Analytical approximate solutions for two-dimensional incompressible Navier-Stokes equations

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Abstract: Analytical approximate solutions of the two-dimensional incompressible Navier-Stokes equations by means of Adomian decomposition method are presented. The power of this manageable method is confirmed by applying it for two selected flow problems: The first is the Taylor decaying vortices, and the second is the flow behind a grid, comparison with High-order upwind compact finite-difference method is made. The numerical results that are obtained for two incompressible flow problems showed that the proposed method is less time consuming, quite accurate and easily implemented. In addition, we prove the convergence of this method when it is applied to the flow problems, which are describing them by unsteady two-dimensional incompressible Navier-Stokes equations.

Keywords: Navier-Stokes equations, Adomian decomposition, upwind compact difference, Accuracy, Convergence analysis,Taylor's decay vortices, flow behind a grid.

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1- Introduction

The problem of real fluid flow is of great complexity due to the many physical effects and a considerable set of non-linear partial differential equations involved. For example, Navier-Stokes equations (NSEs) are one good example and have the widest of application as they govern the motion of every fluid, be it a gas or liquid or a plasticized solid material acted upon by forces causing to change shape. So, specific advanced techniques must be applied to obtain the solutions of this problem. In our considered nonlinear problem, we need good mathematical procedures to simplify or linearize problem and solve it, such as finite difference method and Adomian decomposition method(ADM). At the present time, the need to use ADM in solving partial differential equations became more obvious by using it in solving various problems of different fields such as physics, engineering and applied mathematics [2,6,15,26], especially in the last decade. There is a challenge in using and applying this method to solve the complicated problems that include non-linear differential equations, like fluids flow problems represented by a system of non-linear partial differential equations called NSEs.

In this paper, we apply the numerical method which is known as Adomian decomposition method to obtain analytical approximate solutions for Navier-Stokes equations. These equations are elliptic and non-linear, increase non-linearity with increasing Reynolds number.

The equations of the motion of an incompressible fluid are

$$\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}&=\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y}&=\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\end{aligned}\qquad t>0,(x,y)\in\Omega \qquad (1a)$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \quad \text{in } \Omega \qquad (1b)$$

where Ω is a smooth bounded domain with boundary $\partial\Omega$, u *and* v are velocity components in $x-$direction and $y-$ direction, respectively. p is the pressure, t is the time, x and y are the space coordinates, μ is the kinematic viscosity, ρ is the fluid density, $\nabla = i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}$ is the gradient operator and ∇^2 is the Laplacian operator.

A number of numerical methods for solving several types of multi-dimensional time-dependent incompressible Navier-Stokes equations were given in[5,10-12,17,22,23,26,28,29]. A lot of studies have indicated to the important role of Adomian decomposition method in its application for solving various problems to various scientific models[8,9,13,14-16,19,20,25-27]. The application of the ADM was extended to specific multi-dimensional flow problems subject to a specific data theoretically by Seng et al. [27]. Recently, Sadighi et al.[26], applied the ADM to solve NSEs models, these sources are different from the model of our problem, in case of analysis of convergence theoretically ,and the mathematical formation of a problem. For the simple case of vorticity proportional to the stream function, Taylor obtained an analytical solution for unsteady flow that represented a double infinite array of vortices decaying exponentially with time. Kovasznay extended Taylor's idea by perturbed the stream function by a uniform stream and he was able to linearize the NSEs and obtain an exact solution for steady flow which resembles that downstream of a two-dimensional grid. From the literature review and by depending on our humble knowledge, we observed that the ADM not yet used to study these two problems, this matter was motive for us to use it here to find analytical approximate solution in case of unsteady flows.

The aim of this paper is to extend the application of the ADM proposed by Adomian [1] to solve two-dimensional incompressible NSEs and compare its reliability and efficiency with a high-order upwind compact difference method(UCDM)[29,30] The results that we obtain from using the methods will be saved and compared to prove the efficiency of each method in accuracy, speed of convergence and time.

2- Adomian Decomposition Method

To show the basic ideas of ADM[1], we will study the algorithm application of this method in approximate one-dimensional non-linear initial value problem. This problem is written by using the differential operators, as follows:

$$Lu\,(x,t) + Ru\,(x,t) + Nu\,(x,t) = g \qquad x \in \Re,\ t \geq 0 \tag{2a}$$

$$u_0 = u\,(x,0) \tag{2b}$$

where, $\Re$ is real numbers. The linear terms decomposed into $Lu + Ru$, while the nonlinear terms are represented by Nu , where L is an easily invertible linear operator, R is the remaining linear part and u_0 is initial condition. By taking the invert of linear differential operator L which is denoted by L^{-1}, for the two hands of Equation (2a), we obtain

$$L^{-1}L\,u = L^{-1}(g) - L^{-1}(Ru) - L^{-1}(Nu) \tag{3}$$

Here, for initial values problem (2), L^{-1} for the operator $L = \frac{\partial}{\partial t}$ is defined as;

$$L_t^{-1}(.) = \int_0^t (.)\, d\tau \tag{4}$$

From (4), we have; $L^{-1}L\,u = u - u_0$ (5)

Hence, Equation (3) became

$$u = u_0 + L^{-1}(g) - L^{-1}(Ru) - L^{-1}(Nu) \tag{6}$$

The method consists of decomposing the solution u into sum of an infinite number of components defined by the decomposition series [2, 9] as;

$$u = \sum_{n=0}^{\infty} u_n \tag{7}$$

where the u 's are calculated recurrently.

