# Approximate Solutions of the Non-Relativistic Schrödinger Equation with Inversely Quadratic Yukawa plus Mobius Square Potential via Parametric Nikiforov-Uvarov Method

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**Abstract**
We study Schrödinger equation for inversely quadratic Yukawa plus Mobius square potential using the generalized parametric form of Nikiforov-Uvarov method. We obtain energy eigenvalues and the corresponding normalized wave function expressed in terms of the hypergeometric functions. Three special cases of this potential have been discussed. Numerical values of the energy eigenvalues are also computed for various states.
**Keywords**: Approximate solutions, Schrödinger equation, Nikiforov-Uvarov method, inversely quadratic Yukawa potential, Mobius square potential

## 1. INTRODUCTION

Over the years theoretical physics has been successful in explaining the behavior of different particles in different potentials. This has been made possible through obtaining exact or approximate solutions of the non-relativistic and relativistic wave equations for different physical systems of interest. The exact or approximate solutions of these equations with the central potential play an important role in quantum mechanics (Schiff,1955; Dirac,1958; Landau and Lifshitz,1977 and Greiner,1989). For example, the exact solutions of the hydrogen atom and a harmonic oscillator (Dirac,1958 and Greiner,1989) have been investigated in recent times (Dong, 2003). In fact, exact solutions of a quantum system are significant in Physics. Solving the non-relativistic and relativistic equation is still an interesting work in the existing literature (Ikot *et.al*.,2011; Antia *et.al*., 2013; Ikot *et.al*.,2013; Qiang and Dong,2009; Zhang *et.al*., 2005 and Qiang and Dong,2008).

Recently, the study of exponential-like type potentials have attracted much attention (Qiang and Dong, 2009 and Yahaya *et.al*.,2010). However, the bound state solutions of the Schrödinger equation of some of these potentials are possible for few cases such as Coulumb potential (Dong and Dong, 2002), Woods-Saxon (Berkdemir *et.al*., 2008), Hulthen (Flugge,1971), Manning-Rosen (Ikhdair, 2011) and others

Moreover, when the arbitrary angular momentum quantum number $l$ is present, one can only solve the Schrödinger equation approximately using a suitable approximation scheme (Lu, 2005). Some of such approximations include conventional approximation scheme proposed by Greene and Aldrich (Greene and Aldrich, 1976), improved approximation scheme by Jia e*t.al*., (2009), elegant approximation scheme (Hill,1954) and recently a good approximation by Yazarloo *et al.* (2012). These approximation are used to deal with the centrifugal term and many authors have investigated approximately the bound state solutions of the Schrödinger equation with exponential-like potentials. For further details readers can refer to most recent works (Arda *et.al*., 2010; Arda and Sever, 2011 and Yi *et.al*., 2004).

Various methods have been used to solve Schrödinger Klein-Gordon and Dirac equations, exactly or approximately. These methods include asymptotic iteration method (AIM) (Yauk and Bahar, 2012), Suppersymmetric quantum mechanics (SUSYQM) (Hassanabadi *et.al.,* 2012), Nikiforov-Uvarov (NU) method (Nikiforov and Uvarov, 1988) and others (Ikot, 2012 and Hassanabadi *et.al*., 2011).

The main aim of this paper is to use the proposed approximation in ref. (Yazarloo *et.al*., 2012) and the Nikiforov-Uvarov method to obtain the approximate bound state solutions of the Schrödinger equation with inversely quadratic Yukawa plus Mobius square (IQYMS) potential defined as

$$V(r) = -\frac{V_0 e^{-\alpha r}}{r} + \frac{V_1}{r^2} + V_2\left(\frac{A + Be^{-\alpha r}}{C + D' e^{-\alpha r}}\right)^2, \tag{1}$$

where $V_0, V_1, V_2$ are the potential depths and $A, B, C, D$ are the potential parameters and $\alpha$ is the screening parameter (range). The hypothetical IQYMS potential is a short ranged and hadronic. It could be used to describe nucleon-nucleon interactions, meson-meson interaction and has other applications in atomic and nuclear physics, chemical and other related areas. The rest of the paper is organised as follows. In section 2, the parametric Nikiforov-Uvarov method is presented. Factorization method is presented in section 3. In section 4, solutions of the radial part of Schrodinger equation with IQYMS potential is presented. We discuss the results of our work in section 5. Finally, we present a brief conclusion in section 6.

## 2. THE GENERALIZED PARAMETRIC NIKIFOROV-UVAROV (NU) METHOD

The NU method was presented by Nikiforov-Uvarov (Nikiforov and Uvarov, 1988) and has been employed to solve second order differential equations such as the Schrödinger wave equation (SWE), Klein-Gordon equation (KGE), Dirac equation etc. The SWE

$$\Psi''(r)+\left[E-V(r)\right]\Psi(r)=0 \quad (2)$$

can be solved by transforming it into a hypergeometric-type equation through using the transformation, $s=s(x)$ and its resulting equation is expressed as

$$\Psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\Psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\Psi(s)=0, \quad (3)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ must be polynomials of at most second-degree and $\tilde{\tau}(s)$ is a polynomial with at most first-degree and $\Psi(s)$ is a function of the hypergeometric type.

