

On Higher Order Boundary Value Problems Via Power Series Approximation Method

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Abstract
In this work, a relatively new technique called Power Series Approximation Method (PSAM) is applied for the numerical approximate solution of non-linear higher order boundary value problems. Several examples are given to illustrate the efficiency and implementation of the method. The proposed method is efficient and effective on the experimentation as compared with the exact solutions. Numerical results are included to demonstrate the reliability and efficiency of the methods. Graphical representation of the obtained results reconfirms the potential of the suggested method.
Keywords: Power series, nonlinear problems, boundary value problem, numerical simulation

1. Introduction

Higher order boundary value problems in linear and non-linear form have been a major concern in recent years. This is due to its applicability in many areas of Mathematical Physics and other sciences in its precise analysis of nonlinear phenomena such as computation of radio wave attenuation in the atmosphere, interface conditions determination in electromagnetic field, fluid dynamics, astrophysics, hydrodynamic, hydro magnetic stability, astronomy, beam and long wave theory, induction motors, engineering, and applied physics, potential theory and determination of wave nodes in wave propagation. Te boundary value problems of higher order have been examined due to their mathematical importance and applications in diversified applied sciences. Most conventional analytic methods for higher order boundary value problems are prone to rounding-off and computation errors. As a result, the analytics methods are less dependent in seeking the solution of higher order boundary values problems in most cases, especially the non-linear type. Thus, numerical methods have gained momentum in seeking the solution of higher order boundary value problems. Over the years, several techniques have been developed, such as the Variational Iteration Method [1-3], Homotopy Perturbation Method [4-5], Exp-function method [6-7], Spline-Collocation Approximations Method [8], Spline Method [9], etc. that possess an elaborate procedure and structurally complex, which nevertheless yields efficient results. Siddiqi and Iftikhar [10] worked on a numerical solution of higher order boundary value problems. Also, Siddiqi and Iftikhar [11] adopted the technique of variation of parameter methods for the solution of seventh order boundary value problems. Iftikhar et al. [12] solved the thirteenth order value problems by Differential transform method. Akram and Rehman [13] presented a numerical solution of eighth order boundary value problems in reproducing kernel space. Wu *et al*. [14] presented a precise and rigorous work on nonlinear functional analysis of boundary value problems: novel theory, methods and applications. Mamadu and Njoseh [15] have proposed a method which efficiently finds exact solutions and is used to solve linear Volterra integral equations, recently, Bilal [16] introduced a new form of Galerkin weighted residual method with Legendre polynomials and applied successfully on higher order boundary value problems and Tchebychev orthogonal polynomials to approximate the solution of the BVP as a weighted sum of polynomials was used in [17].

In this present work, the Power Series Approximation Method (PSAM) is a new approach developed for the numerical solution of a generalized *N*th order boundary value problems. The proposed method is structurally simple with well posed Mathematical formulae. It involves transforming the given boundary value problems into system of ODEs together with the boundary conditions prescribed. Thereafter, the coefficients of the power series solution are uniquely obtained with a well posed recurrence relation along the boundary, which leads to the solution. The unknown parameters in the solution are determined at the other boundary. This finally leads to a system of algebraic equations, which on solving yields the required approximate series solution. The method is accurate and efficient in obtaining the approximate solutions of linear and non-linear boundary value problems. The method has an excellent rate of convergence as compared with the exact solutions available in the literature.

2. Power Series Approximation Method (PSAM)

Consider the following *N*th order BVP of the form

$$y^{(N)}(x)+f(x)y(x)=g(x) \quad ,\xi_0<x<\xi_1, \tag{1}$$

With boundary conditions

$$y^{(2m)}(\xi_0)=\lambda_{2m} \quad ,m=0,1,2,3,\ldots,(n-1), \tag{2}$$

$$y^{(2m)}(\xi_1)=\beta_{2m} \quad ,m=0,1,2,3,\ldots,(n-1), \tag{3}$$

Where f(x), g(x) and y(x) are assumed real and continuous $\xi_0 \le x \le \xi_1$, λ_{2m}, β_{2m} $, m = 0,12,3\ldots,(n-1)$ are fine real constants.