The nonlinear operator $Nu = \Psi(u)$ is decomposed as:

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n) \tag{8}$$

where A_n are Adomian's polynomials for the specific nonlinearity [6,8,27]. These Adomian's polynomials depended on components of $u_0, u_1, \dots, u_n$ and the fast convergent formula for the series [1,9,8]. The A_n are given as;

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[\Psi(\sum_{i=1}^{n}\lambda^i u_i)]_{\lambda=0} \qquad n = 0,1,2,\dots \tag{9}$$

There are different algorithms to compute Adomian polynomials which have been discovered by the continuous improvement of this method in finding the analytic solutions or similar formations of good acceleration by many researchers [2, 27].

Substituting Equations (7) and (8) into Equation (6), we obtain

$$u = \sum_{n=0}^{\infty} u_n = u_0 - L^{-1}(R(\sum_{n=0}^{\infty} u_n)) - L^{-1}(\sum_{n=0}^{\infty} A_n) \tag{10}$$

Consequently, it can be written as:

$$\begin{aligned} u_0 &= \phi + L^{-1}(g) \\ u_1 &= -L^{-1}(R(u_0)) - L^{-1}(A_0) \\ u_2 &= -L^{-1}(R(u_1)) - L^{-1}(A_1) \\ &\vdots \\ u_n &= -L^{-1}(R(u_{n-1})) - L^{-1}(A_{n-1}) \end{aligned} \tag{11}$$

where ϕ is the initial condition.

Hence all the terms of u are calculated and the general solution obtained according to the ADM as $u = \sum_{n=0}^{\infty} u_n$. The convergent of this series has been proved in [3,13, 14,27,33]. However, for some problems this series can't be determined, so we use an approximation of the solution from truncated series:

$$U_M = \sum_{n=0}^{M} u_n \quad \text{with} \quad \lim_{M \to \infty} U_M = u \tag{12}$$

The acceleration for this convergent means the need to few terms of Equation (12), for obtaining the formula which nearby to the exact solution[6].

3- Algorithm Analysis of ADM for NSEs

After clearing the simple and basic ideas of using and applying Adomian decomposition method algorithm to solve the differential equations, we will extend this application for a system of non-linear equations; that describes the algorithm to NSEs(1a). In order to facilitate the analysis, the following dimensionless variables are considered ;

$$u' = \frac{u}{U_0}, v' = \frac{v}{U_0}, x' = \frac{x}{L}, y' = \frac{y}{L}, t' = \frac{U_0}{L}t, P' = \frac{P}{\rho U_0^2}$$

where U_0 is a reference velocity, and L is a reference length. We then drop the primes. The equations become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{13a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{13b}$$

where $\mathrm{Re} = \frac{U_0 L}{\mu}$ is the Reynolds number. The dimensionless equation of continuity (1b) gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{13c}$$

This equation enables us to define a stream function ψ such that

$$u = \frac{\partial \psi}{\partial y} \quad and \quad v = -\frac{\partial \psi}{\partial x} \tag{13d}$$

Now, we start applying the ADM algorithm for Equations(13a,b) subject to the initial conditions $u_0 = u(x, y, 0)$, $v_0 = v(x, y, 0)$ and $p_0 = p(x, y, 0)$. Following, we define the linear operators $L_t = \frac{\partial}{\partial t}$, $L_x = \frac{\partial}{\partial x}$, $L_y = \frac{\partial}{\partial y}$, $L_{xx} = \frac{\partial^2}{\partial x^2}$ and $L_{yy} = \frac{\partial^2}{\partial y^2}$. Therefore, we rewrite Equations (13a,b) with operator form as

$$L_t u + u\,L_x u + vL_y u + L_x p = \frac{1}{\mathrm{Re}}(L_{xx} + L_{yy})u \tag{14a}$$

$$L_t v + u\,L_x v + vL_y v + L_y p = \frac{1}{\mathrm{Re}}(L_{xx} + L_{yy})v \tag{14b}$$

By defining the inverse operator L_t^{-1} which are given in (4), we can write the Equations (14a) and (14b) as;

$$u(x, y, t) = u(x, y, 0) - L_t^{-1}(u\,L_x + vL_y)u - L_t^{-1}(L_x p) + \frac{1}{\mathrm{Re}}L_t^{-1}(L_{xx} + L_{yy})u \tag{15a}$$

$$v(x, y, t) = v(x, y, 0) - L_t^{-1}(u\,L_x + v\,L_y)v - L_t^{-1}(L_y p) + \frac{1}{\mathrm{Re}}L_t^{-1}(L_{xx} + L_{yy})v \tag{15b}$$

By using Equation (7), the components solutions can be written as;

$$u(x,y,t)=\sum_{n=0}^{\infty}u_n(x,y,t),\ v(x,y,t)=\sum_{n=0}^{\infty}v_n(x,y,t)$$

and the nonlinear operators are

$$Nu=\Psi_1(u,v)=(u\,L_x+vL_y)u\,,\quad Nv=\Psi_2(u,v)=(u\,L_x+v\,L_y)v$$

The associated decomposition method is given by

$$u_0=u(x,y,0),\quad v_0=v(x,y,0) \tag{16a}$$

$$u_{n+1}=-L_t^{-1}(\Psi_1(u_n,v_n))-L_t^{-1}(L_x p_n)+\frac{1}{\mathrm{Re}}L_t^{-1}(L_{xx}+L_{yy})u_n \tag{16b}$$

$$v_{n+1}=-L_t^{-1}(\Psi_2(u_n,v_n))-L_t^{-1}(L_y p_n)+\frac{1}{\mathrm{Re}}L_t^{-1}(L_{xx}+L_{yy})v_n \tag{16c}$$