The parametric generalization of the NU method is given by the generalized hypergeometric-type equation as (Tezcan and Sever, 2009)

$$\Psi''(s)+\frac{(c_1-c_2s)}{s(1-c_3s)}\Psi'(s)+\frac{1}{s^2(1-c_3s)^2}\left[-\xi_1s^2+\xi_2s-\xi_3\right]\Psi(s)=0. \quad (4)$$

Equation (4) is solved by comparing it with Eq. (3) and the following polynomials are obtained:

$$\tilde{\tau}(s)=(c_1-c_2s),\ \sigma(s)=s(1-c_3s),\tilde{\sigma}(s)=-\xi_1s^2+\xi_2s-\xi_3. \quad (5)$$

According to the NU method, the energy eigenvalues equation and eigenfucvntions, respectively satisfy the following sets of equation

$$c_2n-(2n+1)c_5+(2n+1)\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)+n(n-1)c_3+c_7+2c_3c_8+2\sqrt{c_8c_9}=0, \quad (6)$$

$$\Psi(s)=N_{nl}s^{c_{12}}(1-c_3s)^{-c_{12}-(c_{13}/c_3)}P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3s), \quad (7)$$

where

$$c_4=\frac{1}{2}(1-c_1),\ c_5=\frac{1}{2}(c_2-2c_3),\ c_6=c_5^2+\xi_1,\ c_7=2c_4c_5-\xi_2,\ c_8=c_4^2+\xi_3,$$
$$c_9=c_3c_7+c_3^2c_8+c_6,\ c_{10}=c_1+2c_4+2\sqrt{c_8},\ c_{11}=c_2-2c_5+2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right),$$
$$c_{12}=c_4+\sqrt{c_8},\ c_{13}=c_5-\left(\sqrt{c_9}+c_3\sqrt{c_8}\right) \quad (8)$$

and $P_n$ is the orthogonal Jacobi polynomials.

## 3 FACTORIZATION METHOD

In spherical coordinate, the Schrödinger equation with the potential $V(r)$ is given as (Landau and Lifshitz, 1977)

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$
$$\times\psi(r,\theta,\varphi)+V(r)\psi(r,\theta,\varphi)=E\psi(r,\theta,\varphi). \quad (9)$$

Using the common ansatz for the wave function as

$$\psi(r,\theta,\varphi)=\frac{R(r)}{r}Y_{lm}(\theta,\varphi) \quad (10)$$

and substituting Eq. (10) into Eq. (9), we obtain the following sets of equations:

$$\frac{d^2R_{nl}}{dr^2}+\frac{2\mu}{\hbar^2}\left[E-V(r)-\frac{\lambda\hbar^2}{2\mu r^2}\right]R_{nl}=0, \quad (11)$$

$$\frac{d^2\Theta(\theta)}{d\theta^2}+\cot\theta\frac{d\Theta(\theta)}{d\theta}+\left(\lambda-\frac{m^2}{\sin^2\theta}\right)\Theta(\theta)=0, \quad 12$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}+m_l^2\Phi(\varphi)=0, \quad 13$$

where $\lambda=l(l+1)$ and $m_l^2$ are the separation constants. $Y_{ml}(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$ is the solution of Eqs. (12) and (13). sare the spherical harmonic and their solutions are well known (Schiff, 1955). Equation (11) is the radial part of the Schrödinger equation which is subject of discussion in the next section.

## 4. SOLUTIONS OF THE RADIAL PART OF SCHRÖDINGER EQUATION WITH IQYMS POTENTIAL

Substituting potential of Eq. (1) into the radial Schrödinger equation of eq. (11), we obtain

$$\frac{d^2R_{nl}(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[E+\frac{V_0e^{-\alpha r}}{r}-\frac{V_1}{r^2}-V_2\left(\frac{A+Be^{-\alpha r}}{C+D'e^{-\alpha r}}\right)^2-\frac{l(l+1)\alpha^2\hbar^2}{2\mu r^2}\right]R_{nl}(r)=0 \quad 14$$

It is well known that the Schrödinger equation of eq. (14) cannot be solved exactly for $l\neq 0$ by any known method. The way out is to use approximation for the centrifugal term. On this note, we invoke a good approximation for the centrifugal barrier $\frac{1}{r^2}$ as (Yazarloo *et.al.*, 2012)

$$\frac{1}{r^2}\approx\alpha^2\left(\frac{C}{C+D'e^{-\alpha r}}\right)^2$$
$$=\lim_{\alpha\to 0}\left(\frac{1}{r^2}+\frac{\alpha}{r}+\frac{5}{12}\alpha^2+\frac{1}{12}\alpha^3r+\frac{1}{240}\alpha^4r^2-\frac{1}{720}\alpha^5r^3-\frac{1}{6045}\alpha^6r^4+O(r^5)\right), \quad 15$$

which is valid for $\alpha r\leq 1$ when $C=1$ and $D'=-1$ (Yazarloo *et.al.,* 2012; Qiang and Dong, 2007; Dong *et.al.,* 2007 and Wei *et.al.*, 2008), similar to other related work (Greiner, 2000 and Dong, 2011). In addition, when performing a power series expansion and setting $\alpha\to 0$, Eq. (15) gives the desired $r^{-2}$ suggested by Greene and Aldrich (Greene and Aldrich, 1976). By substituting Eq. (15) into Eq. (14) and using the transformation $s=e^{-\alpha r}$, we have

$$\frac{d^2R_{nl}(s)}{ds^2}+\frac{\left(1+\frac{D'}{C}s\right)}{s\left(1+\frac{D'}{C}s\right)}\frac{dR_{nl}(s)}{ds}+\frac{1}{s^2\left(1+\frac{D'}{C}s\right)^2}\left[-Q_1s^2+Q_2s-Q_3\right]R_{nl}(s)=0, \quad 16$$

