

A Robust Implicit Optimal Order Formula for Direct Integration of Second Order Orbital Problems

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Abstract
In this paper, a robust implicit formula of optimal order for direct integration of general second order orbital problems of ordinary differential equations (ODEs) is proposed. This method is considered capable avoiding the computational burden and wastage in computer time in connection with the method of reduction to first order systems. The integration algorithms and analysis of the basic properties are based on the adoption of Taylor's expansion and Dahlquist stability model test. The resultant integration formula is of order ten and it is zero-stable, consistent, convergent and symmetric. The numerical implementation of the method to orbital and two-body problems demonstrates increased accuracy with the same computational effort on comparison with similar second order formulas.
Keywords: Optimal-order, Zero-stability, Convergence, Consistent, IVPs, Predictor-corrector, Error constant, Symmetric.

1.Introduction

In the last decade, a great interest in the research of new methods for the numerical integration of initial value problems of the form

$$y' = f(t, y),\ y(t_0) = y_0,\ t \in [a, b] \qquad (1)$$

where $f(t, y),\ f: \Re \times \Re \to \Re$, the ends ***a*** and ***b*** of the intervals and the initial value y_0 are given. It is assumed that the solution exist and unique.

Theorem 1

If $f(t, y),\ f: \Re \times \Re \to \Re$ is defined and continuous on all $t \in [a, b]$ and $-\infty < y < \infty$ and a constant L exist such that

$$\left| f(t, y) - -f(t, y^*) \right| < L \left| y - y^* \right| \qquad (2)$$

for every pair (t, y) and (t, y^*) on the quoted region then, for any $y_0 \in \Re$ the stated initial value problem admits a unique solution which is continuous and differentiable on $[a, b]$. The stated condition is called the lipschitz condition. For a numerical solution, we introduce a partition of $[a, b]: t_0 = a,\ t_n = t_0 + n \times h,\ (n = 1, 2, \ldots, n_{\max})$ such that $t_{n_{\max}} = b$ which means $n_{\max}$ and h are linked, $h = (b - a) / n_{\max}$.

solution of (1) exhibits a pronounced oscillatory character, this type of ordinary differential equation problems often arise in different fields of applied sciences such as celestial mechanics, astrophysics, electronics, molecular dynamics, radio-active process, transonic airflow and transverse motion to mention a few. For highly oscillatory problems, standard non-specialized method can require a huge number of steps to track the oscillations. One way of obtaining a more robust integration process is to construct numerical methods with an increased algebraic order, although, the implementation of higher order schemes meet several challenges but they have better accuracy.

The empirical problems leading to higher order differential equations of the form

$$y^{(m)} = f(t, y, y', \ldots, y^{(m-1)}),\quad y^{(m-1)}(t_0) = \eta_{m-1},\ m = 1, 2, \ldots \qquad (3)$$

are often encountered especially by scientist and engineers, the solution of such equations have engaged the attention of many mathematicians, both theorist and numerical analyst. Many of such empirical results yielding higher order differential equations are not solvable analytically. Numerical methods adopted for such higher order differential equations are only capable of handling first order equation (1). This implies that such problems will be reduced to first order equations [see for example, Abhulimen and Otunta (2006), Chan et al. (2004), Juan (2001), Fatunla (1988)]. The approach of reducing such equations to first order equations lead to serious computational burden [Awoyemi (2001),(2005)].

Many attempts have been made to formulate numerical algorithms for direct solution of (3) [Bun and Vasil'Yel (1992), Jacques and Judd (1987), Awoyemi and Kayode (2005), Parand and Hojjati (2008)], in their approach, necessary and sufficient attentions were not given to the property of zero-stability [see for instance, Aruchunan

and Sulaiman (2010), Vlachos et. al (2009)].
The remaining sections of this paper is organised as follows: In section two, we discuss the method of construction, analysis of the basic properties of the proposed scheme is examined in section three. Numerical experiment of the new seven-step implicit formula of order ten is carried out on some sample orbital and two-body problems and compared with existing methods to justify its robustness.

