New analytical approximate solutions of Fifth-order KdV equation

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Abstract

In this paper, we have exposed a process of how to implement a new splitting Adomian decomposition homotopy perturbation method to solve fifth-order KdV equations. The new methodology is applied on two kinds of fifth-order KdV equations with initial data: The first is Sawada-Kotera equation and the second its Lax equation. The numerical results we obtained from solutions of two kinds of fifth-order KdV equations, have good convergent and high accuracy comparison with other methods in literature. The graphs and tables of the new analytical approximate solutions show the validity, usefulness, and necessity of the process.

Keywords: Splitting scheme, Adomian decomposition, homotopy perturbation method, fifth-order KdV equation, convergence analysis.

Mathematics Subject Classifications 2010 [MSC]: 76S05, 65N99, 35Q35

1- Introduction

The KdV equation is the nonlinear partial differential equation and it's an important equation describe a large number of physical phenomena for example; shallow water waves, ion acoustic plasma waves, bubble-liquid mixtures. It name is taken from the world who discovered its Kortewge-de Vries in 1895. Many scientists and researchers attempted to find solution of fifth-order KdV equation by using different methods for examples; Sawada and Kotera [1] use inverse scattering method and Lax[2] travelling wave solution to solve 5th-order KdV equation analytically. Bakodah [3] used modified Adomian decomposition method for solving fifth-order KdV equation. Wazwaz [4] applied sine-cosine and tanh method to solve fifth-order KdV equation. Ghasemi et. al. [5] suggested numerical solutions for fifth-order KdV equations by applying homotopy perturbation method. While, Adomian decomposition method(ADM) and homotopy perturbation method(HPM) are active and strong in finding solutions for physical and mathematical problems, so many authors apply them to solve linear or non-linear initial-boundary value problems. The first method's name is taken from the scientist who discovered it ; namely, G. Adomian[6], and the second was found for first time by the Chinese Mathematician He[7]. In many works[8,9,10], the authors used ADM to find analytic and approximate solutions for different problems. In the same direction, the HPM is active to find solutions for non-linear equations[11,12,13].

Depending on the above literature review of the researchers' attempts to expand and develop ADM and HPM to solve linear and nonlinear boundary value problems, and depending on our simple information about applications of these methods to solve the problems that are under consideration study. We adopt our new method based on combining Adomian decomposition and Homotopy perturbation methods with the splitting time scheme for differential operators, namely splitting decomposition homotopy perturbation method (SDHPM) [14], to solve fifth-order KdV equation. The power of this manageable method is confirmed by applying it for two selected problems as a test to be illustrated by the effectiveness and validity of new methodology in finding solution of fifth-order KdV equation. The numerical results we obtained have shown that the efficiency, activity and high accuracy of the new method in comparison with standard ADM and HPM[5].

2-The main idea of the SDHPM method:

The basic idea of **SDHPM** depends on the algorithms of ADM and HPM those will be discussed in this section. The main idea of the standard two methods Adomian decomposition method and Homotopy perturbation method will be explained to the general initial value problem as in the following differential operators form:

$$Lu + Ru + Nu = g \tag{1a}$$

$$u_0 = u(x,0) \tag{1b}$$

The linear terms decomposed into $Lu + Ru$,while the nonlinear terms are represented by Nu , where L is an easily invertible linear differential operator, R is the remaining linear part, $u = u(x,t)$ is exact solution of Equation(1), and $g = g(x,t)$ is known analytic function.

2.1 Algorithm of ADM: The principle of the Adomian decomposition method when applied for Equation(1) is in the following form (Celik et al.,2006[15]; Seng et al.,1996[16]).

$$u = L^{-1}(g) - L^{-1}(Ru) - L^{-1}(Nu) \tag{2}$$

Where $$L^{-1}(.) = \int_0^t (.)dt \tag{3}$$

is the inverse operator of L .

The decomposition method represents the solution of Equation (2) as the following infinite series:

$$u = \sum_{n=0}^{\infty} u_n \tag{4}$$

The nonlinear operator $Nu = \Psi(u)$ is decomposed as:

$$Nu = \sum_{n=0}^{\infty} A_n \tag{5}$$

where A_n are Adomian 's polynomials , which are defined as (Seng et. al. 1996[16]):

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\Psi\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0} \qquad n = 0,1,2,... \tag{6}$$

Substituting Equations (4) and(5) into Equation (2), we have

$$u = \sum_{n=0}^{\infty} u_n = u_0 - L^{-1}(R(\sum_{n=0}^{\infty} u_n)) - L^{-1}(\sum_{n=0}^{\infty} A_n) \tag{7}$$

Consequently , it can be written as:

$$\left.\begin{aligned} u_0 &= \phi + L^{-1}(g) \\ u_1 &= -L^{-1}\left(R(u_0)\right) - L^{-1}(A_0) \\ u_2 &= -L^{-1}\left(R(u_1)\right) - L^{-1}(A_1) \\ &. \\ &. \\ &. \\ u_n &= -L^{-1}\left(R(u_{n-1})\right) - L^{-1}(A_{n-1}) \end{aligned}\right\} \tag{8}$$

where $\phi = u(x,0)$ is the initial condition.

Hence all the terms of u are calculated and the general solution obtained according to ADM as $u = \sum_{n=0}^{\infty} u_n$. The convergent of this series has been proved in[16]. However ,for some problems this series can't be determined [15], so we use an approximation of the solution from truncated series

$$U_M = \sum_{n=0}^{M} u_n \quad \text{with} \quad \lim_{M\to\infty} U_M = u \tag{9}$$

2.2 Algorithm of HPM: To illustrate the basic idea of the homotopy technique [17,18] for Equation(1) ,with the boundary condition :

$$\mathrm{B}\left(u, \frac{\partial u}{\partial n}\right) = 0 \tag{10}$$

where, B is a boundary operator, we construct a homotopy $v(r,p): \Omega \times [0,1] \to R$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + R(v) + N(v) - g] = 0, \quad p \in [0,1] \tag{11a}$$

or $$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[R(v) + N(v) - g] = 0 \tag{11b}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximate of Equation (1) , which satisfies the boundary conditions. Obviously , from Equation(11) we have;

$$H(v,0) = L(v) - L(u_0) = 0 \tag{12}$$

$$H(v,1) = L(v) + R(v) + N(v) - g = 0 \tag{13}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation , and $L(v) - L(u_0)$, $L(v) + R(v) + N(v) - g$ are called homotopic. And assume that the solution of Equation (11) can be written as a power series in p :

$$v = v_0 + pv_1 + p^2 v_2 + .. \tag{14}$$

Setting $p = 1$ results in the approximate solution of Equation (1):

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + ... \tag{15}$$

The converge of this method is studied in [19].