where $\varepsilon^2=\frac{2\mu E_{nl}}{\alpha^2\hbar^2}$,

$$\xi_1=Q_1=\frac{D'^2}{C^2}\varepsilon^2+\frac{2\mu}{\alpha\hbar^2C}\left(-V_0D'+\frac{V_2B^2}{\alpha C}\right),$$

$$\xi_2=Q_2=2\frac{D'}{C}\varepsilon^2+\frac{2\mu}{\alpha\hbar^2}\left(V_0-\frac{2V_2AB}{\alpha C^2}\right),$$

$$\xi_3=Q_3=-\varepsilon^2+\frac{2\mu}{\hbar^2}\left(V_1+\frac{V_2A^2}{\alpha^2C^2}\right)+l(l+1) \quad 17$$

Comparing Eq. (4) with eq. (16), we obtain the following parameters:

$$c_1 = 1,\ c_2 = c_3 = -\frac{D'}{C},\ c_4 = 0,\ c_5 = \frac{1}{2}\frac{D'}{C},\ c_6 = \frac{D^{1^2}}{4c^2} - \frac{D^{1^2}}{c^2}\varepsilon^2 + \frac{2\mu}{\alpha\hbar^2 c}\left(-V_0 D^1 + \frac{V_2 B^2}{\alpha c}\right)$$

$$c_7 = -\frac{2D'}{C}\varepsilon^2 - \frac{2\mu}{\alpha\hbar^2}\left(V_0 - \frac{2V_2 AB}{\alpha C^2}\right),\ c_8 = -\varepsilon^2 + \frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2 A^2}{\alpha C^2}\right) + l(l+1)$$

$$c_9 = -\frac{D'}{C}\frac{2V_2 AB}{\alpha C^2} + \frac{D'^2}{C^2}\frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2 A^2}{\alpha C^2}\right) + \frac{D'^2}{C^2}l(l+1) + \frac{D'^2}{4C^2} + \frac{2\mu}{\alpha\hbar^2 C}\frac{V_2 B^2}{\alpha C}$$

$$c_{10} = 1 + 2\sqrt{-\varepsilon^2 + \frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2 A^2}{\alpha C^2}\right) + l(l+1)}$$

$$c_{11} = -\frac{2D'}{C} + 2\left(\begin{array}{l}\left[-\frac{D'}{C}2\frac{V_2 AB}{\alpha C^2} + \frac{D'^2}{C^2}\frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2 A^2}{\alpha C^2}\right) + \frac{D'^2}{C^2}l(l+1) + \frac{D'^2}{4C^2} + \frac{2\mu}{\alpha\hbar^2 C}\frac{V_2 B^2}{\alpha C}\right]^{\frac{1}{2}} \\ -\frac{D'}{C}\left[-\varepsilon^2 + \frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2 A^2}{\alpha C^2}\right) + l(l+1)\right]^{\frac{1}{2}}\end{array}\right)$$

$$c_{12} = \left[-\varepsilon^2 + \frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2 A^2}{\alpha C^2}\right) + l(l+1)\right]^{\frac{1}{2}}$$

$$c_{13} = \frac{D'}{2C} - \left(\begin{array}{l}\left[-\frac{D'}{C}2\frac{V_2 AB}{\alpha C^2} + \frac{D'^2}{C^2}\frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2 A^2}{\alpha C^2}\right) + \frac{D'^2}{C^2}l(l+1) + \frac{D'^2}{4C^2} + \frac{2\mu}{\alpha\hbar^2 C}\frac{V_2 B^2}{\alpha C}\right]^{\frac{1}{2}} \\ -\frac{D'}{C}\left[-\varepsilon^2 + \frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2 A^2}{\alpha C^2}\right) + l(l+1)\right]^{\frac{1}{2}}\end{array}\right) \qquad 18$$

Substituting Eq. (18) into energy eigenvalues equation of Eq. (6) we obtain the energy equation for this system as

$$-\frac{D'}{C}n^2 - (2n+1)\frac{D'}{C} + (2n+1)\left(\begin{array}{l}\left[-\frac{D'}{C}2\frac{V_2 AB}{\alpha C^2} + \frac{D'^2}{C^2}\frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2 A^2}{\alpha C^2}\right) + \frac{D'^2}{C^2}l(l+1) + \frac{D'^2}{4C^2} + \frac{2\mu}{\alpha\hbar^2 C}\frac{V_2 B^2}{\alpha C}\right]^{\frac{1}{2}} \\ -\frac{D'}{C}\left[-\varepsilon^2 + \frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2 A^2}{\alpha C^2}\right) + l(l+1)\right]^{\frac{1}{2}}\end{array}\right)$$

$$-\frac{2\mu}{\alpha\hbar^2}\left(V_0 - \frac{2V_2 AB}{\alpha C^2}\right) - \frac{2D'}{C}\left(\frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2 A^2}{\alpha C^2}\right) + l(l+1)\right) + 2\left[\begin{array}{l}\left(-\varepsilon^2 + \frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2 A^2}{\alpha C^2}\right) + l(l+1)\right) \\ \left(\begin{array}{l}-\frac{D'}{C}2\frac{V_2 AB}{\alpha C^2} + \frac{D'^2}{4C^2}\frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2 A^2}{\alpha C^2}\right) + \\ \frac{D'^2}{4C^2}l(l+1) + \frac{D'^2}{4C^2} + \frac{2\mu}{\alpha\hbar^2 C}\frac{V_2 B^2}{\alpha C}\end{array}\right)\end{array}\right]^{\frac{1}{2}} = 0 \qquad 19$$

