Advances in Physics Theories and Applications
ISSN 2224-719X (Paper) ISSN 2225-0638 (Online)
Vol 4, 2012

On a Non-variational Viewpoint in Newtonian Mechanics

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Abstract

Certain features of the works of Dyson (Am. J. Phys. 58(1990) 209), Hojman and Shepley (J. Math. Phys. 32(1991) 142), and, Hughes (Am.J. Phys. 60(1992) 142) are integrated with our definition of the Poisson bracket on a tangent bundle (velocity phase space) to address the Noether theorem inversion and inverse problem in the Newtonian mechanics from a novel viewpoint without assuming apriori a Lagrangian or Hamiltonian . From this standpoint we study the existence, uniqueness and form of the Lagrangian, and, how to pass from conservation laws to Noether symmetries.

Keywords: Noether theorem, Conservation laws, Symmetry

1. Introduction

From the earliest times philosophers and scientists have tended to reduce the manifold phenomena of nature to a minimum of laws and principles. In this spirit Feynman way back in 1949 wanted to explore as widely as possible the Newtonian mechanics without assuming the existence of Lagrangian, canonical momenta and Hamiltonian. Feynman's work published by Dyson [1] in 1990 has received worldwide attention in the works of several authors such as Hojman and Sheply [2], Hughes [3],etc. In Reference [2], Feynman 's argument of avoiding the Lagrangian for a differential system is shown to deal with the problem of the existence of a Lagrangian itself. Particularly, Hojman and Shepley [2] demonstrate a close connection between the Feynman assumptions and Helmholtz conditions, i.e. , the necessary and sufficient conditions under which a system of differential equations may be derived from a variational principle. It is further shown in Reference [4] that quantum mechanics is not a necessary ingredient of the Feynman procedure. Hughes [4] has developed a version of the Feynman procedure entirely within non-relativistic classical mechanics on cotangent bundle (phase space of coordinates and momenta). The aim of present work is to combine these features with our notion of a modified version of the standard Poisson bracket (section 2) in a manner which helps us to study a classical mechanical system with n degrees of freedom without assuming apriori existence of a Lagrangian or Hamiltonian. As an application of this idea we consider several issues bearing on the inverse problem in Newtonian mechanics [9] in section 3 and the correspondence between symmetries and conservation laws implied by Noether Theorem inversion [8] in section 4 .

2 Poisson bracket on a tangent bundle

The usual framework of Hamiltonian system is the cotangent bundle $T^{\star}$M of an n-dimensional smooth manifold M (the configuration space). For a Lagrangian dynamical system (M,L) one considers the associated tangent bundle TM (the velocity phase space of coordinates and velocities) and a smooth function L : TM → **R** called Lagrangian . For a dynamical system with n degrees of freedom , let $(x^i) = x$ denote the local coordinates on M and $(x^I) = (x^i, \dot{x}^i) = (x, \dot{x})$ denote the associated coordinates on TM, where i = 1,2,…,n, I = 1,2,…,2n. Since evolution in symplectic mechanics is given by a vector field (called the Hamiltonian vector field) it is of interest to consider the Lie algebra χ(TM) of vector fields on TM . For a classical mechanical formulation which does not require apriori that the differential equations of motion follow from a variational principle we may proceed as follows.
Define a dynamical system by the flow system (I =1,2,…,2n)

$$\dot{x}^I = S^I \tag{2.1}$$

for the vector field $S \epsilon \chi(TM)$ with

$$S = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + f^i(x, \dot{x}, t) \frac{\partial}{\partial \dot{x}^i} \tag{2.2}$$

and by the following Poisson bracket relations between positions *x* and velocities $\dot{x}$

$$\{x^i, x^j\} = 0 \tag{2.3}$$

$$\{x^i, \dot{x}^j\} = g^{ij}(x, \dot{x}, t) \tag{2.4}$$

We assume also that the matrix (g^{ij}) is invertible , i.e. $det(g^{ij}) \neq 0$. Such a dynamical system is called regular . In this case denote by (w_{ij}) the inverse of g^{ij} . Here the Poisson bracket {,} denotes an internal binary operation on $C^\infty(TM)$, the ring of real- valued smooth functions on TM . The Poisson bracket satisfies the following axioms :

PB 1 : {, } is real bilinear and antisymmetric

PB 2 : Leibnitz Rule I : $\{f, gh\} = \{f, g\}h + g\{f, h\}$

PB 3 : Jacobi identity : $J\{f, g_1, g_2\} = \{f, \{g_1, g_2\}\} - \{\{f, g_1\}, g_2\} - \{g_1, \{f, g_2\}\} = 0$

PB 4 : Leibnitz Rule II : $L_s\{g_1, g_2\} = \{L_s g_1, g_2\} + \{g_1, L_s g_2\}$

Here L_s denotes the Lie derivative with respect to the vector field S. Restricting oneself to be on the flow of S it is apparent that acting on $\varphi \epsilon C^\infty(TM)$ the Lie derivative yields the total time derivative , i.e. $L_s(\emptyset) = \dot{\emptyset}$. This is the reason why L_s is also known as the on shell time derivative.