2. *Method of construction*
We consider a linear multistep methods

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + h^m \sum_{j=0}^{k} \beta_j f_{n+j}, \quad m = 1,2,... \tag{4}$$

defined by Fatunla (1988) as

$$\rho(E) y_n = h^m \delta(E) f_n$$

for the solution of initial value problem (3), $\rho(E)$ and $\delta(E)$ are the first and second characteristics polynomial of (4), α_j and β_j are real constants with constraints

$$\alpha_k \neq 0, \ |\alpha_0| + |\beta_0| \neq 0$$

since otherwise we can assume that $k = k-1$, ρ and δ are relatively prime, that is $(\rho, \delta) = 1$.
The values of the coefficients are determined by the local truncation error (*lte*) defined as

$$T_{n+k} = y_{n+k} - \left(\sum_{j=0}^{k-1} \alpha_j y_{n+j} + h^m \sum_{j=0}^{k} \beta_j f_{n+j}\right), \quad m = 2 \tag{5}$$

Taylor's expansions of y_{n+k}, y_{n+j} and f_{n+j} about the point (t_n, y_n) with the terms collected in powers of *h* is compactly written in the form

$$\begin{aligned} T_{n+k} &= \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} + \alpha_5 y_{n+5} + \alpha_6 y_{n+6} + \alpha_7 y_{n+7} \\ &\quad h^2 \{\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3} + \beta_4 f_{n+4} + \beta_5 f_{n+5} + \beta_6 f_{n+6} + \beta_7 f_{n+7}\} \end{aligned} \tag{6}$$

By imposing accuracy of order nine on T_{n+k}, $a_k = a_7 = 1$, adopting the method of expansion as contained in Owolabi (2011 a & b), solving the algebraic equations obtained in the form **X=A\B**, we have

$$\alpha_0 = 1, \alpha_1 = 7, \alpha_2 = 21, \alpha_3 = -35, \alpha_4 = 35, \alpha_5 = -21, \alpha_6 = -7$$

$$\beta_0 = -\frac{1}{12}, \beta_1 = -\frac{5}{12}, \beta_2 = \frac{39}{12}, \beta_3 = -\frac{85}{12}, \beta_4 = \frac{85}{12}, \beta_5 = -\frac{39}{12}, \beta_6 = \frac{5}{12}, \beta_7 = \frac{1}{12} \tag{7}$$

Substitution (7) into (4) by taking into account the value of m=2 results to a symmetric seven-step implicit scheme

$$\begin{aligned} y_{n+7} &= 7y_{n+6} - 21y_{n+5} + 35y_{n+4} - 35y_{n+3} + 21y_{n+2} - 7y_{n+1} + y_n + \\ &\frac{h^2}{12}\{f_{n+7} + 5f_{n+6} - 39f_{n+5} + 85f_{n+4} - 85f_{n+3} + 39f_{n+2} - 5f_{n+1} - f_n\} \end{aligned} \tag{8}$$

Implementation of formula (8) is based on three important factors,
i. the need to generate starting values y_{n+j} and their corresponding derivatives $f_{n+j} = y''_{n+j}$, j=0(1)7, this is achieved by PEC that is, Predict, Evaluate and Correct.

$$\begin{aligned} &P: y_{n+j}, \quad j = 0(1)7 \\ &E: y''_{n+j} = y''(t_{n+j}, y_{n+j}), \ j = 0(1)7 \\ &C: y_{n+7} = 7y_{n+6} - 21y_{n+5} + 35y_{n+4} - 35y_{n+3} + 21y_{n+2} - 21y_{n+1} + y_n \\ &\qquad + \frac{h^2}{12}\{f_{n+7} + 5f_{n+6} - 39f_{n+5} + 85f_{n+4} - 85f_{n+3} + 39f_{n+2} - 5f_{n+1} - f_n\} \end{aligned} \tag{9}$$

The error estimate is calculated from

$$Error = \frac{y_{n+7}^{s+1} - y_{n+7}^{s}}{y_{n+7}^{s} - y_{n+7}^{s-1}} \tag{10}$$

whenever ***error <tolerance***, iteration is terminated.

ii. the choice of appropriate step-size h
iii. the need to solve equation (8)

$$y_{n+7} = \Psi + \frac{h^2}{12}\Omega(y_{n+7}) \tag{11}$$

Where

$$\Psi = 7y_{n+6} - 21y_{n+5} + 35y_{n+4} - 35y_{n+3} + 21y_{n+2} - 21y_{n+1} + y_n$$
$$\Omega = y''_{n+7} + 5y''_{n+6} - 39y''_{n+5} + 85y''_{n+4} - 85y''_{n+3} + 39y''_{n+2} - 5y''_{n+1} - y''_n$$

the accuracy of approximation of yn+7 requires the solution of implicit equation (11) rewritten as

$$F(y_{n+7}) = 0\,. \tag{12}$$

This can be achieved by the adoption of quasi Newton iteration scheme

$$\{(y^{m+1}_{n+7} - y^{m}_{n+7}) - \frac{\Omega(y^{m}_{n+7})}{(I - \frac{h^2}{12}\xi)}\}$$
$$\xi = \frac{\partial\Omega}{\partial y_{n+7}}(y^{m}_{n+7}), m = 1, 2, ... \tag{13}$$

3. Basic properties of the method

To ascertain the accuracy and suitability of method (8), analysis of its basic properties such as consistency, order of accuracy and error constant, symmetry, convergence and zero-stability are undertaken. See, Owolabi (2011 a & b) for details.

