Advances in Physics Theories and Applications
ISSN 2224-719X (Paper) ISSN 2225-0638 (Online)
Vol.46, 2015

Approximate bound state solutions of nonrelativistic Schrödinger equation with q-deformed Hulthen plus modified inversely quadratic Yukawa potential within the framework of Nikiforov-Uvarov method

Akaninyene D. Antia[1]* and Ita O. Akpan[2]
[1]Theoretical Physics Group, Department of Physics, University of Uyo, Nigeria.
[2]Department of Physics, University of Calabar, Nigeria.

Abstract
We study the nonrelativistic Schrödinger equation for q-deformed Hulthen plus modified inversely quadratic Yukawa potential using the generalized parametric form of NIkiforov-Uvarov method. The energy eigenvalues and the corresponding normalized wave functions expressed in terms of hypergeometric function are obtained. We have also discussed four (4) special cases of this potential, i.e. Woods-Saxon, Hulthen, Morse and inversely quadratic potentials.
Keywords: Nonrelativistic Schrödinger equation, q-deformed Hulthen potential, modified inversely quadratic Yukawa potential, parametric Nikiforov-Uvarov method.

1.0 Introduction

The exact solutions of the nonrelativistic and relativistic equations with the central potentials play important role in quantum physics and solving these equations is still of interest and have been successful over the years [1-10]. Recently, the study of exponential-type potential has attracted a lot of interest by many authors [11-12]. However, it is well known that the exact solutions of Schrödinger equation are possible only for a few set of quantum systems [13-14].

Moreover, when arbitrary angular momentum quantum number l is present, one can only solve the Schrödinger equation approximately using suitable approximation schemes [15]. Some of such approximations include conventional approximation scheme proposed by Greene and Aldrich [16], improved approximation scheme by Jia *et al.* [17], elegant approximation scheme [18] etc. These approximations are used to deal with the centrifugal term or potential barrier arising from the problem.

In solving nonrelativistic or relativistic wave equation whether for central or noncentral potential, various methods are used. These methods include asymptotic iteration method (AIM) [19], supersymmetric quantum mechanics (SUSYQM) [20], shifted $\frac{1}{N}$ expression [21], factorization method [22, 23], Nikiforov-Uvarov (NU) [24] and others [25, 26].

The aim of this paper is to apply the parametric generalization of Nikiforov-Uvarov (NU) method to study the nonrelativistic Schrödinger equation under the q-deformed Hulthen plus modified inversely quadratic Yukawa (qDHMIQY) potential defined as [27, 28]

$$V(r) = -\frac{V_0 e^{-2\alpha r}}{1-qe^{-2\alpha r}} - \frac{V_1 e^{-2\alpha r}}{r^2}, \qquad 1$$

where V_0, V_1 are the potential depth, α is the screening parameter and q is the deformation parameter. This potential is short ranged and hadronic which could be used to describe nucleon-nucleon interactions, meson-meson interaction and also in various branches of nuclear physics and quantum chemistry.

2.0 The generalized parametric Nikiforov-Uvarov (NU) method

The NU method was presented by Nikiforov and Uvarov [24] and has been employed to solve second order differential equations such as the Schrödinger wave equation (SWE), Klein-Gordon equation (KGE), Dirac equation (DE) etc. The SWE

$$\psi''(r) + [E - V(r)]\psi(r) = 0 \qquad 2$$

can be solved by transforming it into a hypergeometric type equation through using the transformation, $s = s(x)$ and its resulting equation is expressed as

$$\psi''(s) + \frac{\bar{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \qquad 3$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ must be polynomials of at most second degree and $\bar{\tau}(s)$ is a polynomial with at most first degree and $\psi(s)$ is a function of the hypergeometric type.

The parametric generalization of the NU method is given by the generalized hypergeometirc-type equation as [29]

$$\psi''(s)+\frac{(c_1-c_2s)}{s(1-c_3s)}\psi'(s)+\frac{1}{s^2(1-c_3s)^2}\left[-\xi_1 s^2+\xi_2 s-\xi_3\right]\psi(s)=0. \qquad 4$$

Equation (4) is solved by comparing it with Eq. (3) and the following polynomials are obtained:

$$\tilde{\tau}(s)=(c_1-c_2s),\ \sigma(s)=s(1-c_3s),\tilde{\sigma}(s)=-\xi_1 s^2+\xi_2 s-\xi_3. \qquad 5$$

