New numerical scheme for solving Troesch's Problem

Abdelmajid El hajaji [1*], Khalid Hilal [1], El ghordaf Jalila [1] and El merzguioui Mhamed [2]

[1] Department of Mathematics, Faculty of Science and Technology,

University Sultan Moulay Slimane, Beni-Mellal, Morocco

[2] Department of Mathematics, Faculty of Science and Technology,

University Abdelmalek Essaadi, Tanger, Morocco

[*]E-mail: a_elhajaji@yahoo.fr

Abstract

In this paper, we will manipulate the cubic spline to develop a collocation method (CSCM) and the generalized Newton method for solving the nonlinear Troesch problem. This method converges quadratically if a relation-ship between the physical parameter λ and the discretization parameter h is satisfied. An error estimate between the exact solution and the discret solution is provided. To validate the theoretical results, Numerical results are presented and compared with other collocation methods given in the literature.

Keywords: Troesch problem, Boundary value problems, Cubic spline collocation method.

1. Introduction

Consider a two-point boundary value problem, Troesch's problem, as follows:

$$\begin{cases} \Delta u = \lambda \sinh(\lambda u) \quad \text{on} \ (0,1), \\ u(0) = 0, \ u(1) = 1, \end{cases}$$

where λ is a positive constant. Troesch's problem is discussed by Weibel and arises in the investigation of the confinement of a plasma column by radiation pressure [1] and also in the theory of gas porous electrodes [2,3]. The closed form solution to this problem in terms of the Jacobian elliptic function has been given [4] as

$$u(x) = \frac{2}{\lambda} \sinh^{-1}\left[\frac{u'(0)}{2} sc(\lambda x \,|\, 1 - \frac{1}{4}u'^2 (0))\right], \tag{1}$$

where $u'(0)$, the derivative of u at 0, is given by the expression $u'(0) = 2\sqrt{1-m}$, with m being the solution of the transcendental equation

$$\frac{\sinh\left(\dfrac{\lambda}{2}\right)}{\sqrt{1-m}} = sc(\lambda \,|\, \mathrm{m}).$$

where the Jacobian elliptic function $sc(\lambda \,|\, \mathrm{m})$ is defined by $sc(\lambda \,|\, \mathrm{m}) = \dfrac{\sin(\phi)}{\cos(\phi)}$, where ϕ, λ are related

by the integral

$$\lambda = \int_0^\phi \frac{1}{\sqrt{1-m\sin^2\theta}} \, d\theta.$$

From (1), it was noticed that a pole of $u(t)$ occurs at a pole of $sc(\lambda x \,|\, 1 - \frac{1}{4} u'^2 (0))$. It was also noticed that

the pole occurs at $x \approx \dfrac{1}{2\lambda} \ln\left(\dfrac{16}{1-m}\right)$.

It also has an equivalent definition given in terms of a lattice.

Troesch's problem has been solved by another method. M. Zarebnia et al. [4] have introduced an sinc-Galerkin method for solving this problem; their method is based on the modified homotopy perturbation technique. They have compared this method with homotopy perturbation method (HPM), Laplace method, perturbation method and spline method. The discontinuous Galerkin finite element (DG) method is applied for solving Troesch's problem by H. Temimi [5]. Mohamed El-Gamel [6] have introduced an efficient algorithm based on the sinc-collocation technique, they have compared this method with the modified homotopy perturbation technique (MHP), the variational iteration method and the Adomian decomposition method. M. Zarebnia, M. Sajjadian [7] have introduced an efficient algorithm based on The sinc-Galerkin method, they have compared this method with [5, 6]. In another article, Mustafa Inc, Ali Akgül [13] have introduced the reproducing kernel Hilbert space method (RKHSM) is applied for solving Troesch's problem. They have compared this method with the homotopy perturbation method (HPM), the Laplace decomposition method (LDM), the perturbation method (PM), the Adomian decomposition method (ADM), the variational iteration method (VIM), the B-spline method and the nonstandard finite difference scheme (FDS).

In this paper we develop a numerical method for solving a one dimentional Troesch's problem by using the CSCM and the generalized Newton method. First, we apply the spline collocation method to approximate the solution of a boundary value problem of second order. The discret problem is formulated as to find the cubic spline coefficients of a nonsmooth system $\varphi(Y) = Y$, where $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^m$. In order to solve the nonsmooth equation we apply the generalized Newton method (see [15, 16, 17], for instance). We prove that the CSCM converges quadratically provided that a property coupling the parameter λ and the discretization parameter h is satisfied.