Solving Eq. (19) explicitly, we obtain the energy eigenvalues of the system as

 

$$E_{nl} = -\frac{\alpha^2\hbar^2}{8\mu}\left[\frac{\beta + \frac{D'}{C}\left(n+\frac{1}{2}-\gamma\right)^2}{\frac{D'}{C}\left(n+\frac{1}{2}-\gamma\right)}\right]^2 + \alpha^2\left(V_1 + \frac{V_2A^2}{\alpha C^2}\right) + \frac{\alpha^2\hbar^2}{2\mu}l(l+1), \quad 20$$

where

$$\gamma = \frac{C}{D'}\left[\frac{D'^2}{4C^2} - \frac{D'}{C}2\frac{V_2AB}{\alpha C^2} + \frac{D'^2}{C^2}\frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2A^2}{\alpha C^2}\right) + \frac{D'^2}{C^2}l(l+1) + \frac{2\mu}{\alpha\hbar^2 C}\frac{V_2B^2}{\alpha C}\right]^{\frac{1}{2}}$$

$$\beta = \frac{2\mu}{\alpha\hbar^2}\left(V_0 - 2\frac{V_2AB}{\alpha C^2}\right) + \frac{D'}{C}\frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2A^2}{\alpha C^2}\right) + \frac{D'}{C}l(l+1) + \frac{2V_2AB}{\alpha C^2} - \frac{C}{D'}\frac{2\mu}{\alpha\hbar^2 C}\frac{V_2B^2}{\alpha C}. \quad 21$$

Using Eqs. (7) and (18), the wave function of this system is obtained as

$$R_{nl}(r) = N_{nl}s^{\in}\left(1 + \frac{D'}{C}s\right)^{v+\frac{1}{2}} P_n^{(2\in,2v)}\left(1 + \frac{2D'}{C}s\right), \quad 22$$

where

$$\in = \sqrt{-\varepsilon^2 + \frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2A^2}{\alpha C^2}\right) + l(l+1)}$$

$$v = -\frac{C}{D'}\left[-\frac{D'}{C}2\frac{V_2AB}{\alpha C^2} + \frac{D'^2}{C^2}\frac{2\mu}{\hbar^2}\left(V_1 + \frac{V_2A^2}{\alpha C^2}\right) + \frac{D'^2}{C^2}l(l+1) + \frac{D'^2}{4C^2} + \frac{2\mu}{\alpha\hbar^2 C}\frac{V_2B^2}{\alpha C}\right]^{\frac{1}{2}}$$

Furthermore, the relation between the hypergeometric function and the Jacobi polynomials are (Dong, 2007)

$$P_n^{(a,b)}(z) = \frac{\Gamma(n+a+1)}{n!\Gamma(a+1)}\ {}_2F_1\left(-n, n+a+b+1; 1+a; \frac{1-z}{2}\right) \quad 23$$

with $a = 2\in > -1, b = 2v > -1$ under the transformation $z = 1 + \frac{2D'}{C}e^{-\alpha r}$.

The normalization constant $N_{nl}$ can be found from normalization condition as (Ikot *et.al.*, 2010)

$$\int_0^\infty |R(r)|^2 dr = \alpha^{-1}\int_0^1 \frac{1}{s}|R_{nl}(r)|^2 ds = 1 \quad 25$$

By using the following integral formula (Ikhdair and Sever, 2010)

$$\int_0^1 (1-z)^{2(\delta+1)} z^{2\lambda-1}\{{}_2F_1(-n, n+2(\delta+\lambda+1); 1+2\lambda;\ z\}^2 dz$$
$$= \frac{(n+\delta+1)n!\Gamma(n+2\delta+2)\Gamma(2\lambda)\Gamma(2\lambda+1)}{(n+\delta+\lambda+1)\Gamma(n+2\lambda+1)\Gamma(2(\delta+\lambda+1)+n)}. \quad 26$$

With the help of Eq. (26) and after some calculations the normalization constant $N_{nl}$ under special condition that $C = 1$ and $D' = -1$ is obtained as

$$N_{nl} = \sqrt{\frac{n!2\in\left(n+v+\frac{1}{2}+\in\right)\Gamma\left(2(v+\frac{1}{2}+\in)+n\right)}{\alpha\left(n+v+\frac{1}{2}\right)\Gamma(n+2\in+1)\Gamma(n+2v+1)}} \quad 27$$

Finally, the total normalized wave function $\Psi(r,\theta,\varphi)$ of the inversely quadratic Yukawa plus Mobius square potential is obtained using Eq. (10) as

 

$$\Psi(r,\theta,\varphi)=\sqrt{\frac{n!2\in\left(n+v+\frac{1}{2}+\in\right)\Gamma\left(2(v+\frac{1}{2}+\in)+n\right)}{\alpha\left(n+v+\frac{1}{2}\right)\Gamma(n+2\in+1)\Gamma(n+2v+1)}}$$
$$\times\frac{1}{r}\left(e^{-\alpha r}\right)^{\in}\left(1+\frac{D'}{C}e^{-\alpha r}\right)^{v+\frac{1}{2}}\frac{\Gamma(n+2\in+1)}{n!\Gamma(2\in+1)}$$
$$\times{}_2F_1\left(-n,n+2\in+2v+1;1+2\in;\frac{1-s}{2}\right)Y_{lm}(\theta,\varphi) \qquad 28$$

