Relativistic Treatment of Spinless Particles Subject to the Hulthen plus Yukawa Potential with Arbitrary l States

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Abstract
We have obtained the bound state energy eigenvalues and the corresponding wave functions of the Klein – Gordon particles for the unequal scalar and vector Hulthen plus Yukawa potential. The generalized parametric form of Nikiforov-Uvarov (NU) method is used in the calculation. By approximate choice of potential parameters our potential reduces to three well known potentials: Hulthen, Yukawa and Coloumb potentials and their corresponding energies are evaluated. Our results under limiting cases are consistent with those available in the literature. We have also obtained the numerical values of our results.
Keywords: Klein – Gordon equation, Hulthen potential, Yukawa potential, Nikiforov-Uvarov (NU) method.

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1. Introduction

The exact solutions of the non-relativistic and relativistic equations with the central potential play an important role in quantum mechanics [1-4]. In fact, exact solution of quantum system is significant in physics and solving the non – relativistic and relativistic equations is still an interesting work in the existing literature [6-12].

In the relativistic limit, the particle motions are commonly described using either the Klein – Gordon or the Dirac equations [8,13] depending on the spin character of the particles. The spin-zero particles like the mesons are described by the Klein-Gordon equation and the spin half particles such as electrons are described satisfactorily by the Dirac equation. One of the interesting problems in nuclear and high energy physics is to obtain exact solution of the Klein-Gordon and the Dirac equations. In recent years, many studies have been carried out to explore the relativistic energy eigenvalues and the corresponding wave functions of these wave equations [14-17].

Recently, the study of exponential-type potentials has attracted much attention to many authors both in non – relativistic quantum mechanics and relativistic quantum mechanics [18-27]. These potentials include the Manning-Rosen potential [28], Eckart-type potential [25,26], multi-parameter exponential-type potentials [20-23] and others. It should be mentioned that most contributions appearing in the literature are concerned with the s-wave case. However, for the l-wave, one can only solve approximately by using a suitable approximation scheme [29].

Hulthen potential is one of the short-range potentials that has been studied in physics [30]. This potential has applications in different branches of physics such as atomic physics [31], nuclear and high- energy physics [32], solid-state physics [33] and chemical physics [34]. The Hulthen potential has been studied by some authors in both the non-relativistic and relativistic quantum mechanics [11,35-38]. Another potential of great important is Yukawa potential. This potential is also a short range potential and has applications in high energy physics [39]. Also, in atomic and molecular physics, Yukawa potential is used as a screened coulomb potential and it is also referred to as Debye – Huckel potential in plasma physics [40]. Thus, it is worth investigating the solution of the Klein – Gordon equation with combined Hulthen and Yukawa potentials.

In this work, we attempt to find the approximate bound state solutions of the Klein-Gordon equation with unequal scalar and vector Hulthen plus Yukawa potentials with arbitrary l-state. The paper is organized as follows. In section 2, the review of Nikiforov – Uvarov method is presented. Factorization method is presented in section 3. We give the bound state solution of Klein – Gordon equation for Hulthen plus Yukawa potentials in section 4. The special cases of our potential are discussed in section 5. Finally, a brief conclusion is given in section 6.

2. Nikiforov – Uvarov (NU) method

To solve second order differential equations, the NU method can be used with an appropriate coordinate transformation $s = s(r)$ [43]

$$\psi_n''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s)+\frac{\tilde{\tau}(s)}{\sigma^2(s)}\psi_n(s)=0, \qquad 1$$

Where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second – degree, and $\tilde{\tau}(s)$ is a first – degree polynomial. The following equation is a general form of the Schrödinger – like equation written for any potential [44, 45].

$$\left[\frac{d^2}{ds^2}+\frac{c_1-c_2 s}{s(1-c_3 s)}\frac{d}{ds}+\frac{1}{s^2(1-c_3 s)^2}\left(-\xi_1 s^2+\xi_2 s-\xi_3\right)\right]\psi_n(s)=0. \qquad 2$$