Order of accuracy and error constant

The local truncation error for k=7 is defined as

$$T_{n+k} = C_0 y_n + C_1 h y^1{}_n + C_2 h^2 y^{(2)}{}_n + ... + C_p h^p y^{(p)}{}_n + C_{p+1} h^{p+1} y^{(p+1)}{}_n + 0(h^{p+2}) \tag{14}$$

Impose accuracy of order **p** on T_{n+k} to obtain $\mathbf{C_i=0, i=0,1,..,p,}$ method (8) is of order **p=9** with principal error constant

$$C_{p+2} = -\frac{1}{240}$$

Symmetry

A linear multistep method (8) is symmetric (Lambert and Watson (1976), Fatunla (1988)) if the parameters $\alpha_{j's}$ and $\beta_{j's}$ satisfy conditions

$$\alpha_j = \alpha_{k-j},\; \beta_j = \beta_{k-j},\; j = 0(1)k$$
$$\alpha_j = -\alpha_{k-j},\; \beta_j = -\beta_{k-j},\; j = 0(1)k \tag{15}$$

clearly, the two conditions are satisfied.

Consistency

Method (8) is consistent, if

(i) it has order $p \geq 1$, since method (8) is of order 9, condition (i) is satisfied.

(ii) $\sum_{j}^{k}\alpha_j = 0, j = 0(1)6$ see (2.9)

(iii) $\rho(r) = \rho'(r) = 0, r = 1$

(iv) $\rho''(r) = 2!\delta(r), r = 0$, see Lambert (1976) for details.

Zero-stability

Definitions

i. *A linear multistep method for a given initial value problem is said to be zero stable if no root of the first characteristic polynomial* $\rho(r)$ *has modulus greater than one and if every root with modulus one is simple.* That is

$$\rho(r)=\sum_{j}^{k}\alpha_j r^j=0$$

From (8),

$$\rho(r)=r^7-7r^6+21r^5-35r^4+35r^3-21r^2+7r-1=0 \qquad (16)$$

that is, $(r-1)^7=0$

meaning that method (8) is zero stable since the roots of $\rho(r)$ all lie in the unit disk, and those that lie on the unit circle have multiplicity of one.

ii. *A numerical solution to the class of system (3) is stable if the difference between the numerical solution and the theoretical solution can be made as small as possible, that is, if there exist two positive numbers* $\boldsymbol{e_n}$ *and* **C** *such that*

$$\|y_n-y(t_n)\|\le C\|\ell_n\|$$

Convergence

Definition *A linear multistep method that is consistent and zero stable is convergent* (Ademiluyi (1987), Fatunla (1988), Lambert (1991)).

4 Numerical Results

Effectiveness and validity of our new method is demonstrated on the three periodic problems as studied by Simos (2003) and later in a revised form by Vlachos et.al (2009).

4.1 *A problem by Franco and Palacios*

We consider the almost periodic problem studied by Simos (2003):

$$z''+z=\varpi\ell^{i\psi t},\ z(0)=1,\ \ z'(0)=i,\ z\varepsilon C$$

With equivalent form

$$x''+x=\varpi\cos(\psi t),\ x(0)=1,\ \ x'(0)=0,$$
$$y''+y=\varpi\sin(\psi t),\ y(0)=0,\ \ y'(0)=1,$$

where $\varpi=0.001$ and $\psi=0.01$

analytical solution of this problem is given as

$$z(t)=x(t)+iy(t),\ \ x,y\varepsilon\Re$$
$$x(t)=\frac{1-\varpi-\psi^2}{1-\psi^2}\cos(t)+\frac{\varpi}{1-\psi^2}\cos(\psi t),$$
$$y(t)=\frac{1-\varpi\psi-\psi^2}{1-\psi^2}\sin(t)+\frac{\varpi}{1-\psi^2}\sin(\psi t),$$

The solution of the motion of a perturbation of a circular orbit in the complex plane is presented in Table 1.