## 5. DISCUSSION

The variation of the IQYMS potential with the radial distance of separation between the interacting particles $(r)$ for different screening parameter $(\alpha)$ with $V_2=0.3,5.0$ and $10.0$ in Figs. 1, 2 and 3 respectively. At very low $\alpha$, the curves almost coincide. Showing that the curves become interpretable at large $\alpha$. This implies that the interacting particles are very massive. Alas, the asymptotic nature of the curves cannot be over emphasized. This is subject to the fact that the origin of the interaction coordinate (taken to be the point $r=0$) is a point of singularity of the IQYMS potential. At depth as low as $V_2=0.3$ (as in Fig. 1) the curves all seem to be monotone decreasing on the radial interval $[\delta,\infty)$, where $0\,fm<\delta<0.5\,fm$. This tendency is retained for the very massive hadrons. But at high depth $V_2=5.0$ *and* $10.0$ (see Figs 2 and 3) and for the less massive hadrons, the monotonicity of the curves begin to increase and the interacting potential becomes constant at a certain interaction distance for a particular screening parameter (alpha, $\alpha$).

In order to test accuracy of our work, we compute numerical values. Numerical result of the approximation scheme we employed is displayed in Fig. 4, showing that this approximation is good for the parameter of alpha, $\alpha=0.01$ (short range). The energy spectrum of the IQYMS potential is reported numerically for various states with different screening parameters $(\alpha)$ in table 1. There is no degeneracies for the considered eigen state. The energy eigenvalues increases with $n$ and $l$ values and drop abruptly as $l$ reduces. Thus the particles with lower angular momentum can be easily separated from the IQYMS potential.

Now a few special cases of our results are discussed below. By adjusting some potential parameters some well known potentials can be obtained. Setting $V_1=V_2=0, C=1, D'=-1$ into Eq. (1), Yukawa potential (Yukawa, 1935)

$$V(r)=-\frac{V_1e^{-\alpha r}}{r} \qquad 29$$

is obtained with the corresponding energy

$$E_{nl}^{Y}=-\frac{\alpha^2\hbar^2}{8\mu}\left[\frac{2l(l+1)-\frac{2\mu V_0}{\alpha\hbar^2}+\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)}\right)^2}{n+\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)}}\right]^2-\alpha V_0+\frac{\alpha^2\hbar^2}{\mu}l(l+1) \qquad 30$$

Similarly, we set $V_1=V_2=0, c=1, D^1=-1$ and $\alpha\to 0$ into Eq. (1), the potential reduces to Coulomb potential (Greiner, 2000)

$$V(r)=\frac{-V_0}{r}, \qquad 31$$

the corresponding energy eigenvalues become

$$E_{nl}^{C}=-\left(\frac{\mu V_0^2}{2\hbar^2}\right)\frac{1}{\left(n+\frac{1}{2}-\sqrt{\frac{1}{4}+l(l+1)}\right)^2} \quad 32$$

When $V_0=V_1=0,$ the potential of Eq. (1) reduces to Mobius square potential. The generalized form of Mobius square potential can be written as

$$V(r)=\frac{V_2}{C^2}\left[\frac{A^2+2ABe^{-\alpha r}+B^2e^{-2\alpha r}}{\left(1+\frac{D'}{C}e^{-\alpha r}\right)}\right] \quad 33$$

Comparing Eq. (33) with Deng-Fan potential (Dong, 2011)

$$V(r)=De\left[\frac{1-2(1+b)e^{-\alpha r}+(1+b)^2e^{-2\alpha r}}{\left(1-e^{-\alpha r}\right)^2}\right] \quad 34$$

for $C=1$ and $D'=-1$, we have $V_2A^2=De, V_2AB=-De(1+b), V_2B^2=De(1+b)^2$, where $b=e^{\alpha r_d}-1$ and $r_d$ is equilibrium inter-nuclear distance. Substituting these parameters into Eq. (20) give the energy spectrum of Deng-Fan potentials as

$$E_{nl}^{DF}=-\frac{\alpha^2\hbar^2}{8\mu}\left[\frac{\beta_1+\left(n+\frac{1}{2}+\gamma_1\right)^2}{\left(n+\frac{1}{2}+\gamma_1\right)}\right]^2+\alpha De+\frac{\alpha^2\hbar^2}{2\mu}l(l+1), \quad 35$$

where
$$\gamma_1=\sqrt{\frac{1}{4}-\frac{2De}{\alpha}(1+b)+\frac{2\mu De}{\hbar^2\alpha}+\frac{2\mu}{\alpha^2\hbar^2}De(1+b)^2+l(l+1)}$$

$$\beta_1=\frac{2\mu De}{\alpha\hbar^2}+\frac{2De}{\alpha}(1+b)-\frac{4\mu De}{\alpha^2\hbar^2}(1+b)-\frac{2\mu De}{\alpha^2\hbar^2}(1+b)^2+l(l+1)$$

## 6. CONCLUSION

In this paper, we have obtained the approximate bound state solutions of the Schrödinger equation with inversely quadratic Yukawa plus Mobius square (IQYMS) potential via parametric Nikiforov-Uvarov (NU) method. The energy eigenvlaues and the corresponding total normalized wave functions expressed in terms of hypergeometric functions for the system are also obtained. The numerical values of our result are presented in table 1. The behaviours of our potential are discussed in Figs.1, 2 and 3. Under appropriate choice of some parameters, our potential reduces to few well known potentials in the literature such as Yukawa, Deng-Fan and Coulomb potentials.

