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Non-Relativistics Treatment of Schrodinger Particle under Modified Inversly Quadratic Hellmann Potential Model Via Parametric N-U Method

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Abstract
The non-relativistics study of particles under the modified inversely quadratic Hellmann potential has been studied. The energy eigenvalues and the corresponding wavefunction expressed in terms of the Jacobi polynomial are obtained using the parametric NU method. In obtaining the solutions for this system, we have used an approximation scheme to evaluate the centrifugal term (potential barrier). To test the accuracy of our result, we compared the approximation scheme with the centrifugal term and the result shows that there is an agreement between the centrifugal term and the approximation scheme for a very small screening parameter; showing that our potential is a short range potential. The results obtain from this work, would have many application in inner shield ionization problem, Electron core, Solid state Physics, Alkali-Hydride molecules. Three special cases of this potential have also been discussed.

1.0 INTRODUCTION

Over the years, theoretical physics has been successful in explaining the behaviour of different particles in different potentials. This has been made possible by obtaining exact or approximate solution of non-relativistic and relativistic wave function for different physical systems of interest. The exact or approximate solution of this equation with central potential plays an important role in quantum mechanics [6, 9, 21, 26]. Inthat effect, exact solutions of quantum system are significant in Physics. Solving the non-relativistic and relativistic equation is still an interesting work in the existing literature [2, 16, 18].

Recently, the study of exponential-like type potentials has attracted much attention [25, 28]. However, the bound state solutions of the Schrodinger wave equation (SWE) of some of these potentials are possible for cases such as the Hellman potential [11],the wood-saxon [5, 7].

Moreover, when an arbitrary angular momentum quantum number (l) is present the Schrodinger wave equation (SWE) can only be solved approximately using suitable approximation scheme [22]. Some of such approximation scheme includes; conventional approximation scheme proposed by Greene and Aldrich [8]; improved approximation scheme by [20]. Elegant approximation scheme [14], and Good approximation by [30]. The approximation scheme used in this study was given by [4]. This approximation scheme is used to deal with the centrifugal term.

Various methods have been used to solve the SWE, Klein-Gordon equation and Dirac Equation exactly or approximately. These methods include: Asymptotic Iteration Method (AIM) proposed by [29], supersymmetric quantum mechanics method (SUSYQM) proposed by [12], Shifted $\frac{1}{N}$ expression [23], factorization method [3, 15],Nikiforov-Uvarov (N-U) method proposed by [24] and others [13, 17].

The main aim of this study is to use the approximation scheme proposed by [4] and the Nikiforov-Uvarov method to obtain the approximate bound state solutions of SWE with modified inversely quadratic Hellmann potential model (MIQH) defined as:

$$V(r) = -V_0\left(\frac{a+be^{-\alpha r}}{r}\right)+\frac{V_1 e^{-\alpha r}}{r^2} \tag{1}$$

Where V_0 and V_1 are the potential depths, a and b are the strength of the Coulomb and Yukawapotential respectively and α is the screening parameter(range).The MIQH potential is a short ranged and central potential. It could be used to describe nucleon-nucleon interaction, meson-meson interactions and has other applications in atomic and nuclear physics, chemical and other related areas.

2. THE GENERALIZED PARAMETRIC NIKIFOROV-UVAROV (N-U) METHOD

The N-U method was presented by Nikiforov and Uvarov[24] and has been employed to solve second order differential equations such as the Schrödinger wave equation (SWE), Klein-Gordon Equation (KGE) and Dirac Equation.

The SWE is given by:

$$\psi''(r)+\left(E-V(r)\right)\psi(r)=0 \tag{2}$$

And can be solved by transforming it into a hypergeometric type equation using the transformation

$$s\to s(r) \tag{3}$$

And the resulting equation is given by:

$$\psi''(s)+\frac{\breve{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\breve{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \tag{4}$$

Where σ and σ^2 are polynomial of atmost second degree and $\breve{\tau}$ is a polynomial of first degree and $\psi(s)$ is a polynomial of hypergeometric type.