The given nth order BVP are transformed to system of ODEs such that we have

$$\begin{cases} \dfrac{dy}{dx} = y_1, \\ \dfrac{dy_1}{dx} = y_2, \\ \dfrac{dy_2}{dx} = y_3, \\ \vdots \\ \dfrac{dy}{dx} = g(x) - f(x)y(x), \end{cases} \tag{4}$$

with boundary conditions

$$\begin{aligned} & y_1(\xi_0) = \lambda_0, \\ & y_2(\xi_0) = \lambda_1, \\ & y_3(\xi_0) = \lambda_2, \\ & \vdots \\ & y_{2n}(\xi_0) = \lambda_{2n-1} \end{aligned} \tag{5}$$

and

$$\begin{aligned} & y_1(\xi_1) = \beta_0, \\ & y_2(\xi_1) = \beta_1, \\ & y_3(\xi_2) = \beta_2, \\ & \vdots \\ & y_{2n}(\xi_1) = \beta_{2n-1} \end{aligned} \tag{6}$$

Let the series approximation of Eq.(1),(2) and (3) be given as

$$y_N(x) = \sum_{i=0}^{N} a_i x^i, \quad N < \infty, \tag{7}$$

where $a_i, i = 0(1)N$ are known constants to be determined and x $\in [\xi_0, \xi_1]$.

Now, we determine the unknown constants $a_i, i = 0(1)N,$ at $x = \xi_0$ by substituting Eq. (7) in (4), which is as follows

$$\begin{aligned} & \frac{dy_N}{dx} = y_1 \\ & \frac{d}{dx}\sum_{i=1}^{N} a_i x^i = y_1 \\ & i\sum_{i=1}^{N} a_i x^{i-1} = y_1 \\ & a_1 + i\sum_{i=1}^{N} a_i (x)^{i-1} = y_1 \end{aligned} \tag{8}$$

at $y_1(\xi_0) = \lambda_0$, we have

$$a_1 + i\sum_{i=2}^{N} a_i(\xi_0)^{i-1} = \lambda_0,$$

$$a_1 = \lambda_0 - i\sum_{i=2}^{N} a_i(\xi_0)^{i-1},$$

So, equation(8) becomes

$$y_1 = \lambda_0 - i\sum_{i=2}^{N} a_i(\xi_0)^{i-1} + i\sum_{i=2}^{N} a_i x^{i-1}$$

Next, $\frac{dy_1}{dx} = y_2$

$$\frac{d[a_1 + i\sum_{i=2}^{N} a_i x^{i-2}]}{dx} = y_2$$

$$i(i-1)\sum_{i=2}^{N} a_i(x)^{i-2} = y_2$$

$$2a_2 + i(i-1)\sum_{i=3}^{N} a_i(x)^{i-2} = y_2 \tag{9}$$

$y_2(\xi_0) = \lambda_1$, we obtain

$$a_2 = \frac{1}{2}[\lambda_1 - i(i-1)\sum_{i=3}^{N}[a_i(\xi_0)^{i-2}]$$

So, Equation (9) becomes

$$y_2 = \lambda_1 - i(i-1)\sum_{i=3}^{N} a_i(\xi_0)^{i-2} + i(i-1)\sum_{i=3}^{N} a_i x^{i-2}$$

For n[th] order ,we obtain the following recursive formulae at $x = \xi_0$,

$$a_n = \frac{1}{n\dot{\iota}}[\lambda_n - n\dot{\iota}\sum_{i=n+1}^{N}[a_i(\xi_0)^{i-n}], \qquad n \geq 0 \tag{10}$$

$$y_n = \lambda_n - n\dot{\iota}\sum_{i=n+1}^{N} a_i(\xi_0)^{i-n} + n\dot{\iota}\sum_{i=1}^{N} a_i x^{i-n}, \qquad n \geq 0 \tag{11}$$

Hence, the choice of N is equivalent to the order of the BVP consider.

3. Numerical Applications

In this section, four numerical examples are provided to show the accuracy of the present method.

Example 3.1 We consider a fourth order non-linear problem with boundary conditions [18]

$$y^4(x) = \sin x + \sin^2(x) - (y''(x))^2 \quad ; 0 < x < 1$$

$$y(0) = 0 \; ; y'(0) = 1; \; y(1) = \sin(1); \; y'(1) = \cos(1). \tag{12}$$

The given fourth order BVP is transformed to system of ODEs such that

$$\frac{dy}{dx} = y_1, \frac{dy_1}{dx} = y_2, \frac{dy_2}{dx} = y_3, \frac{dy_3}{dx} = y_4, \tag{13}$$

with boundary conditions at *x=0*

$$y_1(0) = 0, y_2(0) = 1, y_3(0) = a, y_4(0) = b. \tag{14}$$

The series approximation of given problem is given as

$$y_N(x) = \sum_{i=0}^{N} a_i x^i \; ; \quad N < \infty, \tag{15}$$

Where the unknown

$$a_i \,, i = 0(1)N$$

are uniquely determined by this equation

$$a_n = \frac{\lambda_n}{n!} \;, n \geq 0. \tag{16}$$

Using this equation for n=0(1)N ,we have the following

$$aa_0 = 0 \;; a_1 = 1 \;; a_2 = \frac{a}{2} \;; a_3 = \frac{b}{6} \tag{17}$$

Now, by power series approximation method

$$y(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3. \tag{18}$$

From Eq. (17) and (18), we get

$$y(x) = x + \frac{a}{2}x^2 + \frac{b}{6}x^3. \tag{19}$$

As by using the boundary conditions at *x=1* in Eq. (19), we obtain the values of *a* and *b* as *a= 0.03177864, b= -0.8558382.*