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Table 1: Energy eigenvaluesE(eV) of the IQYMS potential for $D'=-1, V_0=0.2, V_1=0.1, V_2=0.3, A=B=C=\mu=1,$ with $\alpha=0.01$ and $\alpha=0.02$

| $\|n, l\rangle$ | E(eV) | |
|---|---|---|
| | $\alpha=0.01$ | $\alpha=0.02$ |
| $\|0, 0\rangle$ | 3.95023073400 | 1.01502341400 |
| $\|0, 1\rangle$ | 4.08047936200 | 1.08126032600 |
| $\|0, 2\rangle$ | 4.34577613800 | 1.21853223000 |
| $\|0, 3\rangle$ | 4.75572010200 | 1.43643528600 |
| $\|0, 4\rangle$ | 5.32470981500 | 1.74936373400 |
| $\|1, 0\rangle$ | 4.38905007500 | 0.11277021670 |
| $\|1, 1\rangle$ | 0.45337706840 | 0.12012983800 |
| $\|1, 2\rangle$ | 0.48285447020 | 0.13538220070 |
| $\|1, 3\rangle$ | 0.52840377280 | 0.15959354470 |
| $\|2, 0\rangle$ | 0.15799894460 | 0.04058994168 |
| $\|2,1\rangle$ | 0.16320888010 | 0.04323937976 |
| $\|2, 2\rangle$ | 0.17382073200 | 0.04873017912 |
| $\|2, 3\rangle$ | 0.19021846170 | 0.05744618616 |
| $\|3, 0\rangle$ | 0.08060645179 | 0.02070351490 |
| $\|3, 1\rangle$ | 0.52840377280 | 0.02205524943 |
| $\|3, 2\rangle$ | 0.08867877758 | 0.02485663849 |
| $\|3, 3\rangle$ | 0.09704495154 | 0.02930352208 |

 