The generalized parametric N-U method was introduced by [27] and is given by:

$$\psi''(s)+\frac{c_1-c_2s}{s(1-c_3s)}\psi'(s)+\frac{1}{s^2(1-c_3s)^2}\left[-\xi_1s^2+\xi_2s-\xi_3\right]\psi(s)=0 \tag{5}$$

Eq. (5) is solved by Comparing it with (4), we have

$$\breve{\tau}(s)=c_1-c_2s$$

$$\sigma(s)=s(1-c_3s)$$

$$\breve{\sigma}^2(s)=s^2(1-c_3s)^2$$

$$\breve{\sigma}(s)=-\xi_1s^2+\xi_2s-\varepsilon_3$$

According to the NU method, the energy eigenvalues equation and eigenfunctionrespectively satisfy the following sets of equation

$$c_2n-(2n+1)c_5+(2n+1)\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)+n(n-1)c_3+c_7+2c_3c_8+2\sqrt{c_8c_9}=0 \tag{6}$$

$$\psi(s)=N_{nl}S^{c_{12}}(1-c_3s)^{-c_{12}-(c_{13}/c_3)}P_n^{[c_{10}-1;c_{11}/c_3-c_{10}-1]}(1-2c_3s) \tag{7}$$

$$c_4=\frac{1}{2}(1-c_1),c_5=\frac{1}{2}(c_2-2c_3),c_6=c_5^2+\xi_1,c_7=c_4c_5-\xi_2,$$

$$c_8=c_4^2+\xi_3,c_9=c_3c_7+c_3^2c_8+c_6,c_{10}=c_1+2c_4+2\sqrt{c_8},$$

$$c_{11}=c_2-2c_5+2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right),c_{12}=c_4+\sqrt{c_8},c_{13}=c_5-\sqrt{c_9}+c_3\sqrt{c_8}. \tag{8}$$

3. FACTORIZATION METHOD

In spherical polar coordinate, the SWE with potential V(r) is given as [28]

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial^2\phi}\right]\psi(r,\theta,\phi)$$

$$+\left[\frac{2\mu}{\hbar^2}\left(E-V(r)\right)\right]\psi(r,\theta,\phi)=0 \tag{9}$$

Using the common Ansatz for the wave function

$$\psi(r,\theta,\phi)=\frac{R_{nl}(r)}{r}\gamma_{lm}(\theta,\phi) \tag{10}$$

The above simplify to

$$\frac{d^2R_{nl}(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E-V(r)\right)-\frac{\lambda}{r^2}\right]R_{nl}=0 \tag{11}$$

Where $\lambda = l(l+1)$

4. THE SOLUTION OF RADIAL PART OF SWE WITH MIQH POTENTIAL

Substituting the potential of (1) into the redial Schrödinger equation of (11), we obtain

$$\frac{d^2R_{nl}(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E+\frac{V_0}{r}\left(a+be^{-\alpha r}\right)-\frac{V_1e^{-\alpha r}}{r^2}\right)-\frac{\lambda}{r^2}\right]R_{nl}=0 \tag{12}$$

It is well known that the Schrodinger equation of (8) cannot be solved exactly for $\ell \neq 0$ by any known method. The way out is to use approximation for the centrifugal term. On this note, we invoke the approximation used by [4]

$$\frac{1}{r^2}\approx\frac{\alpha^2}{\left(1-e^{\alpha r}\right)^2} \tag{13}$$

And using the transformation $s\to e^{-\alpha r}$ we have,

$$\frac{d^2R(s)}{ds^2}+\frac{1(1-s)}{s(1-s)}\frac{dR(s)}{ds}+\frac{1}{s^2(1-s^2)}\begin{bmatrix}-(\varepsilon+Ab)s^2\\+(2\varepsilon-Aa+Ab-B)s-\\(\varepsilon-Aa+\lambda)\end{bmatrix}R(S)=0 \tag{14}$$