2.3 Algorithm of SDHPM: Now, to illustrate the basic idea of the new methodology, we decomposed the linear differential operator in Equation(1a) into two parts of differential operators as the form :

$$L(u) = \alpha L(w) + \beta L(h) \text{ ,} \tag{16}$$

where $\alpha + \beta = 1,\ \alpha, \beta \in [0,1]$. By this definition, we can split Equation(1a) into two kinds of differential operator equations; one is linear and the other is non-linear as;

$$L(w)+R(w)=0 \tag{17}$$

$$L(h)+N(h)-g=0 \tag{18}$$

We apply ADM as explained above on Equation(17) to find the solution as series $w_n, \quad n=1,2,..$. depending on the initial condition u_0, then using the result as an initial condition for the series solution $h_n, \quad n=1,2,..$ that is obtained by using algorithm of HPM for Equation (18) respectively. Repeating this iterative procedure between Equation(17) and Equation(18) by exchange, in order to reach to the original series solution $u_n, \quad n=1,2,..$, then use (9) to obtain on the solution u .

2.3.1 Algorithm Analysis of SDHPM for fifth-order KdV equation:

Consider the fifth-order KdV initial value problem as the form:

$$u_t + a u^2 u_x + b u_x u_{xx} + c u u_{xxx} + d u_{xxxxx} = 0 \tag{19}$$

$$u(x,0)=u_0 \tag{20}$$

where u represent the velocity and a,b,c and d are real parameters.
Now, we start applying the ADM algorithm for Equation (19) with initial condition(20). Following, we define the differential operators $L_t=\frac{\partial}{\partial t}$, $L_x=\frac{\partial}{\partial x}$, $L_{xx}=\frac{\partial^2}{\partial x^2}$, $L_{xxx}=\frac{\partial^3}{\partial x^3}$, $L_{xxxxx}=\frac{\partial^5}{\partial x^5}$. Therefore, we rewrite Equation (19) in operator form as;

$$L_t u + a u^2 L_x u + b L_x u L_{xx} u + c u L_{xxx} u + d L_{xxxxx} u = 0 \tag{21}$$

By defining the inverse operator L^{-1} that is given in(3), we can write Equation (21) as;

$$u(x,t) = u(x,0) - a L_t^{-1}(Nu) - b L_t^{-1}(Mu) - c L_t^{-1}(Ku) - d L_t^{-1}(L_{xxxxx} u) \tag{22}$$

where $Nu = u^2 L_x u = \sum_{n=0}^{\infty} A_n$, $Mu = L_x u L_{xx} u = \sum_{n=0}^{\infty} B_n$, $Ku = u L_{xxx} u = \sum_{n=0}^{\infty} C_n$ are the nonlinear operators can be calculated by using Adomian's polynomial which is define in Equation(6).

The components solutions can be written as; $u(x,t)=\sum_{n=0}^{\infty} u_n(x,t)$

The associated decomposition method is given by

$$u_0 = u(x,0) \tag{23a}$$

$$u_{n+1} = -a L_t^{-1}(A_n) - b L_t^{-1}(B_n) - c L_t^{-1}(C_n) - d L_t^{-1}(L_{xxxxx} u_n) \tag{23b}$$

where;

$$\left.\begin{aligned}
&A_0 = u_0{}^2 \frac{\partial u_0}{\partial x} \\
&A_1 = 2u_0 u_1 \frac{\partial u_0}{\partial x} + u_0{}^2 \frac{\partial u_1}{\partial x} \\
&\vdots \\
&So\ on.and, \\
&B_0 = \frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2} \\
&B_1 = \frac{\partial u_0}{\partial x}\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial u_1}{\partial x}\frac{\partial^2 u_0}{\partial x^2} \\
&\vdots \\
&So\ on.and, \\
&C_0 = u_0 \frac{\partial^3 u_0}{\partial x^3} \\
&C_1 = u_0 \frac{\partial^3 u_1}{\partial x^3} + u_1 \frac{\partial^3 u_0}{\partial x^3} \\
&\vdots \\
&So\ on.
\end{aligned}\right\} \tag{24}$$

Consequently the iterative solutions are;

$$\left.\begin{aligned}
&u_0 = u(x,0) = \phi \\
&u_1 = L_t^{-1}\left(-d\left(\frac{\partial^5 u_0}{\partial x^5}\right)\right) - a\,L_t^{-1}(A_0) - b\,L_t^{-1}(B_0) - c\,L_t^{-1}(C_0) \\
&u_2 = L_t^{-1}\left(-d\left(\frac{\partial^5 u_1}{\partial x^5}\right)\right) - a\,L_t^{-1}(A_1) - b\,L_t^{-1}(B_1) - c\,L_t^{-1}(C_1) \\
&\vdots
\end{aligned}\right\}. \tag{25}$$

and so on.