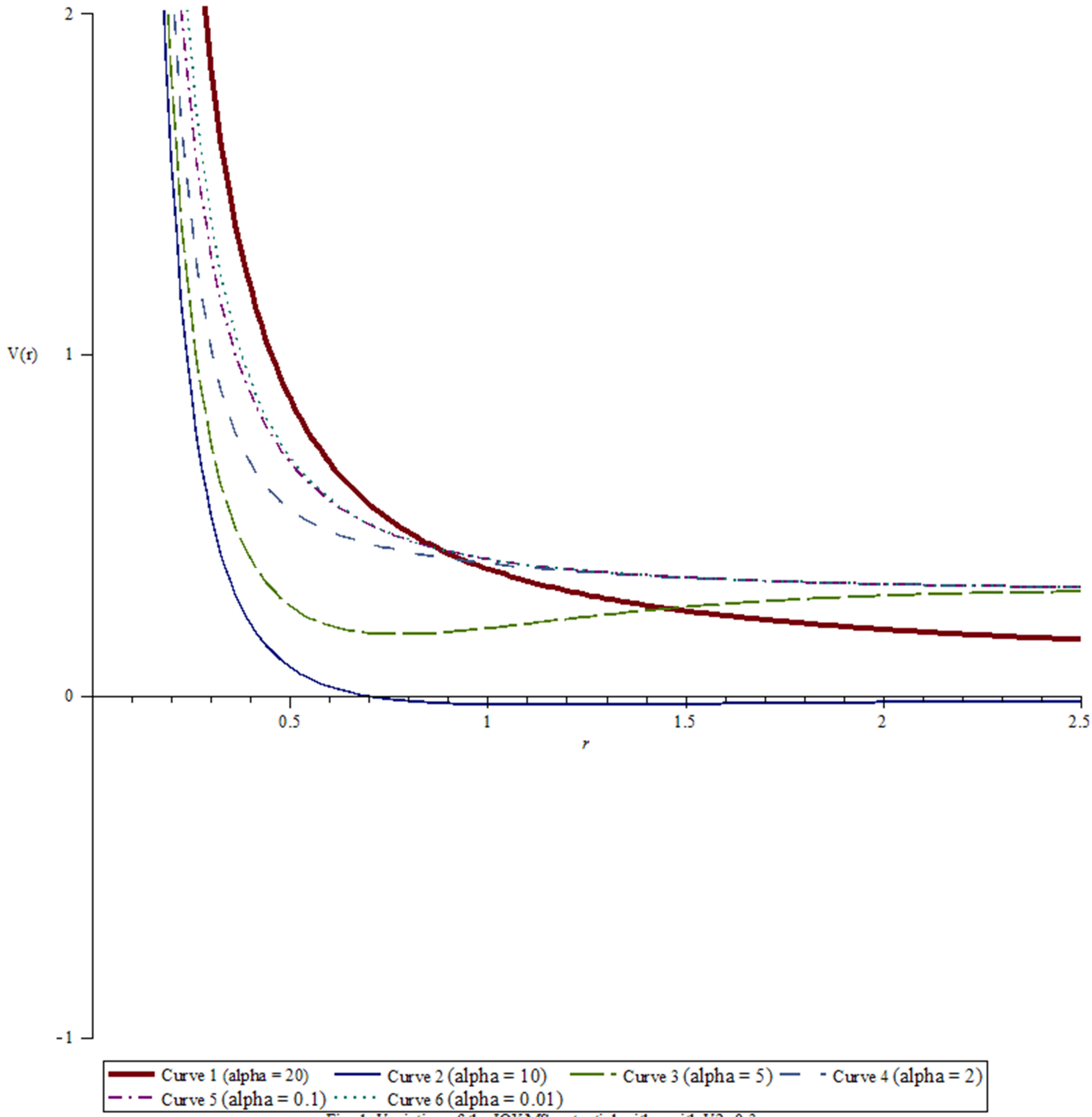


Fig. 1: Variation of the IQYMS potential with r with V2=0.3

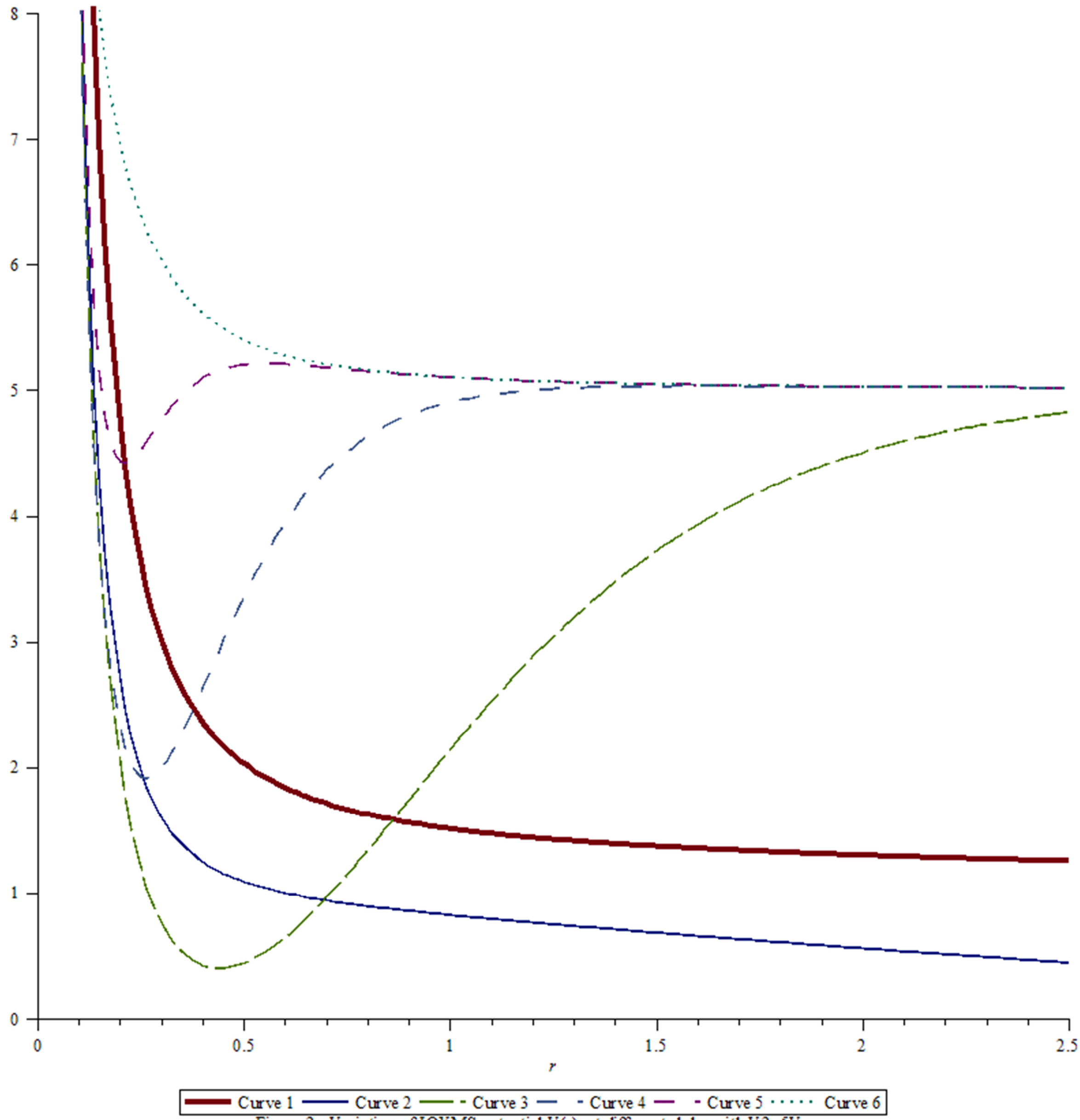


Figura 2: Variation of IQYMS potential V(r) at different alpha with V2=5V