Where the following dimensionless quantities have been defined as;

$$-\varepsilon=\frac{2\mu E}{\alpha^2\hbar^2},\ A=\frac{2\mu V_0}{\alpha\hbar^2};\ B=\frac{2\mu V_1}{\hbar^2}.$$

comparing with equ. (14) withequ. (5) we have

$c_1=1$, $c_2=1$, $c_3=1$, $\xi_1=\varepsilon+Ab$, $\xi_2=2\varepsilon+Aa+Ab-B$, $\xi_3=\varepsilon-aA+\lambda.$

Evaluating equ. (8) we obtain the following parameters

$$c_1=c_2=c_3=1,c_4=0,c_5=-\frac{1}{2},c_6=\frac{1}{4}+\varepsilon+Ab,c_7=-2\varepsilon+Aa-Aa+B,$$

$$c_8=\varepsilon-Aa+\lambda,c_9=B+\lambda+\frac{1}{4},c_{10}=1+2\sqrt{\varepsilon-Aa+\lambda},$$

$$c_{11}=2+2\left(B+\lambda+\frac{1}{4}+\sqrt{\varepsilon-Aa+\lambda}\right),c_{12}=\sqrt{\varepsilon-Aa+\lambda},$$

$$c_{13}=-\frac{1}{2}-\left(\sqrt{B+\lambda+\frac{1}{4}}+\sqrt{\varepsilon-Aa+\lambda}\right). \tag{15}$$

Substituting (15) into (6) we obtain the energy eigenvalue equation for the modified inversely quadratic Hellmann potential as;

$$n+(2n+1)\frac{1}{2}+(2n+1)\left(\sqrt{B+\lambda+\frac{1}{4}}+\sqrt{\varepsilon-Aa+\lambda}\right)+n^2-n-2\varepsilon$$

$$+Aa-Ab+B+2\varepsilon-2Aa+2\lambda+2\sqrt{(\varepsilon-Aa+\lambda)\left(B+\lambda+\frac{1}{4}\right)}=0 \tag{16}$$

Solving (17) explicitly, we obtain the energy eigenvalues for the radial part of Schrödinger wave equation as;

$$E_{nl}=\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{2\mu V_0 a}{\alpha}-\lambda+\left[\frac{\frac{2\mu V_0}{\alpha\hbar^2}(a+b)-l(l+1)-\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{-2\mu V_1}{\hbar^2}+l(l+1)}\right)^2}{2\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{-2\mu V_1}{\hbar^2}+l(l+1)}\right)}\right]^2\right] \tag{17}$$

Table 1: Energy eigenvalues $E(eV)$ of the Modified Inversely Quadratic Hellman Potential

n	l	$E_{nl}(a.u)$	
		$\alpha = 0.01$	$\alpha = 0.02$
1	0	-0.003004	-0.003261
2	0	-0.01691	-0.002182
2	1	-0.00116	-0.00160
3	0	-0.001204	-0.00200
3	1	-0.00100	-0.00200
3	2	-0.009709	-0.001638

Evaluating (6), the corresponding wave function for the radial part is obtained as;

$$\psi(r) = N_{nl} s^{W} (1-s)^{V} p_n^{C,D}(1-2s)$$

And using the transformation $s \rightarrow e^{-\alpha r}$ we have

$$\psi(r) = N_{nl} e^{-\alpha r^{W}} (1-e^{-\alpha r})^{V} p_n^{C,D}(1-2e^{-\alpha r}) \tag{18}$$

The total wavefunction for the system can be express as;

$$\psi(r,\theta,\phi) = N_{nl} e^{-\alpha r^{W}} (1-e^{-\alpha r})^{V} p_n^{C,D}(1-2e^{-\alpha r}) Y_{lm}(\theta,\phi) \tag{19}$$

Where $W = \sqrt{\varepsilon - Aa + \lambda}$, $V = \frac{1}{2} + \sqrt{B + \lambda + \frac{1}{4}}$, $C = 2\sqrt{\varepsilon - Aa + \lambda}$,

$$D = 2\sqrt{B + \lambda + \frac{1}{4}}$$

And N_{nl} is the normalization constant and P_n is the Jacobi polynomials.