Now, By using HPM algorithm to Equation (19) ,we have:

$$H(v,p) = (1-p)\left[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right] + p\left[\frac{\partial v}{\partial t} + a\,v^2\frac{\partial v}{\partial x} + b\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} + c\,v\frac{\partial^3 v}{\partial x^3} + d\frac{\partial^5 v}{\partial x^5}\right] = 0$$

$$or\ \ \frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} = p\left[-a\,v^2\frac{\partial v}{\partial x} - b\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} - c\,v\frac{\partial^3 v}{\partial x^3} - d\frac{\partial^5 v}{\partial x^5} - \frac{\partial u_0}{\partial t}\right] = 0$$

We assume the solution as a power series in p; then we have:

$$\frac{\partial}{\partial t}(v_0 + pv_1 + p^2 v_2 + \ldots) - \frac{\partial u_0}{\partial t} =$$

$$p\left[\begin{aligned}&-a(v_0 + pv_1 + p^2v_2 + \ldots)^2\frac{\partial^2}{\partial x^2}(v_0 + pv_1 + p^2v_2 + \ldots) - b\frac{\partial}{\partial x}(v_0 + pv_1 + p^2v_2 + \ldots)\frac{\partial^2}{\partial x^2}(v_0 + pv_1 + \\ &p^2v_2 + \ldots) - c\left((v_0 + pv_1 + p^2v_2 + \ldots)\frac{\partial^3}{\partial x^3}(v_0 + pv_1 + p^2v_2 + \ldots)\right) - d\frac{\partial^5}{\partial x^5}(v_0 + pv_1 + p^2v_2 + \ldots) - \frac{\partial u_0}{\partial t}\end{aligned}\right]$$

By equal the term which have the same power of p , we get:

$$\left.\begin{aligned}
&p^0: \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \\
&p^1: \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} + a\, v_0^{\,2}\frac{\partial v_0}{\partial x} + b\frac{\partial v_0}{\partial x}\frac{\partial^2 v_0}{\partial x^2} + c\, v_0\frac{\partial^3 v_0}{\partial x^3} + d\frac{\partial^5 v_0}{\partial x^5} = 0 \\
&p^2: \frac{\partial v_2}{\partial t} + a\left(2 v_0 v_1\frac{\partial v_0}{\partial x} + v_0^{\,2}\frac{\partial v_1}{\partial x}\right) + b\left(\frac{\partial v_0}{\partial x}\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial v_1}{\partial x}\frac{\partial^2 v_0}{\partial x^2}\right) + c\left(v_0\frac{\partial^3 v_1}{\partial x^3} + \right. \\
&\qquad \left. v_1\frac{\partial^3 v_0}{\partial x^3}\right) + d\frac{\partial^5 v_1}{\partial x^5} = 0 \\
&\vdots
\end{aligned}\right\} \tag{26}$$

and so on. Then the approximate solution can be found by setting $p = 1$,

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \ldots \tag{27}$$

The application of SDHPM algorithm for Equation(19) is as follows:
Applying (17) and (18) with $\alpha = \beta = 0.5$ on (19) to obtain:

$$L(w) = u = -2L_t^{-1}\left(d\, L_{xxxxx} u\right) \tag{28}$$

$$L(h) = u = -2L_t^{-1}\left(a u^2\, L_x u + b\, L_x u\, L_{xx} u + c\, u\, L_{xxx} u\right) \tag{29}$$

Then applying ADM for (28) to obtain;

$$u_0 = u(x,0) \qquad w_1 = w_0 + L_t^{-1}\left[-2d\left(\frac{\partial^5 u_0}{\partial x^5}\right)\right], where\ w_0 = u_0 \tag{30}$$

and applying HPM for (29) with result of (30) to obtain;

$$h_1 = h_0 + L_t^{-1}\left[-2a\left(w_1^{\,2}\left[\frac{\partial w_1}{\partial x}\right]\right) - 2b\left(\frac{\partial w_1}{\partial x}\left[\frac{\partial^2 w_1}{\partial x^2}\right]\right) - 2c\left(w_1\left[\frac{\partial^3 w_1}{\partial x^3}\right]\right) - \frac{\partial w_1}{\partial t}\right], where\ h_0 = w_1 \tag{31}$$

$$then;\quad u_1 = \alpha w_1 + \beta h_1 \tag{31a}$$

Then, after repeating this procedure between two schemes(ADM &HPM) by exchange, we have:

$$w_2 = w_1 + L_t^{-1}\left[-2d\frac{\partial^5 u_1}{\partial x^5}\right] \tag{32}$$

$$h_2 = h_1 + L_t^{-1}\left[\begin{aligned}&-2\left(2a\, u_1 u_0\frac{\partial u_0}{\partial x}\right) - 2\left(a\, u_0^{\,2}\frac{\partial u_1}{\partial x}\right) - 2\left(b\frac{\partial u_1}{\partial x}\left(\frac{\partial^2 u_0}{\partial x^2}\right)\right) - \\ &2\left(b\frac{\partial u_0}{\partial x}\left(\frac{\partial^2 u_1}{\partial x^2}\right)\right) - 2\left(c\, u_0\left(\frac{\partial^3 u_1}{\partial x^3}\right)\right) - 2\left(c\, u_1\left(\frac{\partial^3 u_0}{\partial x^3}\right)\right)\end{aligned}\right],\ where\ , h_1 = w_2 \tag{33}$$

$$\begin{aligned}&then,\quad u_2 = \alpha w_2 + \beta h_2 \\ &\qquad\vdots \\ &\qquad So\ on.\end{aligned} \tag{33a}$$

Repeated this alternate procedure between two schemes ADM and HPM by using Equations (28) and (29),we obtain successive solutions that can be written as a sum;

$$u(x,t) = u_0 + u_1 + u_2 + \cdots = \sum_{n=0}^{\infty} u_n(x,t)$$

The convergent of this series will be proved in the next section theoretically. However, for some problems this series can't be determined, so we use an approximation of the solution from truncated series:

$$U_M = \sum_{n=0}^{M} u_n \quad \text{with} \quad \lim_{M\to\infty} U_M = u$$

The acceleration for this convergent means the need to few terms of the above equation, for obtaining the formula which nearby to the exact solution.