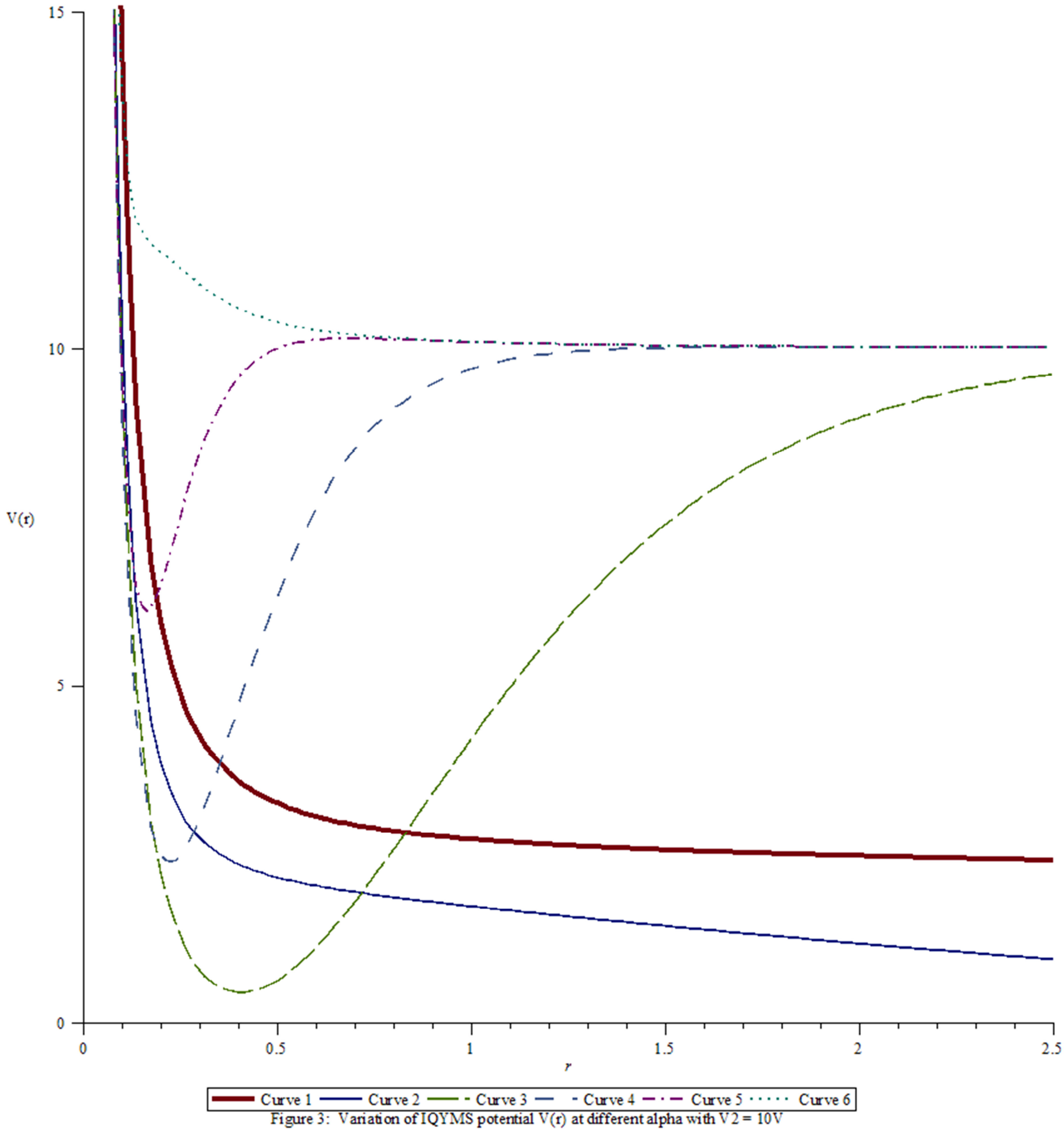

Figure 3: Variation of IQYMS potential V(r) at different alpha with V2 = 10V

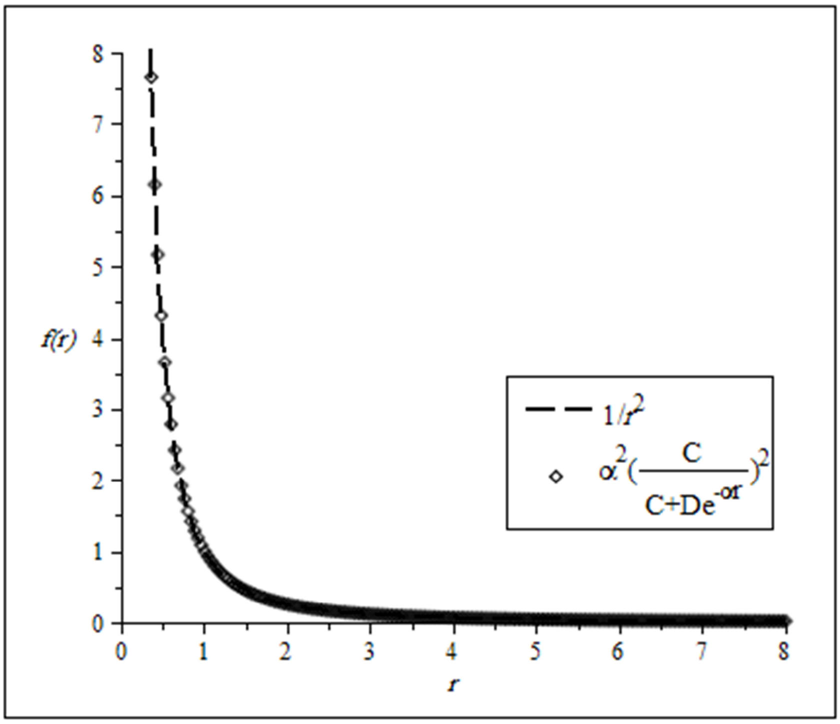


Figure 4: $\frac{1}{r^2}$ and its approximation for α=0.01.