5. DISCUSSION OF RESULTS

The variation of MIQH potential with the radial distance of separation between the interacting particles for screening parameters α = 20, 10, 5, 2, 0.1,0.01, $V_0 = 0.2$, $V_1 = 0.2$ and $a = b = 1$ is Fig.1. At very low α the curves almost coincide showing that the curves are only interpretable at large α, meaning that the interacting particles are very massive.

In order to test accuracy of this study, we compare the approximation scheme and the centrifugal term as shown in Fig. 2. From the graph, it shows that the approximation scheme is good for the screening parameter $\alpha = 0.01$ meaning that the MIQH potential is a short range potential.

The energy spectrum of MIQH potential is reported numerically for various state for $V_0 = 0.1$, $V_1 = 0.5$, $a = b = \mu = \hbar = 1$ with two different screening parameters $\alpha = 0.01$ and 0.02 in Table 1. There is no degeneracies for the considered eigenstate as shown in the table.

Now, a few special case of our results are discussed below. By adjusting some potential parameters. Some well-known potential can be obtained.

5.1 YUKAWA POTENTIAL

Setting $a = 0$, $b = 1$ into Eq.(1), Yukawa potential [31] is obtained as:

$$V(r) = \frac{-V_0 e^{-\alpha r}}{r} \tag{20}$$

Substituting these parameters into Eq. (17) gives the corresponding energy eigenevalues as; [10]

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\frac{\mu V_0}{\alpha\hbar^2} - \left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1)}\right)^2}{2\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1)}\right)}\right]^2 \tag{21}$$

5.2 COULOMB POTENTIAL

By setting $v_1 = 0$, $a = 1$, $b = 0$ into Eq.(1), the coulomb potential is obtained

$$V(r) = \frac{-V_0}{r} \tag{22}$$

With the energy eigenevalue equation as [19]

$$E_{nl}^{c} = -\left(\frac{\mu v_0^{\,2}}{2\hbar^2}\right)\frac{1}{\left[n + \frac{1}{2} - \sqrt{\frac{1}{4} + l(l+1)}\right]^2} \tag{23}$$

5.3 HELLMAN POTENTIAL

If $v_0 = 1$ and $v_1 = 0$, $b = -b$, The Hellmann potential can be obtained as:

$$V(r) = -\frac{a + be^{-\alpha r}}{r} \tag{24}$$

The corresponding energy is given by as found in[11]

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left[\left(\frac{2\mu}{\alpha\hbar^2}(a-b) - \frac{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1)}\right) - l(l+1)}{2\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1)}\right)}\right)^2 - l(l+1) + \frac{2\mu a}{\alpha}\right] \tag{25}$$