3. Numerical Test and Discussion:

The theoretical analysis of SDHPM done in section 2 will be applied here to find the analytical approximate solution of two kinds of KdV initial value problems: The first contains Sawada-Kotera equation and obtained if put $a = 45, b = 15, c = 15, d = 1$ in Equation(19), and the second contains Lax equation and obtained if put $a = 30, b = 30, c = 10, d = 1$ in Equation(19).

Test problem 1 (P1) [5]: Consider Equation (19) as a Sawada-Kotera equation, with the exact solution $u(x,t) = 2k^2 \operatorname{sec}h\big(k(x - 16k^4 t - x_0)\big)^2$, and the initial condition $u(x,0) = 2k^2 \operatorname{sec}h\big(k(x - x_0)\big)^2$. The iterative solutions for this problem by using SDHPM can be obtained after we split the linear operator of time of Sawada-Kotera equation as in Equation(17,18), then by using algorithm of SDHPM for Equations(28,29) that is represented by equations(30-33),we get the successive analytical approximate solution of Equation(19) as the following:

$$u_0 = 2k^2 \operatorname{sec}h\big(k(x - x_0)\big)^2$$

$$u_1 = a + \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d, \quad where, \quad a = \frac{2k^2}{\eta^2}, \quad b = \frac{64k^7 t\zeta\left(17\eta^4 - 60\eta^2\zeta^2 + 45\zeta^4\right)}{\eta^7}$$

$$c = \frac{k^{22}t^4(3762339840\ 000 - \eta^{10}(5702400 - 8294400\ \eta^2) - \eta^2(2926080\ \eta^{12} + 1379524608\ 0000\ ))}{\eta^{22}}$$
$$+ \frac{138240\ k^{12}t^2\eta^{16} + 100k^{22}t^4\eta^4(2023302758\ 40 - 1515287347\ 20\eta^2 + 6123788976\ \eta^4 - 1295607398\ 4\eta^6)}{\eta^{22}}$$
$$+ \frac{10k^{22}t^4\eta^{12}(1303746969\ 6 - 589824000\ \eta^2 + 9437184\ \eta^4) - 1000k^{17}t^3\eta^5\zeta(7962624 - 16257024\ \eta^2)}{\eta^{22}}$$
$$+ \frac{-k^{17}t^3\eta^9\zeta(1114767360\ 0\,k - 2949120000\ \eta^2 + 243793920\ \eta^4 - 5898240\ \eta^6) + 1440\ k^7 t\eta^{15}\zeta}{\eta^{22}}$$

$$d = \frac{k^{12}t^2(544680\ \cosh(3\tau)^6 - 2973510\ \eta + 1380435\ \cosh(5\tau) + 152085\ \cosh(7\tau))}{\eta^{17}}$$
$$+ \frac{k^{12}t^2(128175\ \cosh(9\tau) + 6915\ \cosh(11\tau) - 30\ \cosh(13\tau) - 4534044672\ 0\,k^5 t\zeta)}{\eta^{17}}$$
$$+ \frac{10k^{17}t^3(2279198400\ \sinh(3\tau) - 555187520\ \sinh(5\tau) + 60805376\ \sinh(7\tau))}{\eta^{17}}$$
$$+ \frac{-10k^{17}t^3(2516736\ \sinh(9\tau) - 29632\ \sinh(11\tau) + 64\ \sinh(13\tau))}{\eta^{17}} + \frac{1845\ k^7 t\zeta}{16\ \eta^{17}} + \frac{7695\ k^7 t\sinh(3\tau)}{32\ \eta^{17}}$$
$$+ \frac{375\ k^7 t\sinh(5\tau)}{2\ \eta^{17}} + \frac{2415\ k^7 t\sinh(7\tau)}{32\ \eta^{17}} + \frac{405\ k^7 t\sinh(9\tau)}{32\ \eta^{17}} - \frac{15\ k^7 t\sinh(11\tau)}{16\ \eta^{17}} - \frac{15\ k^7 t\sinh(13\tau)}{32\ \eta^{17}}$$
$$\vdots$$

$$where; \quad \eta = \cosh(\tau), \quad \zeta = \sinh(\tau), \quad \tau = k(x - x_0)$$

Tables (1-2) show the absolute error of present study compared with standard ADM and HPM [5] for problem (1) at different time and space. Figure (1) illustrates the plot of exact and approximate solution at $t=15, k=0.01\, and\, x_0=0$ and the approximate solution at different value of time ($t=0.01,1,5,10,20$).

Table.1: Absolute error $|u_1-u_{Exact}|$ comparison between present study ,HPM,and ADM for p1

x	Method $t \Rightarrow$	0.2	0.4	0.8	1.6	2	4	6	8	10
0.2	Present study	1.64 E-20	6.58 E-20	2.63 E-19	1.05 E-18	1.64 E-18	6.58 E-18	1.48 E-17	2.63 E-17	4.11 E-17
	HPM[5]	1.91 E-15	3.83 E-15	7.67 E-15	1.53 E-14	1.91 E-14	3.83 E-14	5.75 E-14	7.67 E-14	9.59 E-14
	ADM	5.57 E-15	1.15 E-14	2.30 E-14	4.60 E-14	5.57 E-14	1.15 E-13	1.72 E-13	2.30 E-13	2.87 E-13
0.4	Present study	1.64 E-20	6.58 E-20	2.63 E-19	1.05 E-18	1.64 E-18	6.57 E-18	1.48 E-17	2.63 E-17	4.11 E-17
	HPM[5]	3.83 E-15	7.67 E-15	1.53 E-14	3.07 E-14	3.83 E-14	7.67 E-14	1.15 E-13	1.53 E-13	1.91 E-13
	ADM	1.15 E-14	2.30 E-14	4.60 E-14	9.21 E-14	1.15 E-13	2.30 E-13	3.45 E-13	4.60 E-13	5.75 E-13
1	Present study	1.64 E-20	6.56 E-20	2.62 E-19	1.05 E-18	1.64 E-18	6.56 E-18	1.47 E-17	2.62 E-17	4.10 E-17
	HPM[5]	9.59 E-15	1.91 E-14	3.83 E-14	7.67 E-14	9.59 E-14	1.91 E-13	2.87 E-13	3.83 E-13	4.79 E-13
	ADM	2.87 E-14	5.75 E-14	1.15 E-13	2.30 E-13	2.87 E-13	5.75 E-13	8.63 E-13	1.15 E-12	1.43 E-12