Numerical methods to approximate the solution of boundary value problems have been considered by several authors. We only mention the papers [12] and references therein, which use the spline collocation method for solving the boundary value problems.

The cubic B-spline collocation method is widely used in practice because it is computationally inexpensive, easy to implement and gives high-order accuracy. In [12] the authors solved a   the Troesch's problem by using third degree splines, where they considerer the collocation points as the knots of the cubic spline space. In our paper we consider a cubic spline space defined by multiple knots in the boundary and we propose a simple and efficient new collocation method by considering as collocation points the mid-points of the knots of the cubic spline space. It is observed that the collocation method developed in this paper, when applied to some examples, can improve the results obtained by the collocation methods given in the literature (see [12], for instance).

The present paper is organized as follows. In Section 2, we construct a cubic spline to approximate the solution of the boundary problem. Section 3 is devoted to the presentation of the generalized Newton method and we show the convergence of the cubic spline to the solution of the boundary problem and provide an error estimate. Finally, some numerical results are given in Section 4 to validate our methodology.

In this paper, we shall apply CSCM to find the approximate analytical solution of the boundary and initial value problem of the Troesch problem. Comparisons with the exact solution shall be performed.

2. Cubic spline collocation method

In this section, the cubic spline collocation method is developed and implemented for solving the Troesch's problem defined by

$$\begin{cases} \Delta u = J(.,u) \quad \text{on } (0,1), \\ u(0) = 0, \ u(1) = 1, \end{cases} \tag{2}$$

with $J = \lambda \sinh(\lambda u)$, where λ is a positive constant.

It is easy to see that J is a nonlinear continuous function on u; and for any two functions u and v, J satisfies the following Lipschitz condition:

$$| J(x,u(x)) - J(x,v(x)) | \leq \frac{\lambda^2}{2} | u(x) - v(x) | \quad a.e. \ on \ x \in I = (0,1). \tag{3}$$

In order to implement the cubic spline collocation method, we first create a subdivision of the interval $(0,1)$

$$\tau = \{0 = x_{-3} = x_{-2} = x_{-1} = x_0 < x_1 < \cdots < x_{n-1} < x_n = x_{n+1} = x_{n+2} = x_{n+3} = 1\}, \quad \text{without} \quad \text{loss} \quad \text{of}$$

generality, we put $x_i = ih$, where $0 \leq i \leq n$ and $h = 1/n$.

Denote by $\overset{\circ}{S_4}(I,\tau) = \{s \in C^2(I) \cap C^0(\bar{I}), s(0) = s(1) = 0, s_{x_i,x_{i+1}} \in P(x_i, x_{i+1})\}$ the space of piecewise polynomials of degree 3 over the subdivision τ and of class C^2 everywhere on I and class C^0 everywhere on $\bar{I}$. Note that $\overset{\circ}{S_4}(I,\tau) \subseteq H^1(I)$. Let B_i, $i = -3, \cdots, n-1$, be the B-splines of degree 3 associated with τ. These B-splines are positives and form a basis of the space $\overset{\circ}{S_4}(I,\tau)$.

Now, we define the following interpolation cubic spline of the solution u of the nonlinear second order boundary value problem (2).

Proposition 3.1: *Let u be the solution of problem (2). Then, there exists a unique cubic spline interpolant $Sp \in \overset{\circ}{S_4}(I,\tau)$ of u which satisfies:*

$$Sp(t_i) = u(t_i), \quad i = 0, \cdots, n+2,$$

where $t_0 = x_0$, $t_i = \dfrac{x_{i-1} + x_i}{2}$, $i = 1, \cdots, n$, $t_{n+1} = x_{n-1}$ and $t_{n+2} = x_n$.

Proof: Using the Schoenberg-Whitney theorem (see [8]), it is easy to see that there exits a unique cubic spline which interpolates u_ε at the points t_i, $i = 0, \cdots, n+2$.