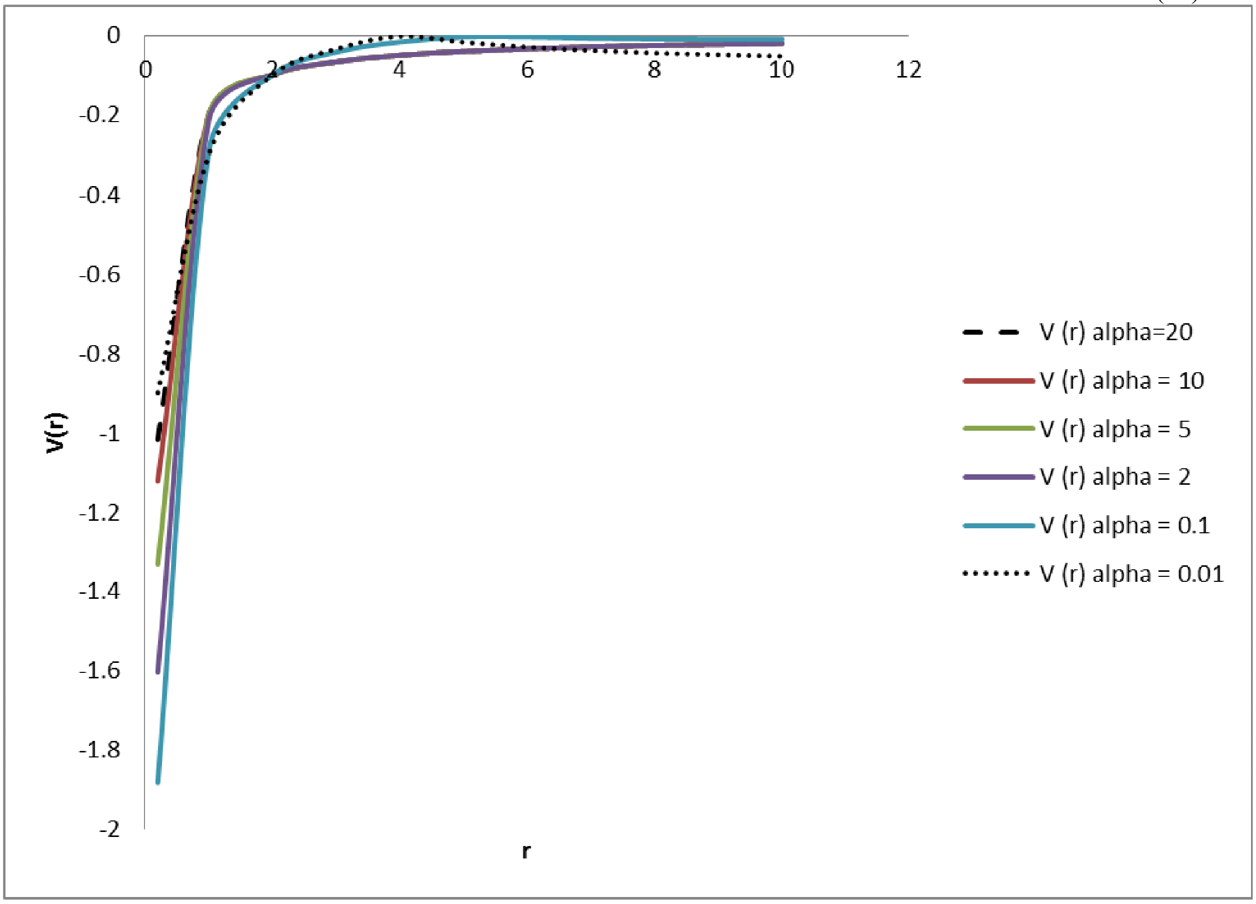

Figure 1. The Variation Of (MIQH) Potential With Radial Distance.