According to the NU method, the energy eigenvalues equation and eigen functions, respectively, satisfy the following sets of equation

$$c_2n-(2n+1)c_5+(2n+1)\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)+n(n-1)c_3+c_7+2c_3c_8+2\sqrt{c_8c_9}=0, \quad 6$$

$$\psi(s)=N_{nl}s^{c_{12}}(1-c_3s)^{-c_{12}-(c_{13}/c_3)}P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3s), \qquad 7$$

where

$$c_4=\frac{1}{2}(1-c_1),\ c_5=\frac{1}{2}(c_2-2c_3),\ c_6=c_5^2+\xi_1,\ \ c_7=2c_4c_5-\xi_2,\ c_8=c_4^2+\xi_3,$$
$$c_9=c_3c_7+c_3^2c_8+c_6,\ c_{10}=c_1+2c_4+2\sqrt{c_8},\ \ c_{11}=c_2-2c_5+2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right), \qquad 8$$
$$c_{12}=c_4+\sqrt{c_8},\ \ c_{13}=c_5-\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)$$

and P_n is the orthogonal Jacobi polynomial.

3.0 Factorization method

In spherical coordinate the Schrödinger equation with central potential of Eq. (1) can be written as follows [30]

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \qquad 9$$
$$\times\psi(r,\theta,\varphi)+V(r)\psi(r,\theta,\varphi)=E\psi(r,\theta,\varphi).$$

The total wave function in Eq. (9) can be defined as

$$\psi(r,\theta,\varphi)=\frac{R(r)}{r}Y_{lm}(\theta,\varphi) \qquad 10$$

and by decomposing the spherical wave function in Eq. (9) using Eq. (10) we obtain the following equations:

$$\frac{d^2R(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}(E-V(r))-\frac{\lambda}{r^2}\right]R(r)=0, \qquad 11$$

$$\frac{d^2\Theta(\theta)}{d\theta^2}+\cot\theta\frac{d\Theta(\theta)}{d\theta}+\left(\lambda-\frac{m^2}{\sin^2\theta}\right)\Theta(\theta)=0, \qquad 12$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}+m^2\Phi(\varphi)=0, \qquad 13$$

where $\lambda=l(l+1)$ and m^2 are the separation constants. $Y_{lm}(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$ is the the solution of Eqs. (12) and (13). $Y_{lm}(\theta,\varphi)$ are the spherical harmonics and their solutions are well known [21]. Equations (11) is the radial part of Schrödinger equation which is subject for discussion in the preceding section.

4.0 Solutions of the radial Schrödinger equation

For eigenvalues and corresponding eigen functions of the radial part of the Schrödinger equation, we substitute Eq. (1) into (11) to obtain

$$\frac{d^2R(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E+\frac{V_0e^{-2\alpha r}}{1-qe^{-2\alpha r}}+\frac{V_1e^{-2\alpha r}}{r^2}\right)-\frac{\lambda}{r^2}\right]R(r)=0. \quad 14$$

Equation (14) has no analytical or exact solution for $l\neq 0$ due to the centrifugal term, but can be solved approximately. Here we make use of the approximation proposed by Greene and Aldrich [16] as

$$\frac{1}{r^2}\approx\frac{4\alpha^2e^{-2\alpha r}}{\left(1-qe^{-2\alpha r}\right)^2}, \quad 15$$

Substituting Eq. (15) into Eq. (14) yields

$$\frac{d^2R(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E+\frac{V_0e^{-2\alpha r}}{\left(1-qe^{-2\alpha r}\right)}+\frac{4\alpha^2V_1e^{-4\alpha r}}{\left(1-qe^{-2\alpha r}\right)^2}\right)-\frac{4\alpha^2\lambda e^{-2\alpha r}}{\left(1-qe^{-2\alpha r}\right)^2}\right]R(r)=0. \quad 16$$

Taking the transformation, $s=e^{-2\alpha r}$, Eq. (16) reduces to

$$\frac{d^2R(s)}{ds^2}+\frac{\left(1-qs\right)}{s\left(1-qs\right)}\frac{dR(s)}{ds}+\frac{1}{s^2\left(1-qs\right)^2}\left[-\left(q^2\varepsilon^2+qA'-B'\right)s^2+\left(2q\varepsilon^2+A'-\lambda\right)s-\varepsilon^2\right]R(s)=0, \quad 17$$

where the following dimensionless quantities have been defined as

$$-\varepsilon^2=\frac{\mu E}{2\alpha^2\hbar^2},\ A'=\frac{\mu V_o}{2\alpha^2\hbar^2},\ B'=\frac{2\mu V_1}{\hbar^2},\lambda=l\left(l+1\right) \quad 18$$