We decomposed Ψ_1 and Ψ_2 according to the series $\sum_{n=0}^{\infty}A_n$ and $\sum_{n=0}^{\infty}B_n$ respectively , where A_n and B_n are calculated by the Adomian's polynomials which are defined in Equation (9), then we obtain

$$\begin{aligned}
A_0&=(u_0\,L_x+v_0\,L_y)u_0\\
A_1&=(u_0\,L_x+v_0\,L_y)u_1+(u_1\,L_x+v_1\,L_y)u_0\\
A_2&=(u_0\,L_x+v_0\,L_y)u_2+(u_1\,L_x+v_1\,L_y)u_1+(u_2\,L_x+v_2\,L_y)u_0\\
&\vdots
\end{aligned} \tag{17a}$$

similarly

$$\begin{aligned}
B_0&=(u_0\,L_x+v_0\,L_y)v_0\\
B_1&=(u_0\,L_x+v_0\,L_y)v_1+(u_1\,L_x+v_1\,L_y)v_0\\
B_2&=(u_0\,L_x+v_0\,L_y)v_2+(u_1\,L_x+v_1\,L_y)v_1+(u_2\,L_x+v_2\,L_y)v_0\\
&\vdots
\end{aligned} \tag{17b}$$

and So on. By using Equation (11), we have

$$\begin{aligned}
u_1&=u_0-L_t^{-1}(A_0)-L_t^{-1}(L_x p_0)+\frac{1}{\mathrm{Re}}L_t^{-1}(L_{xx}+L_{yy})u_0\\
v_1&=v_0-L_t^{-1}(B_0)-L_t^{-1}(L_y p_0)+\frac{1}{\mathrm{Re}}L_t^{-1}(L_{xx}+L_{yy})v_0\\
u_2&=u_1-L_t^{-1}(A_1)-L_t^{-1}(L_x p_1)+\frac{1}{\mathrm{Re}}L_t^{-1}(L_{xx}+L_{yy})u_1\\
v_2&=v_1-L_t^{-1}(B_1)-L_t^{-1}(L_y p_1)+\frac{1}{\mathrm{Re}}L_t^{-1}(L_{xx}+L_{yy})v_1\\
&\vdots
\end{aligned} \tag{18}$$

and so on. From Equations (13) and continuity Equation(1b), we obtain;

$$(L_{xx}+L_{yy})P=-(L_{xx}u)^2-(L_{yy}v)^2-2L_y u\,L_x v \tag{19a}$$

We symbolize the right side by z . Therefore, we rewrite Equations (19a) with operator form as;

$$L_{xx}\, p + L_{yy}\, p = z \tag{19b}$$

Now, from Equation (19b), we calculate the pressure p . Solving for $L_{xx}p$ and inverting the operator L_{xx}, with $L_{xx}^{-1}(.) = \int_0^x\int_0^x (.)\, dx\, dx$, we have

$$p = p\,(t=0) + L_{xx}^{-1}\, z - L_{xx}^{-1}(L_{yy}\, p) \tag{20}$$

where, the nonlinear part z is calculated from the Adomian's polynomial of Equation (9).

Writing $p = \sum_{n=0}^{\infty} p_n$ and identifying $p_0 = p(t=0)$, where the p_n's are calculated recurrently.

Hence $$p_{n+1} = L_{xx}^{-1}\, z - L_{xx}^{-1}(L_{yy}\, p_n) \quad , \quad \text{for } n = 0,1,2,\ldots \tag{21}$$

From Equation (21), we produced;

$$p_1 = p_0 - L_{xx}^{-1}(C_0) - L_{xx}^{-1}(L_{yy}\, p_0) \tag{22}$$

$$p_2 = p_1 - L_{xx}^{-1}(C_1) - L_{xx}^{-1}(L_{yy}\, p_1)$$

$$\vdots$$

where C_n are Adomian's polynomial and calculated by Equation (9) as;

$$C_0 = 2\, L_y u_0\, L_x v_0 + (L_x u_0)^2 + (L_y v_0)^2$$

$$C_1 = 2\,(L_y u_0\, L_x v_1 + L_y u_1\, L_x v_0 + L_x u_0\, L_x u_1 + L_y v_0\, L_y v_1) \tag{23}$$

and we can write n^{th} term approximation for p by $\varphi_n = \sum_{i=0}^{n-1} p_i$ which converges to $\sum_{n=0}^{\infty} p_n$ or p . Similar equations can be written for $L_{yy}p$, where $L_{yy}^{-1}(.) = \int_0^y\int_0^y (.)\, dy\, dy$, we have

$$p_{n+1} = L_{yy}^{-1}\, z - L_{yy}^{-1}(L_{xx}\, p_n) \quad , \quad \text{for} \quad n = 0,1,2,\ldots \tag{24}$$

Almost of previous relations which are made in [1,27],but without the numerical computations and without theoretical prove of convergence. Now, we can summarize the algorithm computing as follows: where u_0 , v_0 and p_0 are obtained through initial conditions, then from Equations (18), u_1 , v_1 are computed depending on u_0 , v_0 and p_0. Also, from Equations (18), the components u_2 and v_2 are computed by dependence on the values of u_1 , v_1 and p_1 , where p_1 is obtained by Equation (22), so on. Moreover, also, by using this algorithm and the results that obtained we can computed the stream function(13d), and the vorticity $\omega = u_y - v_x$.

4- Analysis of Convergence

In this section, we will study the analysis of convergence in the same manner as [3,13, 14] of the decomposition method to the nonlinear Navier-Stokes Equations (13 a or b) and Equation (19a) . Let us consider the Hilbert space H which may be defined as $H = L^2(\Omega\times[0,T])$, the set of applications; $u:\ \Omega\times[0,T]\to\Re$ with $\int_{\Omega\times[0,T]} u^2 d\Omega < +\infty$

And scalar product and induced norm:

$$(u,v) = \int_{\Omega\times[0,T]} uv d\Omega \quad \text{and} \quad \|u\|^2 = (u,u)$$

where, $\Re$ is real numbers.