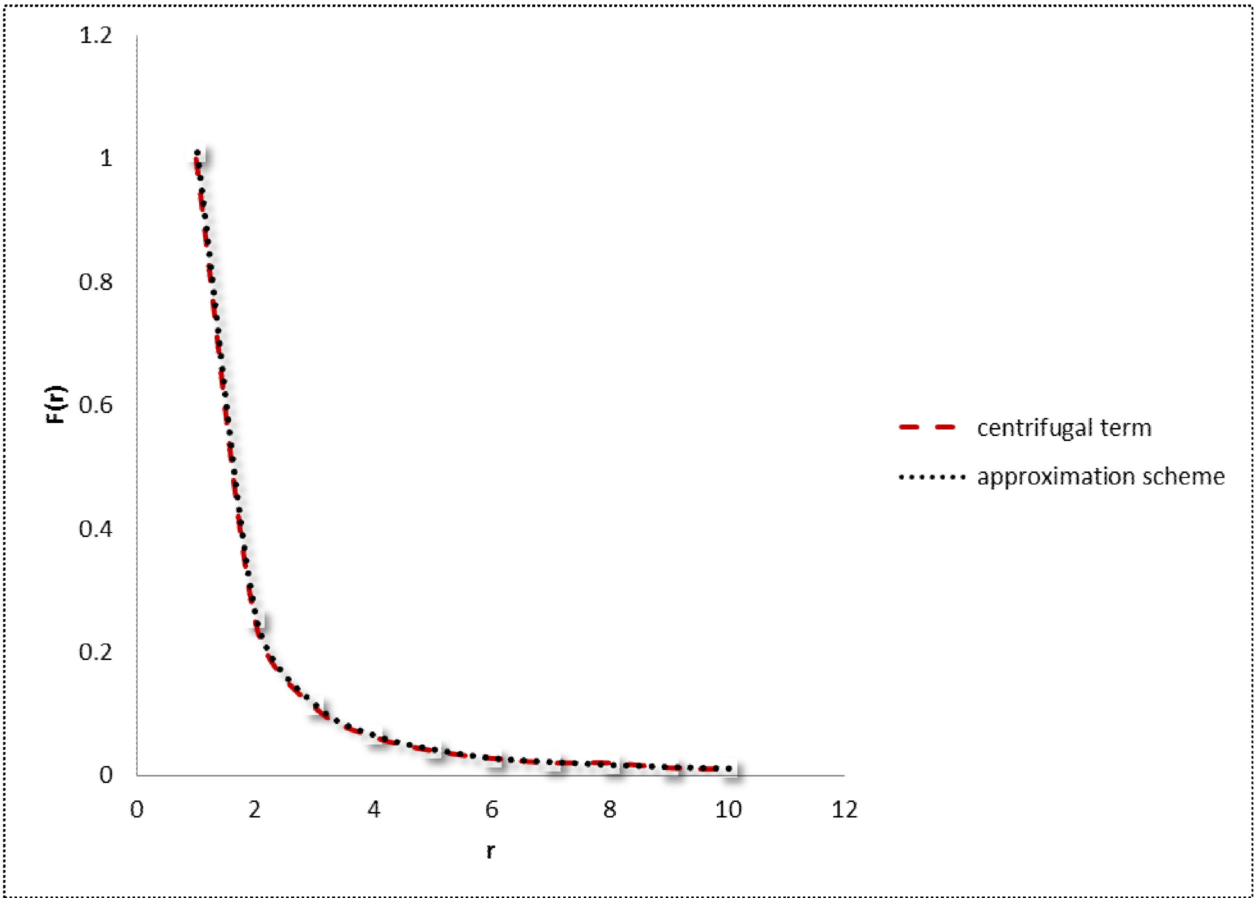

Figure 2. Comparison Of The Centrifugal Term And The Approximation Scheme.

6. CONCLUSION

In this study, the approximate bound state solution of SWE with modified inverselyquadratic Hellmann Potential (MIQH) via parametric Nikiforo-Uvarov (N-U) method was obtained. The energy eigenvalues and the corresponding total normalized wave function were also obtained. The numerical energy eigenvalues obtained in this study is presented in Table 2. The behaviour of the potential was discussed in Figure 1. Our results could be used to study inner shell ionization problem, Electron core, Solid state Physicsand Alkali-hydride molecule[1]. Under appropriate choice of parameters, the potential reduces to few well known potential in literature such as the Yukawa potential, Coulomb potential and the Hellmann potential.

REFERENCES

[1] Amlan K. Roy, Abraham F. Jalbout, Emil I. Proynov (2013) Accurate Calculation of the bound state of Hellmann Potential.
[2] Antia, A. D., Ikot, A. N., Hassanabadi, H. &Maghsoodi, E. (2013).Bound state solutions of Klein-Gordon equation with Mobius square plus Yukawa potentials.Indian J. Phys.87(11), 1133 – 1139.
[3] Antia A. D. Ituen, E. E. Obong, H. P. and Isonguyo, C. N. (2015). Int. J. Recent Advances in Phys. 4(1): 55-65.
[4] Arda A. and Sever R. (2014). Physical Scripta Vol. 89 number 10.
[5] Berkdermir, C., Berkdemir, A. & Sever. R (2008).Polynomial solutions of the Schrodinger equation for thegeneralized Woods-Zaxon potential.Phys. Rev. C 72, 027001.
[6] Dirac, P. A. (1958). The Principles of Quantum Mechanics (USA: Oxford University Press).
[7] Flugge, S. (1971).Practical quantum mechanics II. Berlin: Springer.
[8] Greene, R. L. & Aldrich, C (1976).Phys. Rev. A14, 2363.
[9] Greiner, W. (1989). Quantum Mechanics – An Introduction (Berlin: Springer).
[10] Hamzavi, M. Movahedi, M. Thylwe, K. E. and Rajabi, A. A. (2012). Chin. Phys. Lett. Vol. 29, No. 8(2012) 080302.
[11] Hamzavi, M. Thylwe, K. E. and Rajabi, A. A. (2013). Commun. Theor. Phys. 60(2013) 1-8.
[12] Hassanabadi, H., Maghsoodi, E. &Zarrinkamar, S. (2012). Relativistic symmetries of Dirac equation and theTietz potential.Euro.Phys. J. Plus 127, 31.