Table.2: Absolute error $|u_2-u_{Exact}|$ comparison between present study ,HPM,and ADM for p1

x	Method $t \Rightarrow$	0.2	0.4	0.8	1.6	2	4	6	8	10
0.2	Present study	6.92 E-21	2.77 E-20	1.11 E-19	4.43 E-19	6.93 E-19	2.77 E-18	6.23 E-18	1.11 E-17	1.73 E-17
	HPM[5]	2.56 E-16	5.12 E-16	1.02 E-15	2.04 E-15	2.56 E-15	5.12 E-15	7.69 E-15	1.02 E-14	1.28 E-14
	ADM	4.09 E-15	8.19 E-15	1.63 E-14	3.27 E-14	4.09 E-14	8.19 E-14	1.22 E-13	1.63 E-13	2.04 E-13
0.4	Present study	6.93 E-21	2.76 E-20	1.11 E-19	4.43 E-19	6.93 E-19	2.77 E-18	6.23 E-18	1.11 E-17	1.73 E-17
	HPM[5]	5.12 E-16	1.02 E-15	2.04 E-15	4.09 E-15	5.12 E-15	1.02 E-14	1.53 E-14	2.05 E-14	2.56 E-14
	ADM	8.19 E-15	1.63 E-14	3.27 E-14	6.55 E-14	8.19 E-14	1.63 E-13	2.45 E-13	3.27 E-13	4.09 E-13
1	Present study	6.88 E-21	2.75 E-20	1.10 E-19	4.40 E-19	6.89 E-19	2.75 E-18	6.20 E-18	1.10 E-17	1.72 E-17
	HPM[5]	1.27 E-15	2.55 E-15	5.11 E-15	1.02 E-14	1.27 E-14	2.56 E-14	3.48 E-14	5.12 E-14	6.40 E-14
	ADM	2.04 E-14	4.09 E-14	8.18 E-14	1.63 E-13	2.04 E-13	4.09 E-13	6.14 E-13	8.18 E-13	1.02 E-12

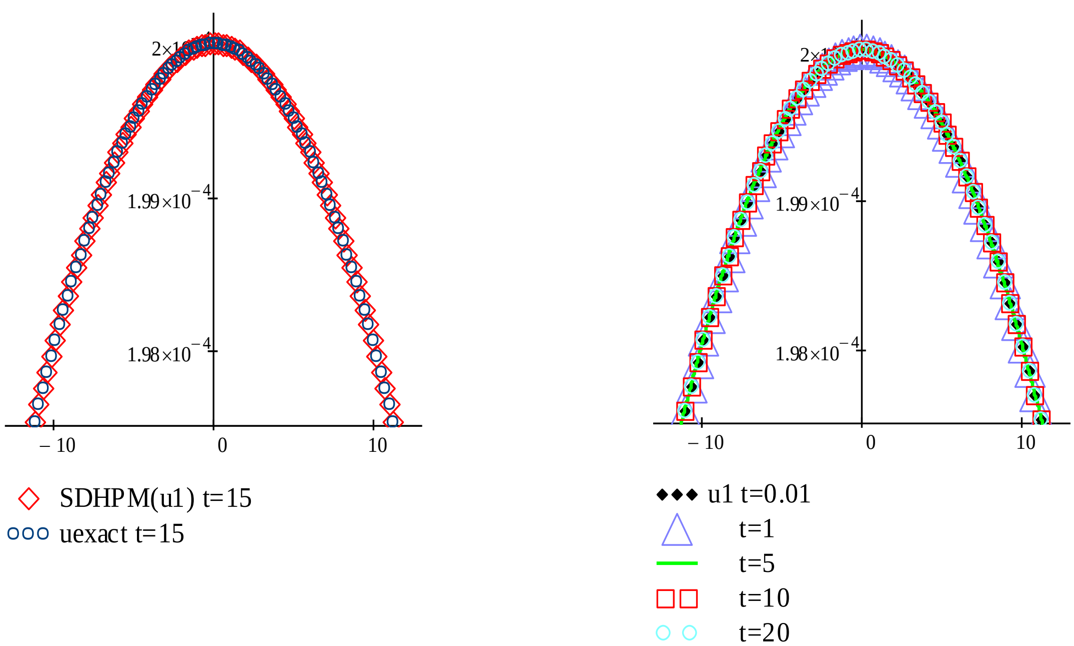

Figure (1): Exact and approximate solution at $t=15$, $k=0.01\, and\, x_0=0$ and the approximate solution at different value of time ($t=0.01,1,5,10,20$) ,for Sawada-Kotera problem.

The result shows that there is a good agreement with the exact solution as shown in Figure(1). Also, we note that the approximate solution converge to the exact solution for large time and its stay governor on the same pattern. Then we can say, that, the new method SDHPM is efficient method with good converge and high accuracy comparing with ADM and HPM[5].

Test problem2 (P2) Lax equation [5]: Consider Equation (19) as in Lax equation with the exact solution $u(x,t) = 2k^2\left(2 - 3\tanh\left(k(x - 56k^4 t - x_0)\right)^2\right)$, and the initial condition $u(x,0) = 2k^2\left(2 - 3\tanh\left(k(x - x_0)\right)^2\right)$.