We consider the nonlinear Navier-Stokes equations, then the operator of a nonlinear Navier-Stokes Equations (13a,b) and Equation (19a) are;

$$L_t(u) = -uL_x u - vL_y u + \frac{1}{\mathrm{Re}}(L_{xx} + L_{yy})u - L_x p \tag{25a}$$

$$L_t(v) = -uL_x v - vL_y v + \frac{1}{\mathrm{Re}}(L_{xx} + L_{yy})v - L_y p \tag{25b}$$

$$Lf(p) = (L_{xx} + L_{yy})p + (L_{xx}u)^2 + (L_{yy}v)^2 + 2L_y u\, L_x v \tag{26}$$

where, $Lf(p)$ is the operator of Equation (19a). Following, we define the difference operator $\Delta z = z - \hat{z}$ for any quantity such as z .The Adomian decomposition method is convergent if the following conditions are satisfied ;

(I_u): $(L_t(\Delta u), \Delta u) \geq k_1\|\Delta u\|^2$ and $(f(\Delta p), \Delta p) \geq k_2\|\Delta p\|^2$, $k_1, k_2 > 0,\ \forall\ u, v, \hat{u}, p, \hat{p} \in H$.

(II_u) : Whatever may be $M > 0$, there exist a constant $C(M) > 0$ such that for $u, \hat{u}, v, p, \hat{p} \in H$ with $\|u\| \leq M$, $\|v\| \leq M$, $\|\hat{u}\| \leq M$ and $\|p\| \leq M$,$\|\hat{p}\| \leq M$, we have:

$(L_t(\Delta u), w) \leq C(M, \mathrm{Re})\|\Delta u\|\|w\|$ and $(Lf(\Delta p), w) \leq I\,\|\Delta p\|\,\|w\|$, for every $w \in H$ and I <0.

(I_v): $(L_t(\Delta v), \Delta v) \geq k_3\|\Delta v\|^2$ and $(f(\Delta p), \Delta p) \geq k_2\|\Delta p\|^2$, $k_3, k_2 > 0,\ \forall\ u, \hat{v}, v, p, \hat{p} \in H$.

(II_v) : Whatever may be $M > 0$, there exist a constant $C(M) > 0$ such that for $u, \hat{v}, v, p, \hat{p} \in H$ with $\|u\| \leq M$, $\|v\| \leq M$, $\|\hat{v}\| \leq M$ and $\|p\| \leq M$,$\|\hat{p}\| \leq M$, we have:

$(L_t(\Delta v), w) \le C(M, \mathrm{Re})\|\Delta v\|\|w\|$ and $(Lf(\Delta p), w) \le I\,\|\Delta p\|\,\|w\|$, for every $w \in H$ and I <0.

Now, we will use the following theorems to satisfy the above conditions as [3,13], in addition, we will verify the pressure hypothesis which included these two hypothesis.

Theorem 1: If (I_u) and (II_u) are satisfied, then ADM of Equations (25a) and (26) is convergent.

Proof: Firstly, we will verify the convergence of condition (I_u) for the operators $L_t(u)$ and $Lf(p)$:

$$L_t(\Delta u) = -(uL_x + vL_y)u + \frac{1}{\mathrm{Re}}(L_{xx} + L_{yy})u - L_x p + (\hat{u}L_x + vL_y)\hat{u} - \frac{1}{\mathrm{Re}}(L_{xx} + L_{yy})\hat{u} + L_x p$$

$$= -[uL_x u - \hat{u}L_x\hat{u}] - [vL_y u - vL_y\hat{u}] + \frac{1}{\mathrm{Re}}([L_{xx}(\Delta u)] + [L_{yy}(\Delta u)])$$

$$= -\frac{1}{2}L_x[\Delta(u^2)] - [vL_y(\Delta u)] + \frac{1}{\mathrm{Re}}([L_{xx}(\Delta u)] + [L_{yy}(\Delta u)]) \tag{27}$$

$$Lf(\Delta p) = (L_{xx} + L_{yy})u + (L_{xx}u)^2 + (L_{yy}v)^2 + 2L_y u L_x v - (L_{xx} + L_{yy})\hat{p} - (L_{xx}u)^2 - (L_{yy}v)^2 - 2L_y u L_x v = (L_{xx} + L_{yy})(\Delta p) \tag{28}$$

Therefore,

$$(L_t(\Delta u), \Delta u) = \frac{1}{2}(-L_x\Delta(u^2), \Delta u) + (-vL_y(\Delta u), \Delta u) - \frac{1}{\mathrm{Re}}((-L_{xx}(\Delta u), \Delta u) - (-L_{yy}(\Delta u), \Delta u)) \tag{29}$$

$$(Lf(\Delta p), \Delta p) = -(-L_{xx}(\Delta p)) - (-L_{yy}(\Delta p)) \tag{30}$$