If we put $Sp = \sum_{i=-3}^{n-1} c_i B_i$, then by using the boundary conditions of problem (2) we obtain $c_{-3} = Sp(0) = u(0) = 0$ and $c_{n-1} = Sp(1) = u(1) = 1$. Hence

$$Sp = \mu + S \quad \text{where} \quad \mu = c_{n-1}B_{n-1} \quad \text{and} \quad S = \sum_{i=-2}^{n-2} c_i B_i.$$

Furthermore, since the interpolation with splines of degree d gives uniform norm errors of order $O(h^{d+1})$ for the interpolant, and of order $O(h^{d+1-r})$ for the rth derivative of the interpolant (see [8], for instance), then for any $u \in C^2(I) \cap C^0(I)$ (see [9]), we have

$$-\Delta S(t_i) = G(t_i,u) + O(1), \quad i = 1, \cdots, n+1. \tag{4}$$

where $G(t_i, u) = J(t_i, u) - \mu''(t_i),$

The cubic spline collocation method, that we present in this paper, constructs numerically a cubic spline $\tilde{S} = \sum_{i=-3}^{n-1} \tilde{c}_i B_i$ which satisfies the equation (2) at the points $t_i, \ i = 0, \cdots, n+2$. It is easy to see that

$$\tilde{c}_{-3} = 0 \ \ and \ \ \tilde{c}_{n-1} = 1,$$

and the coefficients $\tilde{c}_i, \ i = -2, \cdots, n-2$, satisfy the following nonlinear system with $n+1$ equations:

$$-\sum_{i=-2}^{n-2} \tilde{c}_i \Delta B_i(t_j) = G(t_j, \sum_{i=-2}^{n-2} \tilde{c}_i B_i(t_j)), \quad j = 1, \cdots, n+1. \tag{5}$$

Relations (4) and (5) can be written in the matrix form, respectively, as follows

$$\hat{A} C = -F - E,$$
$$\hat{A} \tilde{C} = -F_{\underset{C}{\sim}}, \tag{6}$$

where

$$F = [G(t_1, u(t_1)), \cdots, G(t_{n+1}, u(t_{n+1}))]^T,$$

$$F_{\underset{C}{\sim}} = [G(t_1, \tilde{S}(t_1)), \cdots, G(t_{n+1}, \tilde{S}(t_{n+1}))]^T,$$

and E is a vector where each component is of order $O(1)$. It is well known that $\hat{A} = \dfrac{1}{h^2} A$, where A is a matrix independent of h, with the matrix A is invertible [10].

Then, relation (6) becomes

$$AC = -h^2 F - E,$$
$$A \tilde{C} = -h^2 F_{\underset{C}{\sim}}, \tag{7}$$

Theorem 3.1 *Assume that the penalty parameter λ and the discretization parameter h satisfy the following relation:*

$$h^2 \lambda^2 \| A^{-1} \|_\infty < 2. \tag{8}$$

Then there exists a unique cubic spline which approximates the exact solution u of problem (2).

Proof: From relation (7), we have $\tilde{C} = -h^2 A^{-1} F_{\underset{C}{\sim}}$. Let $\varphi : \mathsf{R}^{n+1} \to \mathsf{R}^{n+1}$ be a function defined by

$$\varphi(Y) = -h^2 A^{-1} F_{\underset{Y}{\sim}}. \tag{9}$$

To prove the existence of cubic spline collocation it suffices to prove that φ admits a unique fixed point.

Indeed, let Y_1 and Y_2 be two vectors of R^{n+1}. Then we have

$$\| \varphi(Y_1) - \varphi(Y_2) \| \leq h^2 \| A \|_\infty \| F_{Y_1} - F_{Y_2} \|_\infty . \tag{10}$$

Using relation (4) and the fact that $\sum_{j=-2}^{n-2} B_j \leq 1$, we get

$$| G(t_i, S_{Y_1}(t_i)) - G(t_i, S_{Y_2}(t_i)) | \leq \frac{\lambda^2}{2} | S_{Y_1}(t_i) - S_{Y_2}(t_i) | \leq \frac{\lambda^2}{2} \| Y_1 - Y_2 \|_\infty .$$

Then we obtain

$$\| F_{Y_1} - F_{Y_2} \|_\infty \leq \frac{\lambda^2}{2} \| Y_1 - Y_2 \|_\infty .$$

From relation (10), we conclude that

$$\| \varphi(Y_1) - \varphi(Y_2) \| \leq \frac{\lambda^2}{2} h^2 \| A^{-1} \|_\infty \| Y_1 - Y_2 \|_\infty .$$

Then we have

$$\| \varphi(Y_1) - \varphi(Y_2) \| \leq k \| Y_1 - Y_2 \|_\infty,$$

with $k = \frac{\lambda^2}{2} h^2 \| A^{-1} \|_\infty$, by relation (8). Hence the function φ admits a unique fixed point.