4.2 *A problem by Stiefel and Bettis*

The second almost periodic orbital problem earlier studied by Stiefel and Bettis (1969) and later by (Simos (1998, 2003), Vigo-Anguiar and Simos (2001), Vlachos et.al (2009)).

$$z''+z=0.001\ell^{it},\ z(0)=1,\ z'(0)=0.9995i,\ \ z\varepsilon C$$

with equivalent form

$$x''+x=0.001\cos(t),\ x(0)=1,\ \ x'(0)=0,$$
$$y''+y=0.001\sin(t),\ y(0)=0,\ \ y'(0)=0.9995.$$

The theoretical solution is

$$z(t)=x(t)+iy(t),\ x,y\varepsilon\Re,$$
$$x(t)=\cos(t)+0.0005t\sin(t),$$
$$y(t)=\sin(t)+0.0005t\cos(t).$$

4.3 *Two-body problem*

Consider the two-body system of coupled differential equations; see Vigo-Anguiar and Simos (2001).

$$x'' = -\frac{x}{r}, x(0) = 1,\ x'(0) = 0.$$

$$y'' = -\frac{y}{r}, y(0) = 0,\ y'(0) = 1,$$

where $r=(x^2 + y^2)^{3/2}$ and whose analytical solution is given by

$$x(t) = \cos(t),$$
$$y(t) = \sin(t).$$

5. Conclusions

The three problems studied in (Simos (1998, 2003), Stiefel and Bettis (1969), Vigo-Anguiar and Simos (2001), Vlachos et.al (2009)) have been re-examined for any step size in the interval [0,1] using the new implicit formula of algebraic order nine. Analysis of the basic properties of our method shows that it is symmetric, consistent and zero-stable. Tables 1-3 present the end point global error. The numerical results obtained with the step sizes equal to $\mathbf{h=2^{-n}}$ for several values of **n** in the interval of integration were compared with the analytical solution.

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Table 1: Solution to problem (4.1) for $h=2^{-n}$, n>0

t	Exact[x]	Computed[x]	Error[x]	Exact[y]	Computed[y]	Error[y]
1/16	*0.9980494633*	*0.9980494622*	*9.7150e-10*	*0.0624593182*	*0.0624593171*	*1.1002e-09*
1/4	*0.9689435093*	*0.9689434682*	*4.1024e-08*	*0.2474039852*	*0.2474039247*	*6..0481e-08*
7/16	*0.9059078696*	*0.9059070518*	*8.1782e-07*	*0.4236763954*	*0.4236736176*	*2.7778e-06*
5/8	*0.8111521558*	*0.8111500424*	*1.1133e-06*	*0.5850976720*	*0.5850234854*	*7.4186e-05*
13/16	*0.6879978749*	*0.6879212076*	*7.6667e-05*	*0.7260095202*	*0.7256493569*	*3.6016e-04*
1	*0.5407619995*	*0.539917176*	*8.3028e-04*	*0.8414725701*	*0.8395507379*	*1.9218e-03*

Table 2: Solution to orbital problem (4.2) for $h=2^{-n}$, n>0

t	Exact[x]	Computed[x]	Error[x]	Exact[y]	Computed[y]	Error[y]
1/16	*0.9980494626*	*0.9980494461*	*1.6428e-08*	*0.0624281289*	*0.0624281088*	*2.0012e-08*
3/16	*0.9824907884*	*0.9824903573*	*4.3110e-07*	*0.1863111898*	*0.1863105947*	*5.9516e-07*
3/8	*0.9305762980*	*0.9305753150*	*9.8301e-07*	*0.3660980589*	*0.3660961928*	*1.8660e-06*
5/8	*0.8111459624*	*0.8111348542*	*1.1108e-05*	*0.5848438470*	*0.5848028445*	*4.1001e-05*
3/4	*0.7319444834*	*0.7318741524*	*7.0331e-05*	*0.6813643767*	*0.6812960855*	*6.8291e-04*
15/16	*0.5921829256*	*0.59126445036*	*9.1842e-04*	*0.8058036996*	*0.8047898881*	*1.0138e-03*

Table 3: Solution to two-body system (4.3) for $h=2^{-n}$, n>0

t	Exact[x]	Computed[x]	Error[x]	Exact[y]	Computed[y]	Error[y]
1/16	*0.9980475107*	*0.9980475084*	*2.2381e-09*	*0.0624593178*	*0.0624593077*	*1.0081e-08*
1/4	*0.9689124217*	*0.9689123336*	*8.8112e-08*	*0.2474039592*	*0.2474033485*	*6.1074e-07*
7/16	*0.9058136834*	*0.9058135820*	*1.0144e-07*	*0.4236762572*	*0.4236739470*	*2.3101e-06*
5/8	*0.8109631195*	*0.8109579881*	*5.1314e-06*	*0.5850972729*	*0.5850524508*	*4.4822e-05*
3/4	*0.7316888688*	*0.7316688520*	*2.0017e-05*	*0.6816387600*	*0.6814397497*	*1.9901e-04*
15/16	*0.5918050751*	*0.5911368239*	*6.6825e-04*	*0.8060811083*	*0.8050792884*	*1.0018e-03*

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