Let us apply Leibnitz rule II to the bracket relations given above. Application of that rule to (2.3) gives

$$\{x^j, \dot{x}^i\} = \{x^i, \dot{x}^j\}$$

That is the matrix (g^{ij}) is symmetric. Application of that rule to (2.4) gives on antisymmetrization

$$\{\dot{x}^i, x^j\} = -\frac{1}{2}\{x^i, f^j\} + \frac{1}{2}\{x^j, f^i\}$$

$$= -\frac{1}{2}\left(g^{ik}\frac{\partial f^j}{\partial \dot{x}^k} - g^{jk}\frac{\partial f^i}{\partial \dot{x}^k}\right) \tag{2.5}$$

where the last result is a consquence of (2.3) and Leibnitz rule I .

On setting aside the acceleration dependent terms on the right hand side in (2.5) we may write $(f^j = g^{jl} f_l)$

$$\{\dot{x}^i, \dot{x}^j\} = -g^{ik}g^{jl}t_{kl} + \frac{1}{2}\left(g^{jk}\frac{\partial g^{il}}{\partial \dot{x}^k} - g^{ik}\frac{\partial g^{jl}}{\partial \dot{x}^k}\right)f_l$$

where $t_{kl} = \frac{1}{2}\left(\frac{\partial f_l}{\partial \dot{x}^k} - \frac{\partial f_k}{\partial \dot{x}^l}\right)$. Consistency demands that

$$g^{ik}\frac{\partial g^{jl}}{\partial \dot{x}^k} = g^{jk}\frac{\partial g^{il}}{\partial \dot{x}^k} \tag{2.6}$$

whereupon $\{\dot{x}^i, \dot{x}^j\} = -g^{ik}g^{jl}t_{kl} = -s^{ij}$

By Leibnitz rule I, {, } acts on each factor as a derivation , whence we can prove that for any Poisson bracket on TM , we have

$$\{F,G\} = \{x^i, x^j\}\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial x^j} + \{x^i, \dot{x}^j\}\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial \dot{x}^j} + \{\dot{x}^i, x^j\}\frac{\partial F}{\partial \dot{x}^i}\frac{\partial G}{\partial x^j} + \{\dot{x}^i, \dot{x}^j\}\frac{\partial F}{\partial \dot{x}^i}\frac{\partial G}{\partial \dot{x}^j}$$

whereupon we are led to the formula

$$\{F,G\} = g^{ij}\left(\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial \dot{x}^j} - \frac{\partial G}{\partial x^i}\frac{\partial F}{\partial \dot{x}^j}\right) - s^{ij}\frac{\partial F}{\partial \dot{x}^i}\frac{\partial G}{\partial \dot{x}^j} \qquad (2.7)$$

For a Lagrangian system the evolution is given in terms of an energy function $h \,\epsilon C^{\infty}(TM)$ such that

$$\dot{x}^i = \{x^i, h\}$$

and

$$\ddot{x}^i = \frac{\partial \dot{x}^i}{\partial t} + \{\dot{x}^i, h\}$$

Some mathematical remarks are in order here. The Jacobi identity also holds for any Poisson bracket on TM. This is the case if and only if the Jacobi identity holds for coordinate functions , i.e. if and only if

$$J(x^i, x^j, x^k) = J(x^i, x^j, \dot{x}^k) = J(x^i, \dot{x}^j, \dot{x}^k) = J(\dot{x}^i, \dot{x}^j, \dot{x}^k) = 0$$

The first identity is trivial.The second one reduces to the validity of (2.6) . For a demonstration of the remaining ones we refer the reader to our work [10] ; suffice it to add here that they can be shown to reduce to the following two identities

$$\frac{\partial t_{ij}}{\partial \dot{x}^k} = \frac{\partial w_{jk}}{\partial x^i} - \frac{\partial w_{ik}}{\partial x^j}$$

$$\frac{\partial t_{ij}}{\partial x^k} + \frac{\partial t_{jk}}{\partial x^i} + \frac{\partial t_{ik}}{\partial x^j} = 0$$

The latter may be derived from the Helmholtz conditions (For a proof of these conditions from Leibnitz rule II for the system under considerations see Reference [10]).