In order to calculate the coefficients of the cubic spline collocation given by the nonsmooth system

$$\tilde{C} = \varphi(\tilde{C}), \tag{11}$$

we propose the generalized Newton method defined by

$$\tilde{C}^{(k+1)} = \tilde{C}^{(k)} - (I_{n+1} - V_k)^{-1}(\tilde{C}^{(k)} - \varphi(\tilde{C}^{(k)})), \tag{12}$$

where I_{n+1} is the unit matrix of order $n+1$ and V_k is the generalized Jacobian of the function

$\tilde{C} \mapsto \varphi(\tilde{C})$, (see [15, 16, 17], for instance).

3. Convergence of the method

Theorem 3.1 *If we assume that the penalty parameter λ and the discretization parameter h satisfy the following relation*

$$h^2 \lambda^2 \| A^{-1} \|_\infty < 1. \tag{13}$$

then the cubic spline $\tilde{S}$ converges to the solution u. Moreover the error estimate $\| u - \tilde{S} \|_\infty$ is of order

$O(h^2)$.

Proof: From (7) and the matrix A is invertible [10], we have

$$C - \tilde{C} = -h^2 A^{-1}(F - F_{\underset{C}{\sim}}) - A^{-1}E.$$

Since E is of order $O(h^2)$, then there exists a constant K_1 such that $\| E \|_\infty \leq k_1 h^2$. Hence we have

$$\| C - \tilde{C} \|_\infty \leq h^2 \| A^{-1} \|_\infty \| F - F_{\underset{C}{\sim}} \|_\infty + K_1 \| A^{-1} \|_\infty h^2. \qquad (14)$$

On the other hand we have

$$\begin{aligned}
| G(t_i, u(t_i)) - G(t_i, \tilde{S}(t_i)) | \;\; &\leq \;\; \frac{\lambda^2}{2} | u(t_i) - \tilde{S}(t_i) | \\
&\leq \;\; \frac{\lambda^2}{2} | u(t_i) - S(t_i) | + \frac{\lambda^2}{2} | S(t_i) - \tilde{S}(t_i) |.
\end{aligned}$$

Since S is the cubic spline interpolation of u, then there exists a constant K_2 such that (see [9]),

$$\| u - S \|_\infty \leq K_2 h^2. \qquad (15)$$

Using the fact that

$$| S - \tilde{S} | \leq \| C - \tilde{C} \|_\infty \sum_{j=-2}^{n-2} B_j \leq \| C - \tilde{C} \|_\infty, \qquad (16)$$

then, we obtain

$$| F - F_{\underset{C}{\sim}} | \leq \frac{\lambda^2}{2} \| C - \tilde{C} \|_\infty + \frac{\lambda^2}{2} K_2 h^2.$$

By using relation (14) and assumption (13) it is easy to see that

$$\begin{aligned}
\| C - \tilde{C} \|_\infty \;\; &\leq \;\; \frac{h^2 \| A^{-1} \|_\infty}{1 - \frac{\lambda^2}{2} h^2 \| A^{-1} \|_\infty} (K_2 \frac{\lambda^2}{2} h^2 + K_1) \\
&\leq \;\; 2 \| A^{-1} \|_\infty (K_2 \frac{\lambda^2}{2} h^2 + K_1) h^2.
\end{aligned} \qquad (17)$$

We have

$$\| u - \tilde{S} \|_\infty \leq \| u - S \|_\infty + \| S - \tilde{S} \|_\infty .$$

Then from relations (15), (16) and (17), we deduce that $\| u - \tilde{S} \|_\infty$ is of order $O(h^2)$. Hence the proof is complete.

Remark 3.1: Theorem 3.1 provides a relation coupling the parameter λ and the discretization parameter h, which guarantees the quadratic convergence of the cubic spline collocation $\tilde{S}$ to the solution u of the Troesch's problem (2).