According to the parametric generalization of the NU method, the wave functions and the energy eigenvlaues equation, respectively, are found as [44, 45]

$$\Psi(s)=s^{c_{12}}(1-c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}}P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3 s) \qquad 3$$

$$c_2 n-(2n+1)c_5+(2n+1)\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)+n(n-1)c_3+c_7+2c_3 c_8+2\sqrt{c_3 c_9}=0, \qquad 4$$

where

$$c_4=\frac{1}{2}(1-c_1), c_5=\frac{1}{2}(c_2-2c_3),$$

$$c_6=c_5^2+\xi_1,\ c_7=2c_4 c_5-\xi_2, c_8=c_4^2+\xi_3,$$

$$c_9=c_6+c_3 c_7+c_3^2 c_8,$$

and

$$c_{10}=c_1+2c_4+2\sqrt{c_8}$$

$$c_{11}=c_2-2c_5+2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)$$

$$c_{12}=c_4+\sqrt{c_8}, c_{13}=c_5-\left(\sqrt{c_9}+c_3\sqrt{c_8}\right). \qquad 5$$

In some cases $c_3=0$ and for this type the composites of the wave functions of Eq. (3) change as

$$\lim_{c_3\to 0}P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3 s)=L_n^{c_{10-1}}(c_{11}s), \qquad 6$$

and

$$\lim_{c_3\to 0}(1-c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}}=e^{c_{13}s}. \qquad 7$$

Hence, the solution given by Eq. (3) becomes as [44]

$$\Psi(s)=s^{c_{12}}e^{c_{13}s}L_n^{c_{10-1}}(c_{11}s). \qquad 8$$

3. Factorization method

The time – independent Klein – Gordon equation with the scalar potential $S(r)$ and vector potential $V(r)$ in the natural unit $\hbar=c=1$ is defined as [5, 7]

$$\left[\nabla^2+(V(r)-E)^2-(S(r)+m)^2\right]\Psi(r,\theta,\varphi)=0, \qquad 9$$

where m is the rest mass, E is the relativistic energy, ∇^2 is the Laplace operator. In the spherical coordinates, the Klein – Gordon equation for a particle in the presence of the Hulthen plus Yukawa potentials $V(r)$ becomes

$$\left[\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\\+2(EV(r)+mS(r))+V^2(r)-S^2(r)+E^2-m^2\end{array}\right]\Psi(r,\theta,\varphi)=0. \qquad 10$$

If one assign the total wave function as $\Psi(r,\theta,\varphi)=\frac{R(r)}{r}Y_{lm}(\theta,\varphi)$ 11

Eq. (10) is separated into variables and the following equations are obtained:

$$\frac{d^2R(r)}{dr^2}+\left[E_{nl}^2-m^2-2(E_{nl}V(r)+mS(r))+V^2(r)-S^2(r)-\frac{\lambda}{r^2}\right]R(r)=0, \qquad 12$$

$$\frac{d^2\Theta(\theta)}{d\theta^2}+\cot\theta\frac{d\Theta(\theta)}{d\theta}+\left(\lambda-\frac{m^2}{\sin^2\theta}\right)\Theta(\theta)=0, \tag{13}$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}+m^2\Phi(\varphi)=0, \tag{14}$$

where $Y_{lm}(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$ (15)

and m^2 and $\lambda=l(l+1)$ are the separation constants. Equations (13) and (14) are spherical harmonic function $Y_{lm}(\theta,\varphi)$ whose solutions are well known [1]

4. Bound state solutions of the radial Klein – Gordon equation with Hulthen plus Yukawa potentials

The Hulthen plus Yukawa scalar and vector potentials are respectively written as [14, 42]

$$S(r)=-\frac{S_0e^{-2\alpha r}}{1-e^{-2\alpha r}}-\frac{S_1e^{-\alpha r}}{r},V(r)=-\frac{V_0e^{-2\alpha r}}{1-e^{-2\alpha r}}-\frac{V_1e^{-\alpha r}}{r}, \tag{16}$$

where $S_0,S_1,V_0 and V_1$ are the potential depths, α is the range of the potential, and r is the distance from the equilibrium position. Substituting Eq. (16) into Eq. (12), we obtain the radial Klein – Gordon equation for the Hulthen plus Yukawa potentials.