Since L_x , L_y , L_{xx} and L_{yy} are differential operators in H , then there exist constants δ_1 , δ_2 , δ_5, δ_6 >0, such that

$$(-L_{xx}(\Delta u), \Delta u) \ge \delta_1\|\Delta u\|^2, (-L_{yy}(\Delta u), \Delta u) \ge \delta_2\|\Delta u\|^2, (-L_{xx}(\Delta p), \Delta p) \ge \delta_5\|\Delta p\|^2, (-L_{yy}(\Delta p), \Delta p) \ge \delta_6\|\Delta p\|^2 \tag{31}$$

and according to the Schwartz inequality, we get

$$(L_x\Delta(u^2), \Delta u) \le \delta_3\|\Delta(u^2)\|\|\Delta u\| = 2\delta_3\eta\|\Delta u\|^2 \le 2\delta_3 M\|\Delta u\|^2 \tag{32}$$

where, $u < \eta < v$, $\|u\|, \|v\|, \|\hat{u}\| \le M$ and δ_3 >0 is the Lipscitzian constant and therefore,

$$(L_x\Delta(u^2), \Delta u) \le 2\delta_3 M\|\Delta u\|^2 \Leftrightarrow (-L_x\Delta(u^2), \Delta u) \ge 2\delta_3 M\|\Delta u\|^2 \tag{33}$$

also, $(vL_y(\Delta u), \Delta u) \le \delta_4\|\hat{u}\|\,\|\Delta u\|^2 \le \delta_4 M\|\Delta u\|^2$, (34)

where, $\|v\| \le M$ and δ_4 >0 is the Lipscitzian constant and therefore,

$$(vL_y(\Delta u), \Delta u) \le \delta_4\|v\|\|\Delta u\|^2 \Leftrightarrow (-vL_y(\Delta u), \Delta u) \ge \delta_4 M\|\Delta u\|^2 \tag{35}$$

Substituting Equations (31)-(35) into Equations (29) and (30) yields

$$(L_t(\Delta u),\Delta u) \geq ((\delta_3+\delta_4)M-\frac{1}{\text{Re}}(\delta_1+\delta_2))\|\Delta u\|^2 = k_1\|\Delta u\|^2 \tag{36}$$

$$(Lf(\Delta p),\Delta p) \geq (-(\delta_5+\delta_6))\|\Delta p\|^2 = k_2\|\Delta p\|^2 \tag{37}$$

where $k_1=(\delta_3+\delta_4)M-\frac{1}{\text{Re}}(\delta_1+\delta_2)$ and $k_2=(\delta_5+\delta_6)$. Then condition (I_u) holds.

Secondly, we will verify the convergence of condition (II_u) for the operators $L_t(u)$ and $Lf(p)$. For that we have:

$$(L_t(\Delta u),\,\Delta u)=\frac{1}{2}\left(-L_x\Delta(u^2),w\right)+\left(-vL_y(\Delta u),w\right)-\frac{1}{\text{Re}}\left(-L_{xx}(\Delta u),w\right)-\frac{1}{\text{Re}}\left(-L_{yy}(\Delta u),w\right)$$

$$\leq M\|\Delta u\|\,\|w\|+M\|\Delta u\|\,\|w\|-\frac{1}{\text{Re}}\|\Delta u\|\,\|w\|-\frac{1}{\text{Re}}\|\Delta u\|\,\|w\|=C(M,\text{Re})\|\Delta u\|\|w\| \tag{38}$$

$$(Lf(\Delta p),\Delta p)=\left(L_{xx}(\Delta p),w\right)-\left(L_{yy}(\Delta p),w\right)\leq I\|\Delta p\|\|w\| \tag{39}$$

where, $C(M,\text{Re})=2M-\frac{2}{\text{Re}}$, $I=-2$ and the condition (II_u) is satisfied. Hence the proof is complete. ■

Theorem 2: If (I_v) and (II_v) are satisfied, then ADM of Equations (25b) and (26) is convergent.

Proof: In the same manner of Theorem 1, we can prove the convergence of Equation(13b),by verifying the convergence of condition (I_v) and (II_v) for the operators $L_t(v)$ and $Lf(p)$. ■

5- Numerical Test and Discussion

The theoretical analysis of ADM done in the previous sections will be applied in this section to find the analytical approximate solutions for two unsteady state flow problems: the first is the Taylor decaying vortices, and the second is the NSEs with a periodicity in one direction(flow behind a grid), in order to test the validity of the present method.

5.1 The Taylor's decay vortices

The problem of Taylor's decay vortices is used so much to pretesting the efficiency of the numerical methods for handling the flow problems[24,29,30,31,32]. To describe the flow of Taylor's decay vortices the NSEs($15_{a,b}$) using with initial conditions $u_0 = u(x, y,0) = -\cos(2x)\sin(2y)$, $v_0 = v(x, y,0) = \sin(2x)\cos(2y)$ and,

$p_0 = p(x, y,0) = -\frac{1}{4}\left[\cos(4x) + \sin(4y)\right]$, for $0 \le x \le 2\pi, 0 \le y \le 2\pi$.

The obtained iterative solutions of decomposition series of Equations ($15_{a,b}$) are using the relations (18,22),where these relations represent the iterative solutions for 2D NSEs**.** The efficiency and a high accuracy in finding the exact and approximate solutions for the initial and boundary values problems are considered positive points for ADM. The numerical computations of test problem, which represent the fluid flow conduct inside square cavity are applied with some Re and t values by using ADM algorithm. Figure 1 : **(a)** Shows Profiles of $u(\pi, y)$ and $v(x,\pi)$ velocities for different times($t = 1,3,7,10$) , **(b)** explained the identification between the exact and numerical solutions and that indication has proved the efficiency of ADM in solve NSEs with good convergence, and **(c)** Shows Contour drawing for vorticity ω at $t = 2$ and Re=100 for ADM and UCDM. In addition, the measurements of maximum error for the velocity and vorticity functions, which are showed in Table1, ensure the ability of suggested method and its accuracy in finding the solutions. From the measures of maximum error, the table shows the required evidence to explain a high accuracy for method, where as the ADM accuracy increase with increasing Reynolds number at $t = 2$. From our computations by using ADM, we notice that the convergence of these computations correlate with the variables (t and Re) in inversely relation, which govern the solution. For example, at ($t = 2$ and $\mathrm{Re} < 50$) or ($t > 5$ and $\mathrm{Re} = 100$)
then convergence of ADM becomes weakly in the solution. All obtained results by the second iteration of ADM represent approximate solution is equivalent and identical to exact solutions for the problem. The other positive points of ADM are storage of time(CPU about 0.007) and effort that is explained from the tabular results in this paper. Figure 1(c) represents the system of eddies arranged in the square pattern, each rotating in the opposite direction to that of its four neighbours, and this fact is confirmed by many authors[3,6,24,29,32].