4. Numerical examples

In this section, we solve Troesch's problem for different values of the parameter λ using the computer algebra system Matlab and make a comparison between our results and those ones reported in the literature to confirm the efficiency and accuracy of our method.

Consider the Troesch's equation as follows, when parameter λ =0.5 and 1.

The maximum absolute errors in solutions of this problem are compared with methods in [7,11,12,13,14] for $h = 1/10$ and tabulated in Tables 1 and 2. The tables show that our results are more accurate.

Table 1.   Absolute errors for λ =0.5.

x	Present	RKHSM [13]	Wavelet[14]	SGM.E[7]	HPM.E[11]	LM.E[12]	PM.E[12]	SM.E[12]
0.1	6.2E-12	7.0E-11	7.6E-04	7.67E-04	7.71E-04	7.7E-04	8.2E-04	7.7E-04
0.2	3.8E-11	3.4E-09	1.5E-03	1.49E-03	1.50E-03	1.5E-03	1.6E-03	1.5E-03
0.3	5.1E-11	4.0E-10	2.1E-03	2.14E-03	2.15E-03	2.1E-03	2.3E-03	2.1E-03
0.4	6.6E-11	7.6E-09	2.7E-03	2.66E-03	2.67E-03	2.7E-03	2.9E-03	2.7E-03
0.5	7.8E-11	5.0E-09	3.0E-03	3.00E-03	3.02E-03	3.0E-03	3.2E-03	3.0E-03
0.6	8.4E-11	2.2E-09	3.1E-03	3.13E-03	3.14E-03	3.1E-03	3.4E-03	3.1E-03
0.7	7.8E-11	2.5E-09	3.0E-03	2.96E-03	2.97E-03	3.0E-03	3.2E-03	3.0E-03
0.8	5.4E-11	5.0E-10	2.4E-03	2.44E-03	2.45E-03	2.4E-03	2.7E-03	2.4E-03
0.9	3.8E-11	6.1E-09	1.5E-03	1.48E-03	1.49E-03	1.5E-03	1.6E-03	1.5E-03

Table 2.   Absolute errors for λ =1.

x	Present	RKHSM [13]	Wavelet[14]	SGM.E[7]	HPM.E[11]	LM.E[12]	PM.E[12]	SM.E[12]
0.1	3.43E-09	4.39E-07	2.8E-03	2.86E-03	3.13E-03	2.9E-03	3.6E-03	2.8E-03
0.2	6.76E-09	9.42E-08	5.6E-03	5.64E-03	6.16E-03	5.9E-03	7.1E-02	5.6E-03
0.3	9.21E-09	8.29E-07	8.2E-03	8.22E-03	8.96E-03	8.2E-03	1.0E-02	8.2E-03
0.4	2.65E-08	2.49E-07	1.0E-02	1.04E-03	1.13E-02	1.0E-02	1.3E-02	1.0E-02
0.5	3.87E-08	3.04E-07	1.2E-02	1.22E-03	1.32E-02	1.2E-02	1.6E-02	1.2E-02
0.6	4.43E-08	9.91E-07	1.3E-02	1.32E-03	1.42E-02	1.3E-02	1.7E-02	1.3E-02
0.7	3.78E-08	4.74E-07	1.3E-02	1.31E-03	1.40E-02	1.3E-02	1.7E-02	1.3E-02
0.8	2.93E-08	3.45E-07	1.1E-02	1.14E-03	1.20E-02	1.1E-02	1.5E-02	1.1E-02
0.9	6.32E-09	9.41E-07	7.4E-03	7.39E-03	7.76E-03	7.4E-03	9.7E-03	7.4E-03

6. Concluding remarks

In this paper, we have consider an approximation of a Troesch equation problem, presented in [2,3]. Then we have developed a numerical method for solving each nonsmooth equation, based on a cubic collocation spline method and the generalized Newton method. We have shown the convergence of the method provided that the physical and the discretization parameters satisfy the relation (13). Moreover we have provided an error estimate of order $O(h^2)$ with respect to the norm $\| \cdot \|_\infty$. The obtained numerical results show the convergence of the approximate solutions to the exact one and confirm the error estimates provided in this paper. The analytical results are illustrated with two numerical examples. The proposed scheme is simple and computationally attractive, and shows a very high precision comparing with many other existing numerical methods.

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