Thus, approximation solution of given BVP is

$$y(x) = x - 0.01588932x^2 - 0.1426397x^3. \tag{20}$$

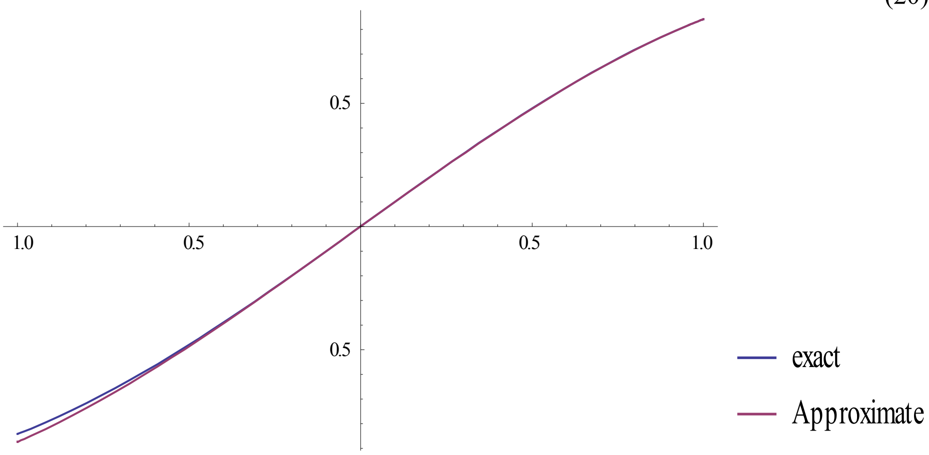

Fig 1 Comparison of exact and approximate solution

Example 3.2 Consider a fifth order non-linear problem [18]

$$y^{(v)}(x) + y^{(iv)}(x) + e^{-2x} y^2(x) = 2e^x + 1 \;,\; 0 \leq x \leq 1 \tag{21}$$

with boundary conditions

$$y(0) = 1, y'(0) = 1, y''(0) = 1, y(1) = e, y'(1) = 0$$

The given 5th order BVP are transformed to system of ODEs,

$$\frac{dy}{dx} = y_1, \frac{dy_1}{dx} = y_2, \frac{dy_2}{dx} = y_3, \frac{dy_3}{dx} = y_4, \frac{dy_4}{dx} = y_5, \tag{22}$$

with boundary conditions at x=0

$$y_1(0) = 1 = \lambda_0,\; y_2(0) = 1 = \lambda_1,\; y_3(0) = 1 = \lambda_2,\; y_4(0) = a = \lambda_3,\; y_5(0) = b = \lambda_4.$$

The series approximation of given problem is given as

$$y_N(x) = \sum_{i=0}^{N} a_i x^i \;; \quad N < \infty, \tag{23}$$

Where the unknown

$$a_i \,, i = 0(1)N$$

are uniquely determined by this equation

$$a_n = \frac{\lambda_n}{n!} \;, n \geq 0. \tag{24}$$

By using this equation, we get

$$a_0 = \frac{\lambda_0}{0!} = 1,\ \ a_1 = \frac{\lambda_1}{1!} = 1, a_2 = \frac{\lambda_2}{2!} = 1,\ a_3 = \frac{\lambda_3}{3!} = \frac{a}{6}, a_4 = \frac{\lambda_4}{4!} = \frac{b}{24} \tag{25}$$

The series solution is

$$y(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4. \tag{26}$$

By putting the values of a_0, a_1, a_2, a_3, a_4 from Eq. (25), we have

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{a}{6}x^3 + \frac{b}{24}x^4. \tag{27}$$

Using boundary conditions at $x=1$, we obtain the values of a and b, as

a=0.92892, b=1.52304

The approximate series solution is

$$y(x) = 1 + x + \frac{1}{2!}x^2 + \frac{0.92892}{3!}x^3 + \frac{1.52304}{4!}x^4. \tag{28}$$