[13] Hassanabadi, H., Yazarloo, B. H., Zarinkamar, S. &Rajabi, A. A. (2011).Duff-Kemmer-Petiau equation under ascalar Coulomb interaction.Phys. Rev. C 84, 064003.
[14] Hill, E. H. (1954). The theory of vector spherical harmonics.Am. J. Phys. 22, 211.Ikhdair, S. M & Sever, R. (2010). Approximate bound state solutions of
[15] Ikhdair, S. M. and Sever, R. (2008). Int. J. mod Phys. 19: 221-9
[16] Ikot, A. N., Akpabio, L. E. &Obu, J. A. (2011).Exact Solutions of Schrodinger equation with five parameterpotentials. J. Vect. Relat. 6,1-14.
[17] Ikot, A. N (2012). Solutions of the Klien-Gordon equation with equal scalar and vector modified hylleraasplusexponential Rosen-Morse potential. Chin. Phys. Lett. 29, 060307.
[18] Ikot, A. N., Antia, A. D., Akpan, I. O. &Awoga, O. A. (2013). Bound state solutions of Schrodinger equationwith modified Hylleraas plus exponential Rosen-Morse potential.Revista Mexicana de Fisica 59, 46 – 53.
[19] Ituen, O. B. and Ituen E. E. (2016). IJRAP Vol 5 No. 2 May 2016.
[20] Jia, C. S., Chen, T. & Cui, L. G. (2009). Approximate analytical solutions of the Dirac equation with thegeneralizedPoschl-Teller potential including the pseudo-spin centrifugal term. Phys. Lett. A 373, 1621.
[21] Landau, L. D. &Lifshitz, E. M. (1977). Quantum Mechanics, Non-relativistic Theory (Canada: Pergamon).
[22] Lu, J. (2005). Approximate spin and pseudospin solutions of the Dirac equation. Phys. Scr. 72, 349.
[23] Mustapha, O. and Sever, R. (1991). Phys. Rev. A44:4142.
[24] Nikiforov, A. F. &Uvarov, V. B. (1988). Special Functions of Mathematical Physics (Basel: Birkhauser).
[25] Qiang, W. C. & Dong, S. H. (2009).The Manning-Rosen potential studied by a new approximation scheme to thecentrifugal term.Phys. Scr. 79, 045004.
[26] Schiff, L. I. (1955). Quantum Mechanics (New York: McGraw Hill).
[27] Tezcan, C. & Sever, R. (2009). A general approach for the exact solution of the schrodinger equation. Int. J.Theor. Phys. 48, 337
[28] Yahaya, W. A., Oyewumi, K. J., Akoshule C. O. & Ibrahim T. T. (2010). Bound state solutions of the relativistic
[29] Yauk, F. &Bahar, M. K. (2012).Approximate solutions of the Dirac equation with position-dependent mass forthe Hulthen potential by the asymptotic iteration method.Phys. Scr. 85, 045004.
[30] Yazarloo, B. H., Hassanabadi, H &Zarinkarmar, S. (2012). Oscillator strengths base on the Mobius squarepotential under Schrodinger equation. Eur. Phys. J. Plus 127, 51.
[31] Yukawa, H. (1935). On the interaction of elementary particles. Proc. Phys. Math. Soc. Japan 17, 48.