The iterative solutions for this problem by using SDHPM as the following form:

$$
\begin{aligned}
u_0 &= 2k^2\left(2 - 3\tanh\left(k(x - x_0)\right)^2\right) \\
u_1 &= \Big[-2k^2(3\delta^2 - 2) - k^7 t\,(\delta^2 - 1)\Big[86\delta\left(45\delta^4 - 60\delta^2 + 17\right) + 48\alpha\left[2351462400\;0\,k^{15} t^3 \delta^{20}\right]\Big]\Big] \\
&\quad - 480\,\alpha k^{22} t^4 \delta^{12}(\delta^2 - 1)\left[-1489259520\;0\,\delta^6 + 4086319104\;0\,\delta^4 - 6347704320\;0\,\delta^2 + 6116821401\;6\right] \\
&\quad - 48\,\alpha k^{22} t^4 \delta^4(\delta^2 - 1)\left[-3762185012\;0\,\delta^6 + 1460055490\;56\,\delta^4 - 3383989862\;\delta^2 + 4101672960\right] \\
&\quad - 48\,\alpha k^{17} t^3 \delta^{11}(\delta^2 - 1)\left[-181112832\;k^{-2} t\,\delta^{-9} + 49766400\;\delta^5 - 232243200\;\delta^3 + 447068160\right] \\
&\quad - 48\,\alpha k^{17} t^3 \delta(\delta^2 - 1)\left[-455731200\;\delta^8 + 261722112\;\delta^6 - 82489344\;\delta^4 + 12498944\;\delta^2 - 591872\right] \\
&\quad - 48\,\alpha k^{12} t^2(\delta^2 - 1)\left[(35640\;\delta^{10} - 106920\,\delta^8 + 119448\;\delta^6 - 59880\;\delta^4 + 12288) + k^{-5} t^{-1}(9\delta^5 + \sigma)\right] \\
&\quad + 1920\,\gamma k^{17} t^3 \delta^9(\delta^2 - 1)\left[13063680\;\delta^6 - 60963840\;\delta^4 + 116702208\;\delta^2 - 1173657600\right] \\
&\quad + 192\,\gamma k^{17} t^3 \delta(\delta^2 - 1)\left[(657331200\;\delta^6 - 198481920\;\delta^4 + 27867136\;\delta^2 - 1079296) + 142560\;k^{-12} t^{-2} \delta^9\right] \\
&\quad + 192\,\gamma k^{12} t^2 \delta^2(\delta^2 - 1)\left[(-427680\,\delta^6 + 474768\;\delta^4 - 234480\;\delta^2 + 46848) - 1984\;k^{-7} t^{-1} \delta^{-2} + 9\delta^5 + \sigma\right] \\
&\quad + 9\,0\beta k^{17} t^3 \delta^7(\delta^2 - 1)^2\left[20321280\;\delta^6 - 74511360\;\delta^4 + 107025408\;\delta^2 - 75543552\right] \\
&\quad + 9\beta k^{17} t^3 \delta(\delta^2 - 1)^2\left[(267423744\;\delta^4 - 42491904\;\delta^2 + 2158592) + k^{-12} t^{-2} \delta^5(83160\;\delta^2 - 166320)\right] \\
&\quad + 9\beta k^7 t(\delta^2 - 1)^2\left[101808\;k^5 t\,\delta^4 - 18672\;k^5 t\,\delta^2 + 408\;k^5 t + 9\delta^3 - 3\delta\right] \\
&\;\vdots
\end{aligned}
$$

$\textit{where};\ \delta = \tanh\left(k(x - x_0)\right),\ \alpha = 30,\ \beta = 30,\ \gamma = 10,\ \sigma = 4\delta(1 - 3\delta^2)$

Tables (3-4) show the absolute error of the present study compared with ADM and HPM [5] for problem (2) at different time and space. Figure (2) illustrates the plot of exact and approximate solution at $t = 15, k = 0.01\, and\, x_0 = 0$ and the approximate solution at different value of time ($t = 1,5,10,20$). The tables of error explained the comparison between the present study and ADM and HPM [5] ,and the figures show the efficiency and the accuracy of the new method SDHPM. In addition, through the figures, we note that there is agreement between the exact and analytical approximate solution and the approximate solution stay converge to the exact solution at large time. Then we can say that the new method is effect method with good convergence and high accuracy comparing with standard ADM and HPM [5].

Table.3: Absolute error $|u_1 - u_{Exact}|$ comparison between present study ,HPM,and ADM for p2

x	Method $t \Rightarrow$	0.2	0.4	0.8	1.6	2	4	6	8	10
0.2	Present study	5.75 E-16	1.15 E-15	2.30 E-15	4.60 E-15	5.75 E-14	1.15 E-14	1.72 E-14	2.30 E-14	2.87 E-13
	HPM[5]	1.20 E-14	2.40 E-14	4.81 E-14	9.63 E-14	1.20 E-13	2.40 E-13	3.61 E-13	4.81 E-13	6.02 E-13
	ADM	2.57 E-14	5.14 E-14	1.02 E-13	2.05 E-13	2.57 E-13	5.14 E-13	7.71 E-13	1.02 E-12	1.28 E-12
0.4	Present study	1.15 E-15	2.30 E-15	4.60 E-15	9.21 E-15	1.15 E-14	2.30 E-14	3.45 E-14	4.60 E-14	5.75 E-13
	HPM[5]	2.40 E-14	4.81 E-14	9.63 E-14	1.92 E-13	2.40 E-13	4.81 E-13	7.22 E-13	9.63 E-13	1.20 E-12
	ADM	5.14 E-14	1.02 E-13	2.05 E-13	4.11 E-13	5.14 E-13	1.02 E-12	1.54 E-12	2.05 E-12	2.57 E-12
1	Present study	2.87 E-15	5.75 E-15	1.15 E-14	2.30 E-14	2.87 E-14	5.75 E-13	8.63 E-13	1.15 E-13	1.43 E-13
	HPM[5]	6.02 E-14	1.20 E-13	2.40 E-13	3.01 E-13	6.02 E-13	1.20 E-12	1.80 E-12	2.40 E-12	3.01 E-12
	ADM	1.28 E-13	2.57 E-13	5.14 E-13	1.02 E-12	1.28 E-12	2.57 E-12	3.85 E-12	5.14 E-12	6.43 E-12