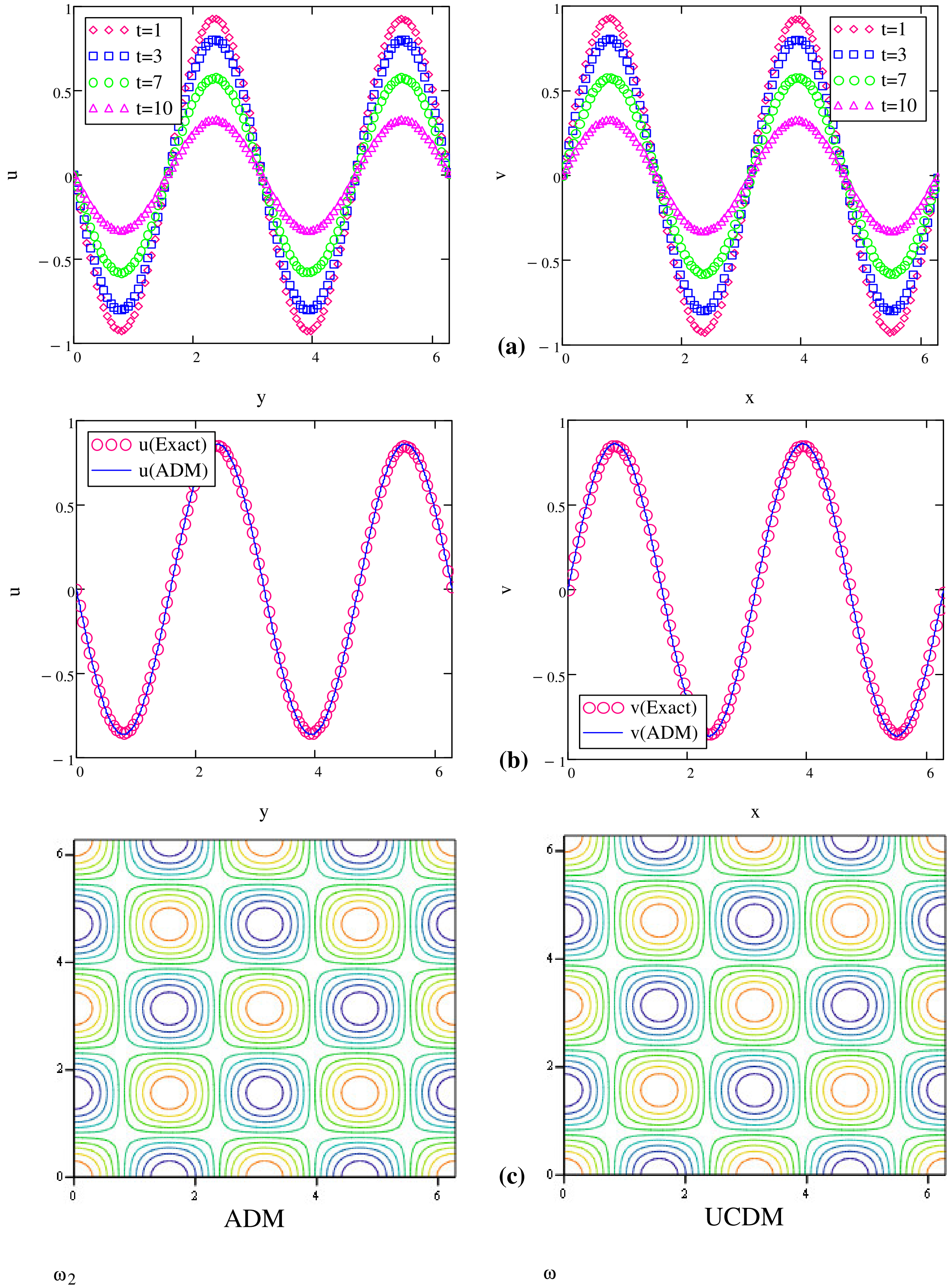

Figure 1. **(a)**Velocities u and v for Re=100, t=1,3,7,10 **(b)** Exact and numerical solutions of u and v for Re=100 and $t=2$. **(c)** Vorticity ω for Re=100 and $t=2$.

Table 1: Errors velocity (u_2, v_2) and vorticity ω_2 of the present study for Taylor's vortices problem.

$t = 2.0 \rightarrow$ $\mathrm{Re} = 100 \rightarrow$	**Re=50** **t=0.1**	**Re=100** **t=1**	**Re=300** **t=5**	**Re=1000** **t=10**
$Max\lvert U\rvert$	2.68×10^{-4} 6.49×10^{-11}	1.34×10^{-5} 7.38×10^{-7}	1.73×10^{-8} 7.20×10^{-6}	1.05×10^{-9} 1.76×10^{-5}
$Max\lvert\omega\rvert$	4.08×10^{-4} 1.16×10^{-12}	6.89×10^{-6} 1.11×10^{-7}	4.69×10^{-10} 1.49×10^{-5}	7.39×10^{-12} 7.99×10^{-4}

For the comparison, addition to exact solution(Figure 1(b)), we select the best approximation for solving NSEs that can be used is high-order upwind compact difference method. Eqs.($13_{a,b}$) and (19_a) can be usually changed into discrete difference equations (for details, see refs.[7,19,21,29,30]), and then be solved through iteratively method. For solving result discrete finite difference equation corresponds to Equations($13_{a,b}$) and (19_a),we used Gauss–Seidel and successive over-relaxation iterative methods ,respectively. We introduced the comparison of the obtained results between ADM and UCDM. The comparison is represented by the study of errors

Table 2: Errors comparison of the present study and UCFDM at $t = 2.0$ for Taylor's vortices problem.