$$\frac{d^2R}{dr^2}+\left\{\begin{array}{l}E_{nl}^2-m^2+2\left[E_{nl}\left(\frac{V_0e^{-2\alpha r}}{1-e^{-2\alpha r}}+\frac{V_1e^{-\alpha r}}{r}\right)+m\left(\frac{S_0e^{-2\alpha r}}{1-e^{-2\alpha r}}+\frac{S_1e^{-\alpha r}}{r}\right)\right]\\+\frac{V_0^2e^{-4\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}+\frac{2V_0V_1e^{-3\alpha r}}{\left(1-e^{-2\alpha r}\right)r}+\frac{V_1^2e^{-2\alpha r}}{r^2}-\frac{S_0^2e^{-4\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}-\frac{2S_0S_1e^{-3\alpha r}}{\left(1-e^{-2\alpha r}\right)r}\\-\frac{S_1^2e^{-2\alpha r}}{r^2}-\frac{l(l+1)}{r^2}\end{array}\right\}R(r)=0. \tag{17}$$

It is well known that Eq. (17) has no analytical solution for $l\neq 0$ due to the centrifugal term. Therefore, we must take a proper approximation to the centrifugal term as [27](see Fig.1).

$$\frac{1}{r^2}\approx\frac{4\alpha^2e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2} \tag{18a}$$

$$\frac{1}{r}\approx\frac{2\alpha e^{-\alpha r}}{\left(1-e^{-2\alpha r}\right)}. \tag{18b}$$

Substituting Eqs. (18a) and (18b) into Eq. (17) yields,

$$\frac{d^2R(r)}{dr^2}+\left\{\begin{array}{l}E_{nl}^2-m^2+2\left[E_{nl}\left(\frac{V_0e^{-2\alpha r}}{1-e^{-2\alpha r}}+\frac{2\alpha V_1e^{-2\alpha r}}{1-e^{-2\alpha r}}\right)+m\left(\frac{S_0e^{-2\alpha r}}{1-e^{-2\alpha r}}+\frac{2\alpha S_1e^{-2\alpha r}}{1-e^{-2\alpha r}}\right)\right]\\+\frac{V_0^2e^{-4\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}+\frac{4\alpha V_0V_1e^{-4\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}+\frac{4\alpha^2V_1^2e^{-4\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}-\frac{S_0^2e^{-4\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}-\frac{4\alpha S_0S_1e^{-4\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}\\-\frac{4\alpha^2S_1^2e^{-4\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}-\frac{4\alpha^2l(l+1)e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}\end{array}\right\}R(r)=0. \tag{19}$$

Simplying Eq.(19) in view of the new variable $s=e^{-2\alpha r}$, we obtain the following hypergeometric – type equation

$$\frac{d^2R(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dR(s)}{ds}+\frac{1}{s^2(1-s)^2}\left[-Q_1s^2+Q_2s-Q_3\right]R(s)=0, \tag{20}$$

where

$$Q_1 = -\frac{\left(E_{nl}^2 - m^2\right)}{4\alpha^2} + \frac{E_{nl}V_0}{2\alpha^2} + \frac{\left(2mS_0 + S_0^2 - V_0^2\right)}{4\alpha^2} + \frac{1}{\alpha}\left(E_{nl}V_1 + mS_1 + S_0S_1 - V_0V_1\right) + S_1^2 - V_1^2, \quad 21$$

$$Q_2 = -\frac{2\left(E_{nl}^2 - m^2\right)}{4\alpha^2} + \frac{E_{nl}V_0}{2\alpha^2} + \frac{mS_0}{2\alpha^2} + \frac{1}{\alpha}\left(E_{nl}V_1 + mS_1\right) - l(l+1), \quad 22$$

$$Q_3 = -\frac{2\left(E_{nl}^2 - m^2\right)}{4\alpha^2} \quad 23$$

Now comparing Eq. (20) with Eq. (2), we obtain the following parameters:

$$\xi_1 = Q_1 = -\frac{\left(E_{nl}^2 - m^2\right)}{4\alpha^2} + \frac{E_{nl}V_0}{2\alpha^2} + \frac{\left(2mS_0 + S_0^2 - V_0^2\right)}{4\alpha^2} + \frac{1}{\alpha}\left(E_{nl}V_1 + mS_1 + S_0S_1 - V_0V_1\right) + S_1^2 - V_1^2$$

$$\xi_2 = Q_2 = -\frac{2\left(E_{nl}^2 - m^2\right)}{4\alpha^2} + \frac{E_{nl}V_0}{2\alpha^2} + \frac{mS_0}{2\alpha^2} + \frac{1}{\alpha}\left(E_{nl}V_1 + mS_1\right) - l(l+1)$$