• Example

Consider the linear gyroscopic system

$$M\ddot{x} + S\dot{x} + Vx = 0$$

where $x \,\epsilon\, R^n$, M is a positive definite symmetric $n\times n$ matrix , S is a skewsymmetic matrix and V is a symmetric $n\times n$ matrix. This system may by formulated on a 2n - dimensional tangent bundle along the lines given above by taking the canonical choice $g = M^{-1}$ for the matrix g^{ij}. This system is Lagrangian with the energy function

$$h(x,\dot{x}) = \frac{1}{2}\dot{x}.M\dot{x} + \frac{1}{2}\,x.V(x)$$

and the Poisson bracket on $\boldsymbol{R^{2n}}$ given by

$$\{F,G\} = \frac{\partial F}{\partial x}M^{-1}\frac{\partial G}{\partial x} - \frac{\partial G}{\partial x}M^{-1}\frac{\partial F}{\partial x} - \frac{\partial F}{\partial x}M^{-1}SM^{-1}\frac{\partial G}{\partial x}$$

whose expression follows straightforwardly from (2.7) .

In this case it is easy to verify that the flow system given by $\dot{x} = \{x, h\}$ and $\ddot{x} = \{\dot{x}, h\}$ reproduces the system under consideration .

3 Problems with the Lagrangian formulation

In many instances Newton's equations of motion can be derived from the Euler Lagrange equations. Despite the fact that the Lagrangian is very useful and convenient to compute with ,there seem to be some difficulties about the existence, uniqueness and form of Lagrangian used to define a given dynamical system. This section will illustrate the so called inverse problem of classical mechanics which is specifically concerned with the following issues .

1. **Whether a Lagrangian exists** : As an illustration consider

$$\ddot{x} = -\dot{y} \quad \text{and} \quad \ddot{y} = -y \tag{3.1}$$

It is easy to prove in this case that there is no energy function $h(x, \dot{x}, y, \dot{y})$ on $C^{\infty}(\boldsymbol{R}^{4})$ such that

$$\dot{x} = \{x, h\} \qquad \dot{y} = \{y, h\},$$
$$\ddot{x} = \{\dot{x}, h\} \qquad \ddot{y} = \{\dot{y}, h\}.$$

Therefore the time evolution in this case cannot be given by a Hamiltonian vector field and the above system of differential equations has no Lagrangian.

As a trial let $\{x, \dot{x}\} = 1$ and $\{x, y\} = 0$

Repeated application of Leibnitz rule II gives in this case

$$\{x, \dot{y}\} = \{\dot{x}, \dot{y}\} = \{\dot{x}, y\} = \{\dot{y}, y\} = 0$$

thereby showing that the Poisson bracket of y vanishes with x, y, $\dot{x}$ and $\dot{y}$ making it impossible to satisfy $\dot{y} = \{y, h\}$ irrespective of our choice for h.

2. **Whether Lagrangian is unique if it exists** : It is possible to have a variety of Lagrangian which yield Euler - Lagrange equations that are equivalent to the given system of differential equations.

• One dimensional example :

Let us consider $\ddot{x} + \rho\,\dot{x} = 0$ (ρ = constant) (3.2)

as an illustration of a dynamical system which admits of two inequivalent Lagrangians. With $g = g_1 = e^{-\rho t}$ we have the energy function on $C^{\infty}(\boldsymbol{R}^{2})$

$$h_1 = e^{\rho t}\frac{\dot{x}^2}{2}$$

so that the flow system $\dot{x} = \{x, h_1\}$

$$\ddot{x} = \frac{\partial x}{\partial t} + \{\dot{x}, h_1\} = -\rho\dot{x} + \{\dot{x}, h_1\} = -\rho\dot{x}$$

agrees with (3.2). On the other hand , with $g_2 = \dot{x}$ we have the energy function on $C^{\infty}(\boldsymbol{R}^{2})$

$$h_2 = \dot{x} + \rho x$$

which also leads to the flow system : $\dot{x} = \{x, h_2\}$ $\ddot{x} = \{\dot{x}, h_2\}$ in agreement with (3.2). With the former choice the Larangian depends explicity on time, whereas with the latter choice the Lagrangian does not involve time explicitly.