Table.4: Absolute error $|u_2 - u_{Exact}|$ comparison between present study ,HPM,and ADM for p2

x	Method $t \Rightarrow$	0.2	0.4	0.8	1.6	2	4	6	8	10
0.2	Present study	7.54 E-17	1.50 E-16	3.01 E-16	6.02 E-16	7.52 E-16	1.50 E-15	2.24 E-15	2.98 E-15	3.72 E-15
	HPM[5]	1.18 E-14	2.36 E-14	4.72 E-14	9.44 E-14	2.68 E-14	5.37 E-14	8.06 E-14	1.07 E-13	1.34 E-13
	ADM	6.71 E-15	1.34 E-14	2.68 E-14	5.37 E-14	6.71 E-14	1.34 E-13	2.01 E-13	2.68 E-13	3.35 E-13
0.4	Present study	1.48 E-16	2.96 E-16	5.93 E-16	1.18 E-15	1.48 E-15	2.96 E-15	4.43 E-15	5.90 E-15	7.36 E-15
	HPM[5]	2.36 E-14	4.72 E-14	9.44 E-14	1.88 E-13	5.37 E-14	1.07 E-13	1.61 E-13	2.15 E-13	2.68 E-13
	ADM	1.34 E-14	2.68 E-14	5.37 E-14	1.07 E-13	1.34 E-13	2.68 E-13	4.03 E-13	5.37 E-13	6.71 E-13
1	Present study	3.52 E-16	7.04 E-16	1.40 E-15	2.81 E-15	3.52 E-15	7.03 E-15	1.05 E-14	1.40 E-14	1.75 E-14
	HPM[5]	5.90 E-14	1.18 E-13	2.36 E-13	4.72 E-13	1.34 E-13	2.68 E-13	4.03 E-13	5.37 E-13	6.72 E-13
	ADM	3.35 E-14	6.71 E-14	1.34 E-13	2.68 E-13	3.35 E-13	6.71 E-13	1.01 E-12	1.34 E-12	1.67 E-12

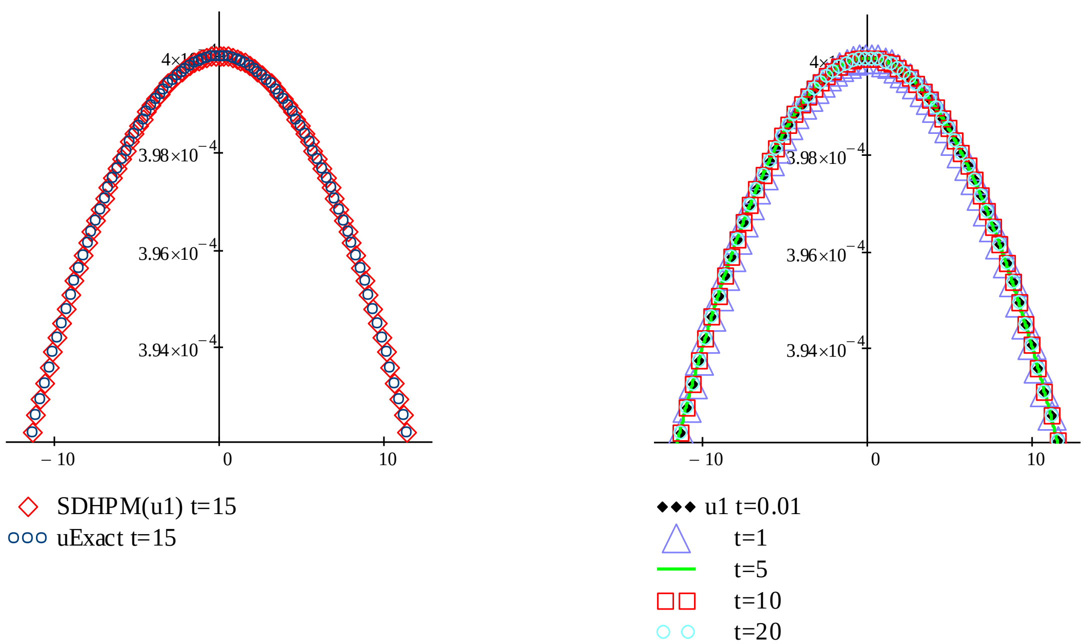

Figure (2)**:** Exact and approximate solution at $t = 15$, $k = 0.01$ *and* $x_0 = 0$ and the approximate solution at different value of time ($t = 0.01, 1, 5, 10, 20$) ,for Lax problem.

4-Convergence analysis of SDHPM:

In this section, we will study the analysis of convergence in the same manner as [20,21,22] for the decomposition method to the nonlinear fifth-order KdV Equation (19). Let as consider the Hilbert space H which may be defined as $H = L^2(\Omega \times [0,T])$, the set of applications ; $u : \Omega \times [0,T] \rightarrow \Re$ with

$$\int_{\Omega\times[0,T]} u^2 d\Omega < +\infty \tag{27}$$

And scalar product and induced norm :

$$(u,v) = \int_{\Omega\times[0,T]} uv d\Omega \quad \text{and} \quad \|u\|^2 = (u,u) \tag{28}$$

where , $\Re$ is real numbers.

The Adomian decomposition method is convergent if the following conditions are satisfied ;

$(\mathrm{I}): (L_t(\Delta u), \Delta u) \geq k_1 \|\Delta u\|^2, k_1 > 0, \forall u, \hat{u} \in H$

(II): Whatever may be $M > 0$, there exist a constant $C(M) > 0$ such that for $u, \hat{u} \in H$ with $\|u\| \leq M, \|\hat{u}\| \leq M$, we have: $(L_t(\Delta u), w) \leq C(M,(b+c+d))\|\Delta u\|\|w\|$ for every $w \in H$.

Now , we will use the following theorem to satisfy the above conditions as [20,21].

Theorem 1: If (I) and (II) are satisfied , then ADM of Equation (21) is convergent.