Comparing Eq. (17) with Eq. (4) and making use of Eq. (8), we obtain the following parameters:

$$\begin{aligned}
&c_1=1,c_2=c_3=q\\
&\xi_1=q^2\varepsilon^2+qA'-B',\ \xi_2=2q\varepsilon^2+A'-\lambda,\ \xi_3=\varepsilon^2\\
&c_4=0,\ c_5=-\frac{q}{2},\ c_6=\frac{q^2}{4}+q^2\varepsilon^2+qA'-B' \qquad 19\\
&c_7=-2q\varepsilon^2-A'+\lambda,\ c_8=\varepsilon^2,\ c_9=\frac{q^2}{4}-B'+q\lambda,\ c_{10}=1+2\varepsilon\\
&c_{11}=2\left(q+q\varepsilon+\sqrt{\frac{q^2}{4}-B'+q\lambda}\right),\ c_{12}=\varepsilon,\ c_{13}=-\left(\frac{q}{2}+q\varepsilon+\sqrt{\frac{q^2}{4}-B'+q\lambda}\right).
\end{aligned}$$

Substituting Eqs.(18) and (19) into Eq. (6), we obtain the energy spectrum for the $q-$deformed Hulthen plus modified inversely quadratic Yukawa potential as

$$qn+\left(2n+1\right)\frac{q}{2}+\left(2n+1\right)\left(\sqrt{\frac{q^2}{4}-\frac{2\mu V_1}{\hbar^2}+ql\left(l+1\right)}+q\sqrt{-\frac{\mu E_{n,l}}{2\alpha^2\hbar^2}}\right)+n\left(n-1\right)q+q\frac{\mu E_{n,l}}{\alpha^2\hbar^2}-\frac{\mu V_0}{2\alpha^2\hbar^2}$$
$$-q\frac{\mu E_{n,l}}{\alpha^2\hbar^2}+l\left(l+1\right)+2\sqrt{\left(-\frac{\mu E_{n,l}}{2\alpha^2\hbar^2}\right)\left(\frac{q^2}{4}-\frac{2\mu V_1}{\hbar^2}+ql\left(l+1\right)\right)}=0.$$
20

Solving Eq.(20) more explicitly, we obtain the energy eigenvalues as

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{q\left(n^2+n+\frac{1}{2}\right)-\frac{\mu V_0}{2\alpha^2\hbar^2}+l(l+1)+(2n+1)\sqrt{\frac{q^2}{4}-\frac{2\mu V_1}{\hbar^2}+ql(l+1)}}{\left(n+\frac{1}{2}\right)q+\sqrt{\frac{q^2}{4}-\frac{2\mu V_1}{\hbar^2}+ql(l+1)}}\right]^2 . \quad 21$$

Using Eqs.(7) and (19), corresponding wave functions for this system is obtained as

$$R(s) = N_{nl}s^{\varepsilon}(1-qs)^{\frac{1}{2}+\nu}P_n^{(2\varepsilon,2\nu)}(1-2qs), \quad 22$$

where $\nu = \frac{1}{q}\sqrt{\frac{q^2}{4}-B^{'}+q\lambda}$.

With $s = e^{-2\alpha r}$, Eq. (22) can also be written as

$$R(r) = N_{nl}e^{-2\alpha r\varepsilon}\left(1-qe^{-2\alpha r}\right)^{\frac{1}{2}+\nu}P_n^{(2\varepsilon,2\nu)}\left(1-2qe^{-2\alpha r}\right) \quad 23$$

where N_{nl} is a normalization constant.

The relation between the hypergeometric function and the Jacobi polynomial are [32]

$$P_n^{(a,b)}(z) = \frac{\Gamma(n+a+1)}{n!\Gamma(a+1)}\ {}_2F_1\left(-n, n+a+b+1; 1+a; \frac{1-z}{2}\right) \quad 24$$

with $a = 2\varepsilon > -1, b = 2\nu > -1$ under the transformation $z = 1-2qe^{-2\alpha r}$.