• Two dimensional example :

Consider the equations of motion for a two dimensional isotropic oscillator

$$m\ddot{x} = -k\,x \qquad \text{and} \qquad m\ddot{y} = -k\,y \qquad (3.3)$$

In this case it is easy to find two energy functions on $C^{\infty}(\boldsymbol{R^4})$

$$h_1' = \frac{1}{2} m\,(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} k\,(x^2 + y^2)$$
$$h_2' = m\,\dot{x}\dot{y} + K\,x\,y$$

with the associated flow systems (i =1,2) both leading to (3.3)

$$\dot{x} = \{x, h_i'\} \qquad \dot{y} = \{y, h_i'\}$$
$$\ddot{x} = \{\dot{x}, h_i'\} \qquad \ddot{y} = \{\dot{y}, h_i'\}$$

which correspond respectively to the canonical choice $g = g_1' = \begin{pmatrix} 1/m & 0 \\ 1 & 1/m \end{pmatrix}$ and , a noncanonical choice $g = g_2' = \begin{pmatrix} 0 & 1/m \\ 1/m & 0 \end{pmatrix}$ for the matrix (g^{ij}) i=1,2 and j=1,2 defining the given system. In other words we have two inequivalent Lagrangians in this case.

3. **What its form is even if the Lagrangian exists and is unique (upto the addition of a total time derivative) :** As an example consider the equations of motion of a two dimensional istropic oscillator in polar coordinates $(r\,,\theta)$.

$$m\ddot{r} = -kr + mr\,\dot{\theta}^2 \quad \text{and} \quad r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0 \qquad (3.4)$$

The flow system on a 4 - dimensional tangent manifold

$$\dot{r} = \{r\,,\bar{h}\} \qquad \dot{\theta} = \{\theta\,,\bar{h}\}$$
$$\ddot{r} = \{\dot{r}\,,\bar{h}\} \qquad \ddot{\theta} = \{\dot{\theta}\,,\bar{h}\}$$

generated by a Hamiltonian vector field associated with an energy function of the form

$$\bar{h} = \tfrac{1}{2} m\left(r^2 + r^2\dot{\theta}^2\right) + \tfrac{1}{2} kr^2$$

leads to (3.4) . In this case the matrix (g^{ij}) assumes the form $g = \overline{g_1} = \begin{pmatrix} m^{-1} & 0 \\ 0 & (mr^2)^{-1} \end{pmatrix}$

To sum up we have considered here a somewhat unconventional viewpoint according to which equations of motion is what really counts. We find that to address the inverse problem in Newtonian mechanics it is not necessary to assume that the given equations of motion can be derived from a variational principle. Although the assumptions made by us may appear to be innocuous , they are none-the-less strong enough to constrain the Newtonian mechanics into a viable Lagrangian theory. A mathematically minded reader is urged to study Reference [5,10] for details.

4 From conservation laws to symmetries

We have illustrated how a dynamical system may be studied classically by means of a formulation based on a tangent bundle which bypasses the Lagrangian. We now turn to the all important subject of conservation laws which provide a powerful restriction on the behaviour of a dynamical systen. The very well - known way to obtain conservation laws for a system of differential equations is Noether theorem which bring out the power of symmetry in physics and associates to every symmety a conservation law.In this section we give a new method to solve the inverse problem , i.e., passage form conservation laws to symmetries. The advantage of this method is that it does not require, as does the Noether theorem, that the differential equations follow from a variational principle.

Let $C(x,\dot{x})$ be an observable which is conserved on the flow of a Hamiltonian vector field S . In Reference [5,10] we found the explicit form of a vector field E associated with C which is a symmetry of S, i.e., $L_s E = 0$, with

$$E = X^i \frac{\partial}{\partial x^i} + \bar{X}^i \frac{\partial}{\partial \dot{x}^i}$$

$$X^i = \{x^i, C\} \qquad \text{and} \qquad \bar{X}^i = \{\dot{x}^i, C\} \tag{4.1}$$

Locally we have $S^J \frac{\partial}{\partial x^J} E^I = E^J \frac{\partial}{\partial x^J} S^I$

Since from our standpoint E generates a symmetry of the dynamical system considered,we need to resolve whether it is Noether or non Noether (A Noether symmetry is one under which the Lagrangian considered is transformed into a total time defivative ; a non - Noether symmetry does not leave the action integral invariant). In this connection we recall that any symmetry of a dynamical system formulated on a tangent bundle with matrix $g^{ij} = \{x^i, \dot{x}^j\}$ determines a constant of the motion

$$\emptyset = -E(\ln D) + \frac{\partial \eta^i}{\partial x^i} + \frac{\partial \eta^i}{\partial \dot{x}^i} \tag{4.2}$$

where $E = \eta^i \frac{\partial}{\partial x^i} + \dot{\eta}^i \frac{\partial}{\partial \dot{x}^i}$ is the symmetry generator and $D = \det(g^{ij})$.

