tamil Nadu, pp.131-134(ISBN: 978-81-8424-649-0).

[16] Nirmala, V., Chenthurpandian, S.(2011), "Numerical Solution of Fuzzy Differential Equation by Fourth Order Runge-Kutta Method with Higher Order Derivative Approximations", *European Journal of Scientific Research*, **62** (2)pp.198-206.

[17]Palligkinis,S.Ch.,Papageorgiou,G.,Famelis,I.TH.(2009),"Runge-Kutta methods for fuzzy differential equations", *Applied Mathematics Computation,* 209,pp.97-105.

[18] Puri, M.L., .Ralescu, D.A (1983), "Differentials of fuzzy functions", *Journal of Mathematical Analysis and Applications*, 91, pp.552-558.

[19] Seikkala, S. (1987), "On the Fuzzy initial value problem", *Fuzzy sets and systems* 24, pp.319-330.

Table: 1 Comparison of Results

r	Exact solution	Euler's Approximation	New RKApproximation	New MSRK Approximation
0	2.0387113,3.0580670	1.9453068,2.9179603	2.0386329,3.0579494	2.0386636,3.0579953
0.2	2.1746254,2.9901100	2.074994,2.8531167	2.1745418,2.989995	2.1745745,2.9900399
0.4	2.3105395,2.9221529	2.2046811,2.7882731	2.3104507,2.9220406	2.3104854,2.9220844
06	2.4464536,2.8541959	2.3343682,2.7234296	2.4463595,2.8540861	2.4463963,2.854129
0.8	2.5823677,2.7862388	2.4640553, 2.658586	2.5822684,2.7861317	2.5823072,2.7861735
1	2.7182818,2.7182818	2.5937425,2.5937425	2.7181773,2.7181773	2.7182181,2.7182181