$$\xi_3 = Q_3 = -\frac{\left(E_{nl}^2 - m^2\right)}{4\alpha^2}$$

$$c_1 = c_2 = c_3 = 1. \quad 24a$$

Other coefficients are determined from Eq. (5) as

$$c_4 = 0, c_5 = -\frac{1}{2}$$

$$c_6 = \frac{1}{4} - \frac{\left(E_{nl}^2 - m^2\right)}{4\alpha^2} + \frac{E_{nl}V_0}{2\alpha^2} + \frac{\left(2mS_0 + S_0^2 - V_0^2\right)}{4\alpha^2} + \frac{1}{\alpha}\left(E_{nl}V_1 + mS_1 + S_0S_1 - V_0V_1\right) + S_1^2 - V_1^2$$

$$c_7 = \frac{2\left(E_{nl}^2 - m^2\right)}{4\alpha^2} - \frac{E_{nl}V_0}{2\alpha^2} - \frac{mS_0}{2\alpha^2} - \frac{1}{\alpha}\left(E_{nl}V_1 + mS_1\right) + l(l+1)$$

$$c_8 = -\frac{\left(E_{nl}^2 - m^2\right)}{4\alpha^2}$$

$$c_9 = \frac{1}{4} + \frac{\left(S_0^2 - V_0^2\right)}{4\alpha^2} + \frac{1}{\alpha}\left(S_0S_1 - V_0V_1\right) + S_1^2 - V_1^2 + l(l+1)$$

$$c_{10} = 1 + 2\sqrt{-\frac{\left(E_{nl}^2 - m^2\right)}{4\alpha^2}}$$

$$c_{11} = 2 + 2\left(\sqrt{\frac{1}{4} + \frac{\left(S_0^2 - V_0^2\right)}{4\alpha^2} + \frac{1}{\alpha}\left(S_0S_1 - V_0V_1\right) + S_1^2 - V_1^2 + l(l+1)} + \sqrt{\frac{-\left(E_{nl}^2 - m^2\right)}{4\alpha^2}}\right)$$

$$c_{12} = \sqrt{\frac{-\left(E_{nl}^2 - m^2\right)}{4\alpha^2}}$$

$$c_{13} = -\frac{1}{2} - \left(\sqrt{\frac{1}{4} + \left(\frac{S_0^2 - V_0^2}{4\alpha^2}\right) + \frac{1}{\alpha}\left(S_0S_1 - V_0V_1\right) + S_1^2 - V_1^2 + l(l+1)} + \sqrt{\frac{-\left(E_{nl}^2 - m^2\right)}{4\alpha^2}}\right) \quad 24b$$

Substituting Eqs. (24a) and (24b) into Eq. (4), we obtain the energy eigenvalues equation of the Hulthen plus Yukawa potentials as

$$n^2+\frac{1}{2}(2n+1)+(2n+1)\left(\sqrt{\frac{1}{4}+\left(\frac{S_0^2-V_0^2}{4\alpha^2}\right)+\frac{1}{\alpha}(S_0S_1-V_0V_1)+S_1^2-V_1^2}+\sqrt{-\frac{(E_{nl}^2-m^2)}{4\alpha^2}}\right)$$
$$-\frac{E_{nl}V_0}{2\alpha^2}-\frac{mS_0}{2\alpha^2}-\frac{1}{\alpha}(E_{nl}V_1+mS_1)+l(l+1) \qquad 25$$
$$+2\sqrt{-\frac{(E_{nl}^2-m^2)}{4\alpha^2}\left(\frac{1}{4}+\left(\frac{S_0^2-V_0^2}{4\alpha^2}\right)+\frac{1}{\alpha}(S_0S_1-V_0V_1)+S_1^2-V_1^2+l(l+1)\right)}=0$$

or explicitly the energy spectrum of this system can be expressed as

$$E_{nl}^2-m^2=-\alpha^2\left[\frac{\delta+\left(n+\frac{1}{2}+\beta\right)^2}{\left(n+\frac{1}{2}+\beta\right)}\right]^2, \qquad 26$$

where

$$\delta=\frac{(V_0^2-S_0^2)}{4\alpha^2}+\frac{1}{\alpha}(V_0V_1-S_0S_1)+V_1^2-S_1^2-\frac{(E_{nl}V_0+mS_0)}{2\alpha^2}-\frac{1}{\alpha}(E_{nl}V_1-mS_1)+l(l+1), \qquad 27$$

and

$$\beta=\sqrt{\frac{1}{4}+\left(\frac{S_0^2-V_0^2}{4\alpha^2}\right)+\frac{1}{\alpha}(S_0S_1-V_0V_1)+S_1^2-V_1^2+l(l+1)} \qquad 28$$