The normalization constant N_{nl} can be found from normalization condition as [33]

$$\int_0^{\infty}|R(r)|^2 dr = \alpha^{-1}\int_0^1\frac{1}{s}|R_{nl}(r)|^2 ds = 1 \quad 25$$

By using the following integral formula [34]

$$\int_0^1(1-z)^{2(\delta+1)}z^{2\lambda-1}\left\{{}_2F_1(-n,n+2(\delta+\lambda+1);1+2\lambda;\ z\right\}^2 dz$$
$$= \frac{(n+\delta+1)n!\Gamma(n+2\delta+2)\Gamma(2\lambda)\Gamma(2\lambda+1)}{(n+\delta+\lambda+1)\Gamma(n+2\lambda+1)\Gamma(2(\delta+\lambda+1)+n)}. \quad 26$$

With the help of Eq. (26) and after some calculations the normalization constant N_{nl} under special condition that $q=1$ is obtained as

$$N_{nl} = \sqrt{\frac{n!2\varepsilon\left(n+\nu+\frac{1}{2}+\varepsilon\right)\Gamma\left(2(\nu+\frac{1}{2}+\varepsilon)+n\right)}{\alpha\left(n+\nu+\frac{1}{2}\right)\Gamma(n+2\varepsilon+1)\Gamma(n+2\nu+1)}} \quad 27$$

Finally, the total normalized wave function $\Psi(r,\theta,\varphi)$ for the q deformed Hulthen plus modified inversely quadratic Yukawa potential is obtained using Eq. (10) as

$$\Psi(r,\theta,\varphi)=\sqrt{\frac{n!2\varepsilon\left(n+v+\frac{1}{2}+\varepsilon\right)\Gamma\left(2(v+\frac{1}{2}+\varepsilon)+n\right)}{\alpha\left(n+v+\frac{1}{2}\right)\Gamma(n+2\varepsilon+1)\Gamma(n+2v+1)}}$$
$$\times\frac{1}{r}\left(e^{-\alpha r}\right)^{\varepsilon}\left(1-2qe^{-2\alpha r}\right)^{v+\frac{1}{2}}\frac{\Gamma(n+2\varepsilon+1)}{n!\Gamma(2\varepsilon+1)} \quad 28$$
$$\times{}_2F_1\left(-n,n+2\varepsilon+2v+1;1+2\varepsilon;\frac{1-s}{2}\right)Y_{lm}(\theta,\varphi)$$

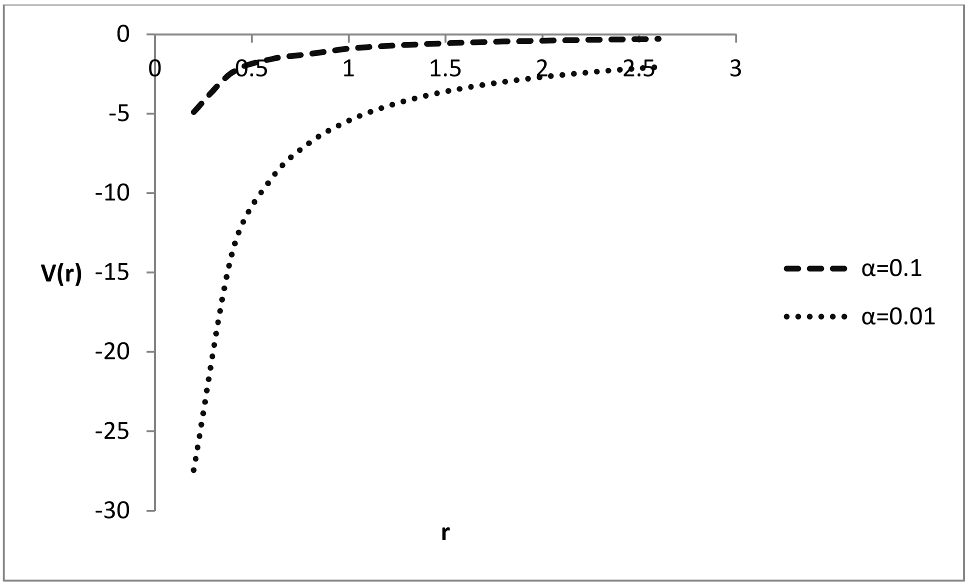

Fig. 1: Variation of Hulthen plus Yukawa potential with r for $V_0=0.1, V_1=0.5$ with $\alpha=0.1$ and $\alpha=0.01$.