Grids	**Method**	**Re=50**	**Re=100**	**Re=300**	**Re=1000**
65×65 Velocities (u, v)	**UCDM** **Present Method**	2.32×10^{-4} 2.68×10^{-4}	1.36×10^{-4} 1.34×10^{-5}	3.09×10^{-5} 1.73×10^{-8}	1.57×10^{-5} 1.05×10^{-9}
65×65 Vorticity(ω)	**UCDM** **Present Method**	1.19×10^{-3} 4.08×10^{-4}	9.10×10^{-4} 6.88×10^{-6}	8.01×10^{-4} 4.69×10^{-10}	8.13×10^{-4} 7.39×10^{-12}

for vorticity (ω) and velocities (u, v) , and also by iterations number and CPU time. We notice that the numerical solutions of vorticity function and the velocities u , v by using ADM and UCDM are correspondent. The suggested two methods confirm its efficiency in solving the two dimensions NSEs. The accuracy of these two methods increases with increasing Reynolds number at $t = 2$. Moreover, From Table 2,we see that accuracy of ADM is higher and better than UCDM for different Reynolds number values ($\mathrm{Re} = 50 - 1000$). Besides that, CPU time(0.007) and iterations number(2) of ADM is better than CPU time(2 < CPU< 26.7) and iterations number(80 < No. of Iterations<1000) of UCDM. We can say that ADM is faster convergence and more accurate than UCDM.

5.2 Unsteady flow behind a grid

we consider the laminar flow of viscous fluid Equation(1) behind a two-dimensional grid, with x-axis normal to the grid and the velocity field is assumed to be such that $u := U_0 + u$ and $v := v$, where U_0 is the mean velocity(reference of velocity) in the x-direction. Thus; the two-dimensional Navier-Stokes equations with a periodicity in one direction, which may represent the wake of a two dimensional grid as the same as Equations($1_{a,b}$) with replacing the coefficients of convective terms in x-direction by $U_0 + u$;that is: $(U_0 + u)\dfrac{\partial u}{\partial x}$ and $(U_0 + u)\dfrac{\partial v}{\partial x}$ and in the non-dimensional NSEs(13ab) these terms become $(1+u)\dfrac{\partial u}{\partial x}$ and $(1+u)\dfrac{\partial v}{\partial x}$.The laminar flow behind a periodic array of medium[29] is used to examine the verification of accuracy of ADM. To compute the numerical results for unsteady state of this problem by using the algorithm (section3) ,the initial values that are adopted is the steady state two-dimensional exact solution of this problem [18,23,24,32];which is given as

$$
\begin{aligned}
u_0 &= u(x,y) = 1 - e^{\frac{x(\mathrm{Re}-\sqrt{\mathrm{Re}^2+16\pi^2})}{2}}\cos(2\pi y),\\
v_0 &= v(x,y) = -\frac{\mathrm{Re}-\sqrt{\mathrm{Re}^2-16\pi^2}}{4\pi}e^{\frac{x(\mathrm{Re}-\sqrt{\mathrm{Re}^2+16\pi^2})}{2}}\sin(2\pi y),\\
P_0 &= P(x,y) = \mathrm{P}_0 - \frac{1}{2}e^{\frac{x(\mathrm{Re}-\sqrt{\mathrm{Re}^2+16\pi^2})}{2}}
\end{aligned}
$$

where P_0 is a reference pressure (an arbitrary constant). We computed the analytical approximate solution by using AD algorithm for unsteady of this problem using recurrence relations(18&22) and the relations are related to its such as stream function and vorticity. The calculations are run by Mathcad 14 software. The computed streamlines and vorticity contours for Re=5,20,40 are shown in Figure 2. The pairs of bound eddies generated behind the single elements of the grids, and at large distance downstream, however, the streamlines become parallel and equidistant as shown by the short lines on the right side of the figure for all values of Re(as in the case of a viscous fluid). From the figure of the streamlines and vorticities , we note that when the Reynold number increases, the whole flow pattern is extended uniformly in the direction of main flow. we observe that the rate of change of the flow is very great, and the length of vortices increase with