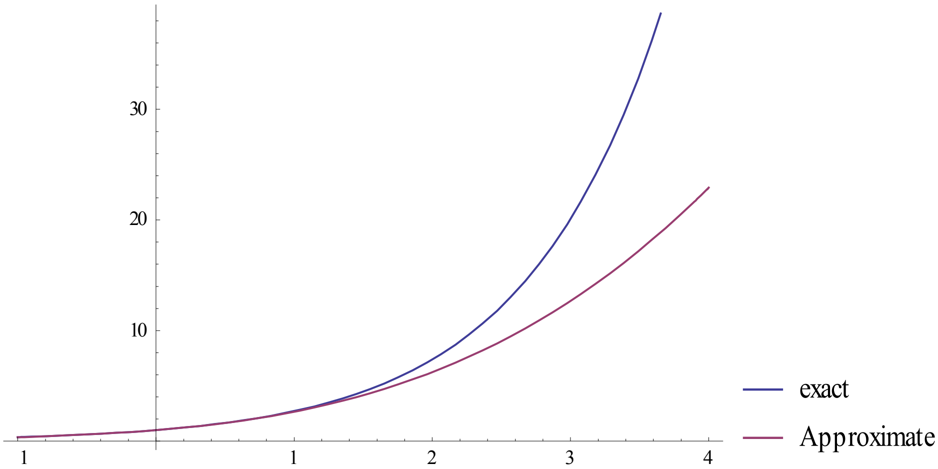

Fig 2 Comparison of exact and approximate solution

Example 3.3 Consider a six order non-linear problem with boundary conditions [18]

$$y^{(vi)}(x) + e^{-x} y^2(x) = e^{-x} + e^{-3x},\ x \in [0,1] \tag{29}$$

$$y(0) = 1; y(1) = \frac{1}{e}; y'(0) = -1; y'(1) = \frac{-1}{e}; y''(0) = 1; y''(1) = \frac{1}{e}$$

The given 6th order BVP are transformed to system of ODEs as

$$\frac{dy}{dx} = y_1, \frac{dy_1}{dx} = y_2, \frac{dy_2}{dx} = y_3, \frac{dy_3}{dx} = y_4, \frac{dy_4}{dx} = y_5, \frac{dy_5}{dx} = y_6, \tag{30}$$

with boundary conditions at $x=0$

$$y_1(0) = 1 = \lambda_0, y_2(0) = -1 = \lambda_1,\ y_3(0) = 1 = \lambda_2,\ y_4(0) = a = \lambda_3, y_5(0) = b = \lambda_4, y_6(0) = c = \lambda_5. \tag{31}$$

According to defined above procedure, we have

$$a_0 = \frac{\lambda_0}{0!} = 1,\ \ a_1 = \frac{\lambda_1}{1!} = -1, a_2 = \frac{\lambda_2}{2!} = \frac{1}{2},\ a_3 = \frac{\lambda_3}{3!} = \frac{a}{6}, a_4 = \frac{\lambda_4}{4!} = \frac{b}{24}, a_5 = \frac{\lambda_5}{5!} = \frac{c}{120}. \tag{32}$$

The approximate series solution is

$$y(x) = 1 - x + \frac{1}{2}x^2 + \frac{a}{6}x^3 + \frac{b}{24}x^4 + \frac{c}{120}x^5. \tag{33}$$

Using boundary conditions at $x=1$, we obtain the values of a, b and c as

a=2.006; b=-23.0727; c=59.388

Finally, the approximate series solution is

$$y(x) = 1 - x + \frac{1}{2!}x^2 + \frac{2.006}{3!}x^3 - \frac{23.072}{4!}x^4 + \frac{59.388}{5!}x^5. \tag{34}$$