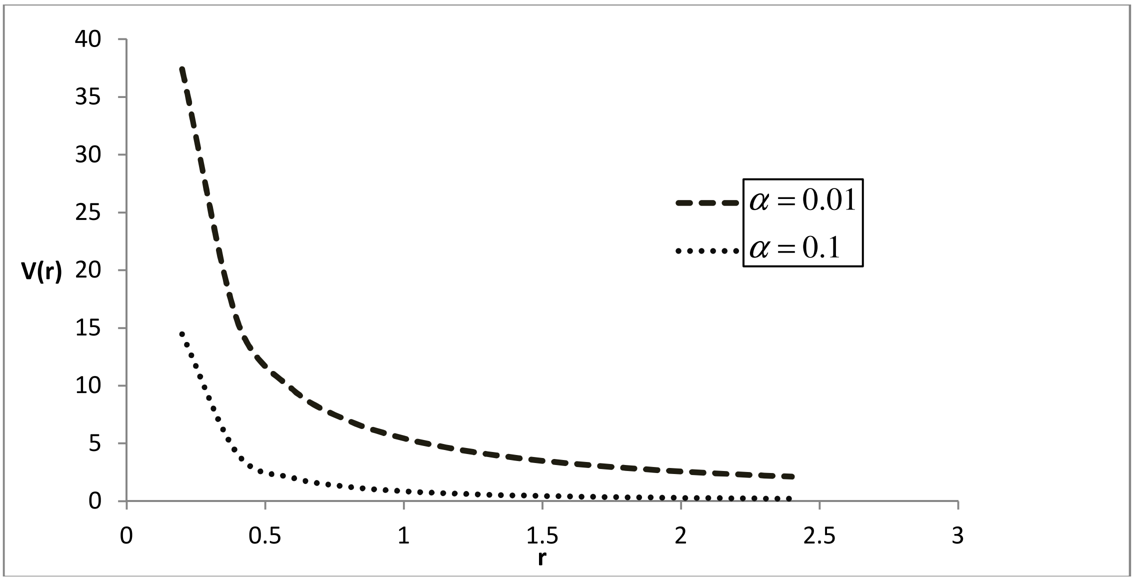

Fig. 2: Variation of qDHMIQY potential with r for $V_0=0.1, V_1=0.5$ with $\alpha=0.1$ and $\alpha=0.01$ for $q=1$.

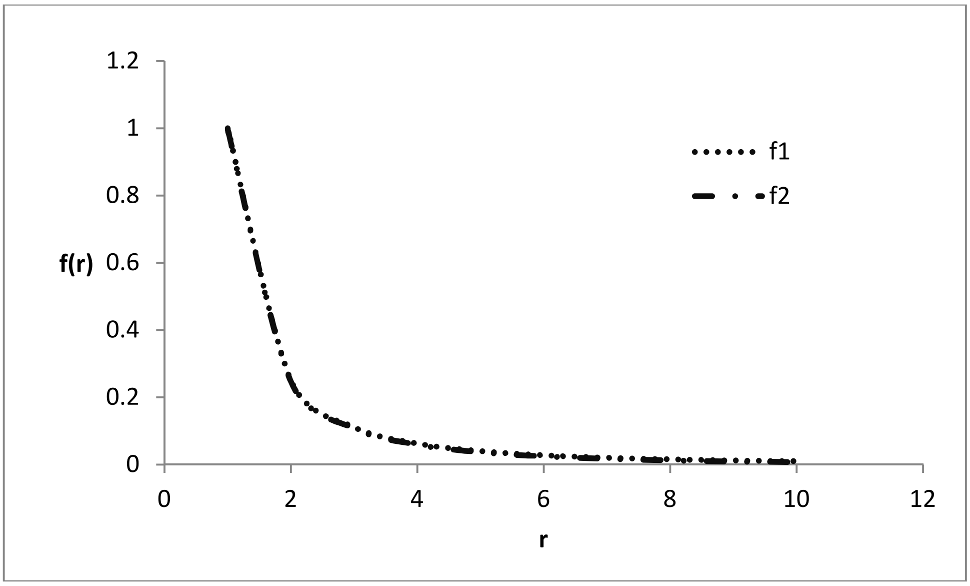

Fig. 3: Comparison of the centrifugal term f1= $\frac{1}{r^2}$ with the approximation f2 for $\alpha = 0.1$ for $q = 1$

5.0 Results and Discussion

The behaviors of Hulthen plus Yukawa potential is presented in Fig.1 while our qDHMIQY potential with r for $q = 1$ is presented in Fig. 2. Comparison of Fig.1 with Fig. 2 shows that our qDHMIQY potential accounted for positive potential unlike the Hulthen plus Yukawa potential that accounted for negative potential. In order to test accuracy of our work, we have compared the approximation of Eq.(15) for $\alpha = 0.1$ denoted as f2 for $q = 1$ with the centrifugal term f1= $\frac{1}{r^2}$ in Fig.3. This shows that the approximation is in good agreement with the centrifugal term.