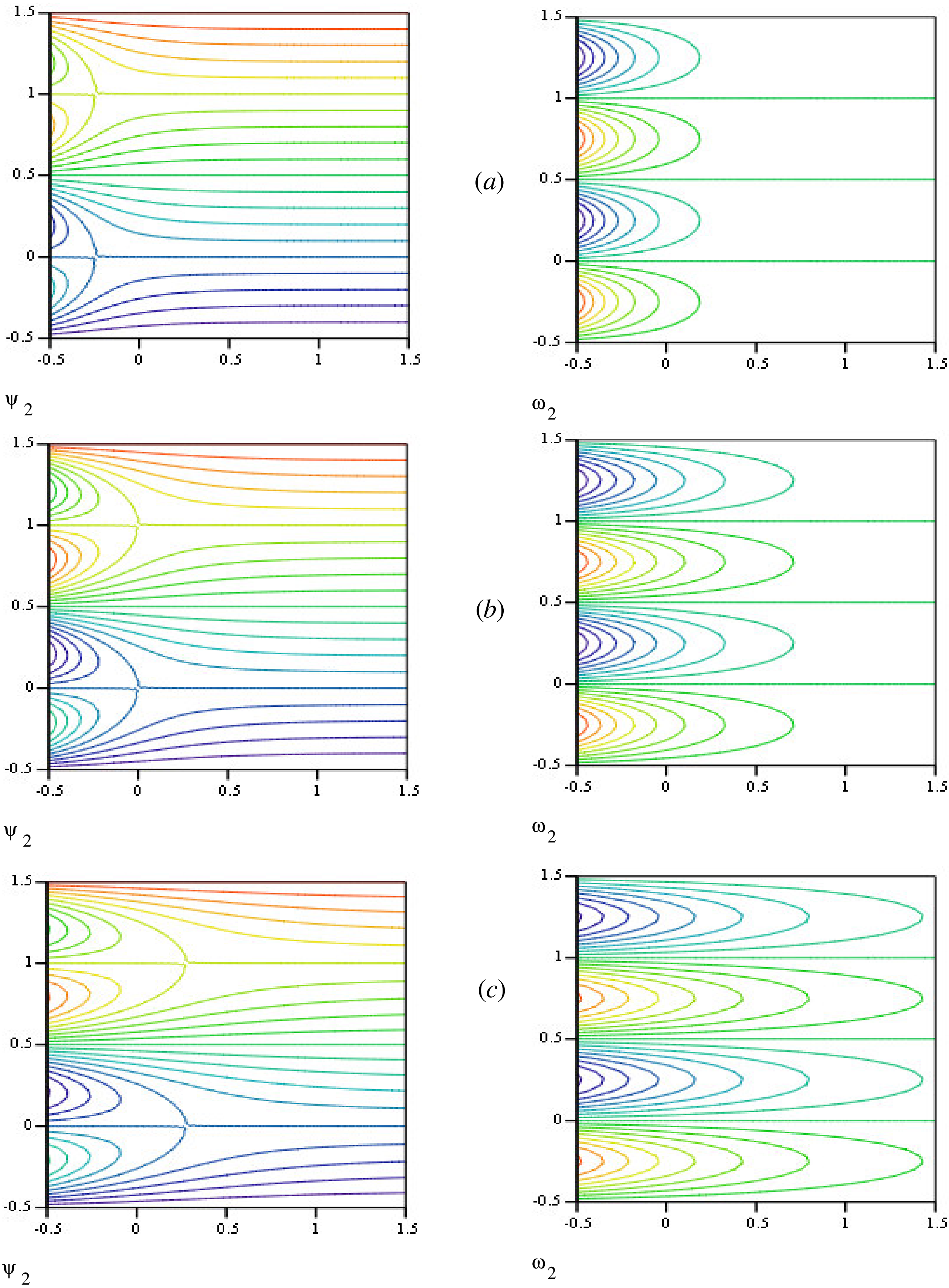

Figure. 2 Streamlines and vorticity contour plots for $t = 0.1$ and (**a**)Re=5,(**b**) Re=20 (**c**)Re=40

the increase in the Reynold number towards the downstream flow. Tables 3 show grid refinement test results for upwind compact difference and Adomian decomposition Methods. The comparison is represented by the study of errors for stream function and vorticity , and also by iterations number and CPU time. It is noted that the magnitudes of the vorticity gradients and streamlines in the present study are similar to those obtained by Shah et al. [29] when solving the unsteady flow and steady flow by[23,24,32]. Moreover, The accuracy of these two methods increase with increasing Reynolds number at $t = 0.1$. From Table(3),we see that the accuracy of ADM is higher and better than UCDM for different Reynolds number values (Re=1,20,40,100,1000) and number of grid points. Besides that, CPU time(0.0003) and iterations number(2) of ADM is less than CPU time(2 < CPU< 12.6) and iterations number(No. of Iterations <100) of UCDM. We can say that ADM is faster convergence and more accurate than UCDM.

Table 3. Errors Comparison of the present study and UCDM for flow behind a grid problem.

Grids	**Method**	**Re=1**	**Re=20**	**Re=40**	**Re=100**	**Re=1000**
41×41	**UCDM**	1.27×10^{-2}	1.73×10^{-3}	9.46×10^{-4}	4.96×10^{-4}	2.26×10^{-4}
	Present Method	3.85×10^{-3}	5.75×10^{-4}	3.82×10^{-4}	2.89×10^{-4}	2.24×10^{-4}
161×161	**UCDM**	2.02×10^{-4}	1.45×10^{-5}	9.57×10^{-6}	5.71×10^{-6}	2.68×10^{-6}
	Present Method	6.26×10^{-6}	1.12×10^{-6}	7.60×10^{-7}	5.78×10^{-7}	4.83×10^{-7}

Conclusions

The Adomian decomposition method is tested for two-dimensional time-dependent incompressible Navier-Stokes equations that describe the Taylor's vortices with low to moderate Reynolds numbers and flow behind grid with compared the results of both of these problems with the UCDM. The application of ADM gives a simple powerful tool to obtain the solutions without a need for large size of computations, unlike UCDM. The results show that ADM has high accuracy and efficiency in finding the exact and approximate solutions with less computation workload. Also, we conclude that ADM is efficient and better than UCDM in iterations number and CPU time, at least in the current cases. There are identification between the approximate solutions of UCDM and ADM for solving NSEs at $\text{Re}=100$ and $t=2$. Beside that the accuracy of solutions by using these two methods increase with increasing Reynolds number with fixed time and the rate of convergence is a very high. Advantages of ADM over the classical techniques. For example, it avoids discretization and provides an efficient analytical approximate solution with high accuracy and low computational load.

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