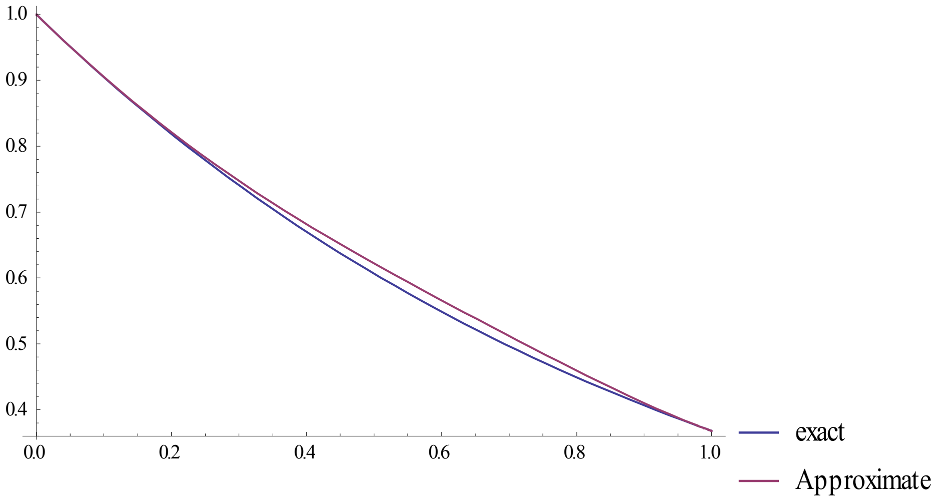

Fig 3 Comparison of exact and approximate solution

Example 3.4 Consider a seven order non-linear problem with boundary conditions [19]

$$y^{(7)}(x)+y^{(4)}(x)-e^{y(x)}y(x)=e^{x}((12-4x+(x-1)e^{(-e^{x}(x-1)\cos x)}\cos x)-8(5+x)\sin x), \tag{35}$$

subject to the boundary conditions

$$y(0)=1\;;y'(0)=0\;;y''(0)=-2\;;y'''(0)=-2\;;y(1)=0\;;y'(1)=-e\cos 1\;;y''(1)=-2e\cos 1+2\sin 1$$

The given seventh order BVP is transformed to system of ODEs as

$$\frac{dy}{dx}=y_1,\frac{dy_1}{dx}=y_2,\frac{dy_2}{dx}=y_3,\frac{dy_3}{dx}=y_4,\frac{dy_4}{dx}=y_5,\frac{dy_5}{dx}=y_6,\frac{dy_6}{dx}=y_7, \tag{36}$$

With boundary conditions at *x=0*

$$y_1(0)=1=\lambda_0, y_2(0)=0=\lambda_1,\; y_3(0)=-2=\lambda_2,\; y_4(0)=-2=\lambda_3, y_5(0)=a=\lambda_4, y_6(0)=b=\lambda_5, \\ y_7(0)=c=\lambda_6. \tag{37}$$

Following the above defined procedure, using boundary conditions at *x=1*, we obtain the values of *a*, *b* and *c* as

a= -1.8671; b=74.5650; c= -309.4747654

Finally the following series approximate solution is determined as

$$y(x)=1-x^2+\frac{2}{3!}x^3-\frac{1.867122}{4!}x^4+\frac{74.5650}{5!}x^5-\frac{309.4748}{6!}x^6. \tag{38}$$

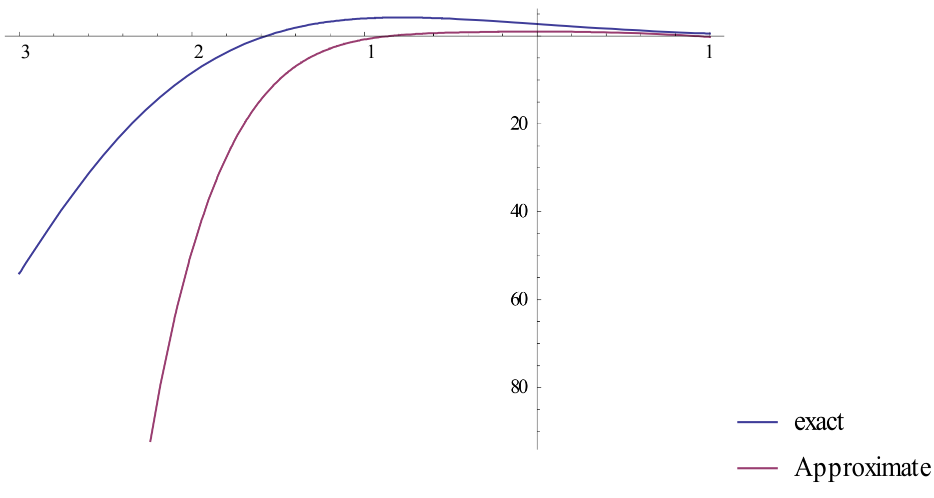

Fig 4 Comparison of exact and approximate solution

4. Conclusion

In this paper, we have used successfully the Power Series Approximation Method for solving higher order nonlinear boundary value problems. The suggested method was applied to get the approximate solutions of higher order boundary value problems. This method is applied in direct way with no restriction of discretization, linearization or perturbation. By increasing the order of approximation more accuracy can be obtained. Comparison of the results obtained with existing exact solution shows that the PSAM is more efficient and accurate while still maintaining the high accuracy of the numerical result, the size reduction amounts to the improvement of performance of approach. Hence, it is easier and more economical to apply PSAM in solving BVPs.

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