Further, we are going to study four special cases of the q-deformed Hulthen plus modified inversely quadratic Yukawa potential and their corresponding energy eigenvalues by making appropriate choice for the potential parameters.

5.1 Woods-Saxon potential

If we set $V_1 = 0, q = -1$, it is found that our potential in Eq. (1) reduces to Woods-Saxon potential of the form [35]

$$V(r) = -\frac{V_0 e^{-2\alpha r}}{1+e^{-2\alpha r}}, \tag{29}$$

Substituting these parameters into Eq. (21) we obtain the corresponding eigenvalues

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\left(n^2+n+\frac{1}{2}\right)+\frac{\mu V_0}{2\alpha^2\hbar^2}-l(l+1)-(2n+1)\sqrt{\frac{1}{4}-l(l+1)}}{\left(n+\frac{1}{2}\right)-\sqrt{\frac{1}{4}-l(l+1)}}\right]^2 . \tag{30}$$

5.2 Hulthen potential

Setting $V_1 = 0, q = 1$ into Eq. (1) , we obtain Hulthen potential of the form [28]

$$V(r) = -\frac{V_0 e^{-2\alpha r}}{1-e^{-2\alpha r}}, \tag{31}$$

with the energy eigenvalue as

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\left(n^2+n+\frac{1}{2}\right)-\frac{\mu V_0}{2\alpha^2\hbar^2}+l(l+1)+(2n+1)\sqrt{\frac{1}{4}+l(l+1)}}{\left(n+\frac{1}{2}\right)+\sqrt{\frac{1}{4}+l(l+1)}}\right]^2 . \quad 39$$

5.3 Morse potential

If we set $V_1 = q = 0$ and map $V_0 \to -V_0$ and $\alpha \to \frac{1}{2b}$ in Eq. (1) we obtain Morse potential of the form [36]

$$V(r) = V_0 e^{-\frac{1}{b}r} \quad 44$$

Substituting these parameters into Eq.(21), the required energy spectrum of Morse potential is obtained accordingly.

5.3 Inversely quadratic potential

Setting $V_0 = 0, q = 1, \alpha = o$ into Eq.(1), we obtain inversely quadratic potential of the form [38]

$$V(r) = -\frac{V_1}{r^2} . \quad 45$$

Substituting these parameters into Eq.(21), the energy eigenvalues the potential in Eq.(45) is obtained.

6.0 Conclusion

In this paper, we have studied the bound state solutions of the nonrelativistic Schrödinger equation q-deformed Hulthen plus modified inversely quadratic Yukawa potential under the frame work of parametric Nikiforov-Uvarov method with the help of approximation scheme in ref.[16] to evaluate the centrifugal term. The bound states energy eigenvalues and the corresponding normalized wave functions in terms of hypergeometric function are obtained. Our results could be used to study the interactions and binding energies of the central potential for diatomic molecules in the nonrelativistic framework. The results will also have many applications in chemical and molecular physics and the recently reported result of neutron-proton pairs in heavy nuclei using perturbation theory [37]. The variation of Hulthen plus Yukawa potential with r for various screening parameter α is discussed in Fig. 1. The behaviors of our potential with r with $\alpha = 0.1$ and 0.01 for $q = 1$ is presented in Fig. 2 while comparison of the approximation used in this work with the centrifugal term is shown in Fig. 3. By appropriate choice of potential parameters our potential in Eq. (1) reduces to four (4) well known potentials: Woods-Saxon, Hulthen, Morse and inversely quadratic potentials and their respective energy eigenvalues are also evaluated.