In order to test the accuracy of our work, we computed the numerical values of our results as presented in tables 1-4 which are consistent with those obtained in the literature. For a special case: Equal scalar and vector potential, $V(r)=S(r)$ (i.e $S_0=V_0$ and $S_1=V_1$), the energy spectrum of Eq. (26) reduces to

$$E_{nl}^2-m^2=-\alpha^2\left[\frac{-\frac{(E_{nl}+m)V_0}{2\alpha^2}-\frac{1}{\alpha}(E_{nl}+m)V_1+l(l+1)+\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)}\right)^2}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}++l(l+1)}\right)}\right]^2 \qquad 29$$

Substituting Eqs. (24a) and (24b) into Eq. (3), we obtain the radial wave function for this system as

$$R(s)=N_{nl}S^{\frac{\mu}{2}}(1-s)^{\frac{1+v}{2}}P_n^{(\mu,v)}(1-2s), \qquad 30$$

where $\mu=2\sqrt{-\frac{(E_{nl}^2-m^2)}{4\alpha^2}}$, $v=2\sqrt{\frac{1}{4}+\left(\frac{S_0^2-V_0^2}{4\alpha^2}\right)+\frac{1}{\alpha}(S_0S_1-V_0V_1)+S_1^2-V_1^2+l(l+1)}$

and N_{nl} is the normalization constant.

5. A few limiting cases

Let us now study some limiting cases of our potential. By choosing appropriate parameters in the Hulthen plus Yukawa potentials, we can obtain some well known potentials.

5.1 Hulthen potential

If we set $V_1=0$ and $S_1=S_0=0$, it is found that the potential in Eq. (16) turns into the Hulthen potential [14]

$$V(r)=-\frac{V_0e^{-2\alpha r}}{1-e^{-2\alpha r}} \qquad 31$$

with the corresponding energy eigenvalues (for pure vector potential) as

$$E_{nl}^2 - m^2 = -\alpha^2 \left[\frac{\frac{V_0^2}{4\alpha^2} - \frac{E_{nl}V_0}{2\alpha^2} + l(l+1) + \left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{V_0^2}{4\alpha^2} + l(l+1)} \right)^2}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{V_0^2}{4\alpha^2} + l(l+1)} \right)} \right]^2 \quad 32$$

5.2 Yukawa Potential

Setting $V_0 = 0, S_0 = S_1 = 0$ into Eq. (16), the potential becomes the Yukawa potential of the form [42]

$$V(r) = -\frac{V_1 e^{-\alpha r}}{r} \quad 33$$

with the energy spectrum (for pure vector potential) obtained as

$$E_{nl}^2 - m^2 = -\alpha^2 \left[\frac{V_1^2 - \frac{1}{\alpha} E_{nl} V_1 + l(l+1) + \left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - V_1^2 + l(l+1)} \right)^2}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - V_1^2 + l(l+1)} \right)} \right]^2 . \quad 34$$

This result is consistent with those found in the literature [42,47] for s-wave as shown in table 1.

5.3 Coulomb potential

If we set $V_0 = 0, S_0 = 0$, $S_1 = V_1$ and $\alpha \to 0$ into Eq. (16), we obtain the Coulomb potential [46] as

$$V(r) = -\frac{V_1}{r} \quad 35$$

The energy eigenvalues corresponding to Eq. (35) is obtained by using the resulting parameters in Eq. (26) as

$$E_{nl}^2 - m^2 = -\left[\frac{(E_{nl} + m)V_1}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1)} \right)} \right]^2 \quad 36$$

6. Conclusion

We have presented approximate analytical bound state solutions to the Klein – Gordon equation for the Hulthen plus Yukawa potentials for arbitrary l - state by using parametric generalization of the NU method. We have explicitly obtained the energy eigenvalues and the corresponding wave function of the Klein – Gordon particles for unequal scalar and vector potentials. We have discussed some special cases of our potential. We have also computed our results numerically and compare it with others. Under limiting cases our results are compatible with those available in the literature.