This result was demonstrated in Reference [5] without recourse to Lagrangian, although the same result can also be demonstrated for a Lagrangian system (M, L) by making use of a conservation law deduced by Hojman [3] and later generalized by Gonzalez- Gascon [6] ; in the Lagrangian case D is the determinant of the matrix which is reciprocal of the Hessian $\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$. It is important to recall that for a Noether symmetry , $\emptyset$ vanishes [7]. Thus the method is effectual in generating a constant of motion only if E generates a non-Noether symmetry of a dynamical system.

Returning to the present case the constant of motion

$$\emptyset = -E(\ln D) + \frac{\partial\{x^i, C\}}{\partial x^i} + \frac{\partial\{\dot{x}^i, C\}}{\partial \dot{x}^i} \tag{4.3}$$

is found to vanish on using Jacobi identities satisfied by the Poisson bracket. This shows that the associated symmetry is Noether.

To summarize the discussion of this section , given a constant of the motion $C(x,\dot{x},t)$ for a dynamical system

$$\ddot{x} = f^i(x,\dot{x},t)$$

$$\{x^i, x^j\} = 0 \qquad \{x^i, \dot{x}^j\} = g^{ij}$$

thus formulated on a tangent bundle with matrix $g^{ij}(x,\dot{x},t)$ the variations

$$\delta x^i = \varepsilon\{x^i, C\} \qquad \delta\dot{x}^i = \varepsilon\{\dot{x}^i, C\}$$

generate a Noether symmetry under which the Lagrangian varies into a total time derivative . This result is Noether theorem inversion. Our viewpoint may be illustrated with the following example of a dynamical symmetry.

• Three dimensional example

Consider the inverse square force with the equations of motion

$$m\ddot{\boldsymbol{x}} = k\boldsymbol{x}x^{-3} = -grad\ kx^{-1} \qquad (4.4)$$

This dynamical system may be formulated on a six-dimensional tangent bundle with $g = I_3$, a 3×3 unit matrix. In this case it is well known that the Lenz vector

$$A = m\dot{\boldsymbol{x}} \times (\boldsymbol{x} \times \boldsymbol{x}) + k\,\boldsymbol{x}\,x^{-1} = \text{ Constant} \qquad (4.5)$$

Taking the Poisson bracket of $\boldsymbol{A^k}$ with $\boldsymbol{x}, \dot{\boldsymbol{x}}$ we arrive at the following variation of the coordinates (i = 1,2,3 and k fixed)

$$\delta \boldsymbol{x^i} = 2\mathrm{m}\epsilon\left(\dot{\mathrm{x}}^{\mathrm{i}}\mathrm{x}^{\mathrm{k}} - \frac{1}{2}\mathrm{x}^{\mathrm{i}}\mathrm{x}^{\mathrm{k}}\right) - \frac{1}{2}\mathrm{x}^{\mathrm{j}}\,\dot{\mathrm{x}}^{\mathrm{j}}\delta^{\mathrm{ik}}$$

and of the associated velocities (i = 1,2,3 and k fixed)

$$\delta x^{\boldsymbol{i}} = \mathrm{m}\varepsilon\left(\mathrm{x}^{\mathrm{i}}\mathrm{x}^{\mathrm{k}} - \dot{\mathrm{x}}^2\delta^{\mathrm{ik}} - \mathrm{k}\,\mathrm{m}^{-1}\mathrm{x}^{-1}\delta^{\mathrm{ik}} + \mathrm{k}\,\mathrm{m}^{-1}\mathrm{x}^{-3}\mathrm{x}^{\mathrm{i}}\mathrm{x}^{\mathrm{k}}\right)$$

where x= |**x**| . For completeness , we may add that in this case a simple calculation yields the variation of the familiar Lagrangian

$\frac{1}{2}\mathrm{m}\dot{\mathbf{x}}^2 - \mathrm{kx}^{-1}$ into a total time derivative , viz, $-2k\epsilon m\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)\left(\frac{\mathrm{x^k}}{\mathrm{x}}\right)$.

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