REFERENCES

1. L. Z. Yi, Y. F. Diao, J. Y. Liu, C. S. Jia . Phys. Lett. A. 333, 212 (2004).
2. A. N. Ikot . Afri. Rev. Phys. 6, 221 (2011).
3. A. D. Antia, I. E. Essien, E. B. Umoren and C. C. Eze . Advances in Physics Theories and Applications 44, 1-13 (2015).
4. A. D. Antia, I. O. Akpan and A. O. Akankpo . International Journal of High Energy Physics 2(4), 50-55 (2015).
5. A. D. Antia, A. N. Ikot, E. E. Ituen and I. O. Akpan . Sri Lankan Journal of Physics 13(1), 27-40 (2012)
6. H. Hassanabadi, S. Zarrinkamar and A. A. Rajabi . Commun. Theor. Phys. 55, 541(2011).
7. W. C. Qiang and S. H. Dong . Phys. Scr. 79, 045004 (2009).
8. A. D. Antia, A. N. Ikot, H. Hassanabadi and E. Maghsoodi . Indian Journal of Physics 87(11), 1133-1139 (2013).
9. G. F. Wei, C. Y. Long, X. Y. Duan and S. H. Dong . Phys. Scr. 77, 035001 (2008).
10. S. Dong, Garcia-Ravalo and S. H. Dong . Phys. Scr. 76, 393 (2007).
11. W.A. Yahaya, K. J. Oyewumi, C.O. Akoshule and T. T. Ibrahim . J. Vect. Relat. 5, 27 – 34 (2010).
12. A.N.Ikot, A. D. Antia, I. O. Akpan and O. A. Awoga . Revista Mexicana de Fisica 59, 46-53 (2013).
13. K. J. Oyewumi and A. W. Ogunsola . J. Pure Appl. Scr. 10, 343 (2004).

14. S. Dong and S. H. Dong . Phys. Scr. 66, 342-344 (2002).
15. J. Lu . Phys. Scr. 72, 349 (2005).
16. R. L. Greene and C. Aldrich . Phys. Rev. A 14, 2363 (1976).
17. C. S. Jia, T. Chen and L. G. Cui . Phys. Lett. A 373, 1621 (2009).
18. E. H. Hill . Am. J. Phys. 22, 1712 (1954).
19. F. Yasuk and M. K. Bahar . Phys. Scr. 85, 045004 (2012).
20. H. Hassanabadi, E. Maghsoodi and S. Zarrinkamar . Euro. Phys. J. Plus 127, 31 (2012)..
21. O. Mustafa and R. Sever . Phys. Rev. A. 44, 4142 (1991).
22. A. D. Antia, E. E. Ituen, H. P. Obong and C. N. Isonguyo . Int. J. Recent advances in Phys. 4(1), 55-65 (2015).
23. S. M. Ikhdair and R. Sever . Int. J. Mod. Phys. 19, 221-229 (2008).
24. A. F. Nikiforov and V. B. Uvarov . Special Functions of mathematical Physics (Basel: Birkhauser) (1988).
25. A. N. Ikot . Chin. Phys. Lett. 29, 060307(2012).
26. H. Hassanabadi, B. H. Yazarloo, S. Zarrinkamar and A. A. Rajabi . Phys. Rev. C 84, 064003 (2011).
27. H. Yukawa . Proceeding of Physical mathematical Society of Japan 17, 48-52(1935).
28. L. Hulthen . Arkiv. Mathematics Astron. Fysik 28A(5), 1-12(1942).
29. C. Tezcan and R. Sever . Int. J. Theor. Phys. 48, 337 (2009).
30. L. D. Landau and E. M. Lifshitz . Quantum Mechanics, Non-relativistic Theory (Canada: Pergamon) (1977).
31. L. I. Schiff . Quantum Mechanics (New York: McGraw Hill) (1955).
32. S. H. Dong (2007). Factorization Method in Quantum Mechanics (Springer, Dordrecht).

33. A. N.Ikot, O.A. Awoga, A.D. Antia, H. Hassanabadi and E. Maghsoodi . Few-Body System Doi 10.1007/s00601-013-0706-1 (2013).
34. S. M. Ikhdair and R. Sever . Appl. Math. Comp. 216,911 (2010).
35. A. Berkdemir, C. Berkdemir and R. Sever . Modern Phys. Lett. A 21,2087 (2006).
36. A. D. Alhaidari . Phys. Lett. B 699(4), 309-313 (2011).
37. K. M.Khanna, G. F. Kanyeki, S.K.Rotich, P. K. Torongey and S. E. Ameka . Indian J. Pure and Appl. Phys. 48, 7(2010).
38. K. O. Oyewumi and E. A. Bangudu . Arab. J. Sci. Eng. 28, 173-182 (2003).