Table 1: Comparison of Klein – Gordon ground state energies for different parameter values ($V_0 = S_0 = S_1 = 0$)

V_1	α	**Ref. 42**	**Ref. 47**	**Our results**
0.125	0.01250	0.998702	0.993484	0.9935159216
	0.06250	0.999999	0.997573	0.9979335649
	0.09375	0.999542	0.999030	0.9994488394
	0.12500	0.998100	0.999808	0.9999979671
0.25	0.0250	0.994130	0.971776	0.9718941536
	0.1250	0.999960	0.998678	0.9901091872
	0.1875	0.998556	0.995030	0.9968269670
	0.2500	0.993138	0.998708	0.9998396706

Table 2: Energy eigenvalues for Hulthen plus Yukawa potentials with $m=1, V_0=0.04, V_1=-0.2, S_0=0.5, S_1=-0.1, \alpha=0.05$ for different values of n and l

n	l	$E_{nl}^{+}(fm^{-1})$	$-E_{nl}^{-}(fm^{-1})$
1	0	0.6979762765	0.8549344258
2	0	0.8513299838	0.9397943420
	1	0.8735089723	0.9485503434
3	0	0.9332620483	0.9805584444
	1	0.9452976759	0.9845515246
	2	0.9635448018	0.9907088959
4	0	0.9764224158	0.9974727131
	1	0.9823817415	0.9986166402
	2	0.9908116077	0.9998353553
	3	0.9975810202	-

Table 3: Energy eigenvalues (pure vector is $S_0=S_1=0$) with $m=1, V_0=0.5, V_1=0.1$ and $\alpha=0.05$ for different values of n and l

n	l	$E_{nl}(fm^{-1})$
1	0	1.065907013
2	0	0.9962348945
	1	0.9940934797
3	0	0.9193762871
	1	0.9187227658
	2	0.9176953201
4	0	0.8702598866
	1	0.8719074514
	2	0.8754586553
	3	0.8814096557

Table 4: Energy eigenvalues (pure scalar i.e. $V_0=V_1=0$) with $m=1, S_0=0.6, S_1=0.3$ and $\alpha=0.05$ for different values of n and l

n	l	$E_{nl}(fm^{-1})$
1	0	0.7550682835
2	0	0.8796404648
	1	0.8920873130
3	0	0.9467019919
	1	0.9534482858
	2	0.9648052731
4	0	0.9822227470
	1	0.9855254878
	2	0.9908162735
	3	0.9961078948

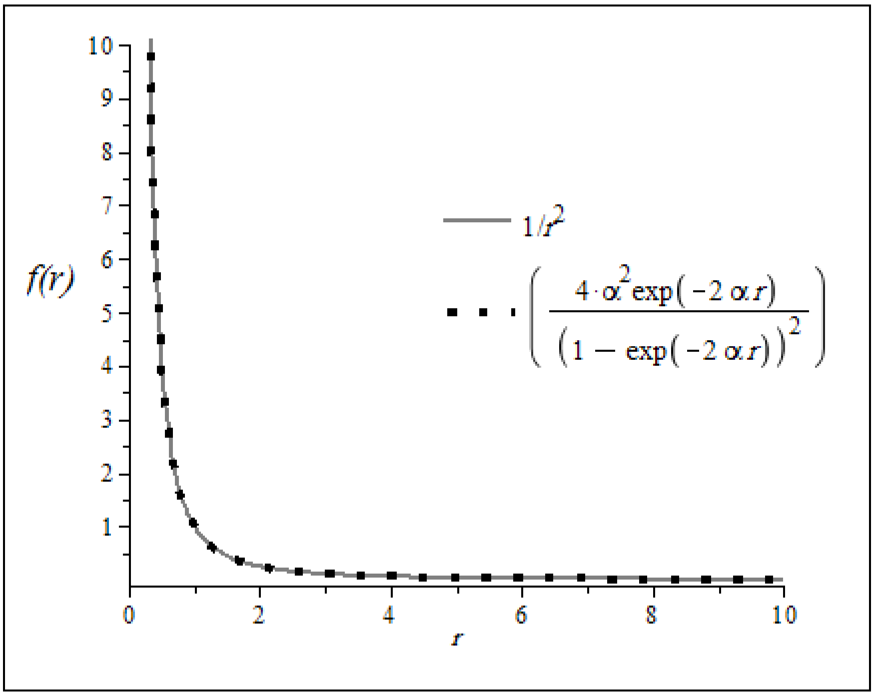

Fig.(1). The centrifugal term ($1/r^2$) and its approximation for $\alpha = 0.1$.

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