Proof : It is easy to prove (I) and (II) as the same manner in[9,20,21] to obtain on the results: Then condition (I) holds with $k_1 = M(2a\delta_1 + b\delta_2 + c\delta_3 + d\delta_4)$, where $\delta_1, \delta_2, \delta_3$ and δ_4 are constants and the condition (II) is satisfied with $C(M,(b+c+d)) = Ma + (b+c+d)$. Hence the prove is complete. ■

Let us consider Equation (1) (after we apply the HPM) in the following form:

$$L(v) = L(u_0) + p(g - N(v) - R(v) - L(u_0)) \tag{29}$$

Applying the inverse operator, L^{-1} to both sides of Equation (29) , we obtain

$$v = u_0 + p(L^{-1}g - L^{-1}N(v) - L^{-1}R(v) - u_0) \tag{30}$$

Suppose that

$$v = \sum_{i=0}^{\infty} p^i u_i \tag{31}$$

Substituting (31) into the right-hand side of Equation (30), yields

$$v = u_0 + p(L^{-1}g - (L^{-1}N)\sum_{i=0}^{\infty} p^i u_i - (L^{-1}R)\sum_{i=0}^{\infty} p^i u_i - u_0) \tag{32}$$

if $p \to 1$, the exact solution may be obtained by using Equation (14) as;

$$u = L^{-1}(g) - \left[\sum_{i=0}^{\infty}(L^{-1}N)(u_i)\right] - \left[\sum_{i=0}^{\infty}(L^{-1}R)(u_i)\right]$$

To study the convergence of this method , let us state the following theorem.

Theorem 2: (Sufficient Condition of Convergence)[19]

Supposes that X and Y are Banach spaces and $N: X \to Y$ is a contractive nonlinear mapping , that is $\forall w, w^* \in X; \|N(w) - N(w^*)\| \leq \gamma \|w - w^*\| \quad ,0 < \gamma < 1.$

Then according to Banach's fixed point theorem N has a unique fixed point u , that is $N(u) = u$.

Assume that the sequence generated by homotopy perturbation method can be written as;

$$W_n = N(W_{n-1}), \qquad W_{n-1} = \sum_{i=0}^{n-1} w_i, \qquad n = 1,2,3,...$$

and suppose that: $W_0 = w_0 \in B_r(w)$ where $B_r(w) = \{w^* \in X \mid \|w^* - w\| < r\}$ (33)

Then we have : $(i) \quad W_n \in B_r(w),.$ $\qquad (ii) \quad \lim_{n\to\infty} W_n = w.$

Proof : We can see the proof in [19].

Depending on the above theorems and their proofs, the converge of SDHPM(sufficient condition of convergence) is to be hold. Also**,** the combination of the two theorems gives us guarantee for convergence of the solutions that are obtained by SDHPM.

We illustrate the convergence of Splitting Adomian decomposition homotopy perturbation method theoretically by applying the sufficient condition of convergence. According to the theorems of convergence, the convergence of splitting Adomian decomposition homotopy perturbation method for the non-linear fifth-order KdV equation (19) and (30-33) will be illustrated as follows respectively. By using definitions (27) and(28) and supposing that

$$W_n = \mathrm{N}(W_{n-1}), \qquad W_{n-1} = u_{n-1}$$

$$u_n = \sum_{j=0}^{n}\int_0^t \left(-d\,\frac{\partial^5 u_j}{\partial x^5} - \sum_{k=0}^{j}\left(a\,u^2{}_k\left(\frac{\partial u_{j-k}}{\partial x}\right) + b\,\frac{\partial u_k}{\partial x}\left(\frac{\partial^2 u_{k-j}}{\partial x^2}\right) + c\,u_k\left(\frac{\partial^2 u_{k-j}}{\partial x^2}\right)\right)\right)dt, \qquad n = 1,2,3,\ldots$$

with the theorem (2) (Sufficient Condition of Convergence) for the nonlinear mapping N, a sufficient condition for convergence of the SDHPM is the strict contraction of N, we have :

$$\|u_0 - u\| = \left\|2k^2 \operatorname{sec}h(k(x-x_0))^2 - 2k^2 \operatorname{sec}h\left(k(x-16k^4t-x_0)\right)^2\right\|$$

$$\|u_1 - u\| \le \|u_0 - u\|\gamma \quad ,\gamma = 0.0000120 < 1$$

$$\|u_2 - u\| \le \|u_0 - u\|\gamma^2 \quad ,\gamma^2 = 0.00000461 < 1$$

$$\vdots$$

$$\|u_n - u\| \le \|u_0 - u\|\gamma^n$$

Therefore, $\lim_{n\to\infty}\|u_n - u\| \le \lim_{n\to\infty}\|u_0 - u\|\gamma^n = 0.$ be hold for the problem 1. Also, for the problem 2,we have:

$$\|u_0 - u\| = \left\|2k^2\left(2 - 3\tanh\left(k(x-x_0)\right)^2\right) - 2k^2\left(2 - 3\tanh\left(k(x-56k^4t-x_0)\right)^2\right)\right\|$$

$$\|u_1 - u\| \le \|u_0 - u\|\gamma \quad ,\gamma = 0.0999 < 1$$

$$\|u_2 - u\| \le \|u_0 - u\|\gamma^2 \quad ,\gamma^2 = 0.0183 < 1$$

$$\vdots$$

$$\|u_n - u\| \le \|u_0 - u\|\gamma^n$$

Therefore, $\lim_{n\to\infty}\|u_n - u\| \le \lim_{n\to\infty}\|u_0 - u\|\gamma^n = 0.$ be hold for the problem 2.

5-Conclusions

In this paper splitting decomposition homotopy perturbation method, which we proposed in the first time in [14] used to solve fifth-order KdV problem successfully. The results which we obtain by using Mathcad.15 for solving two types of fifth-order KdV problems; show that the SDHPM its efficient method with good converge and high accuracy to find analytical approximate solutions of these two problems. In addition, the absolute errors for the velocity explained the high accuracy of the present study. We conclude that the SDHPM is efficient method with good converge and high accuracy to find analytic approximate solutions of fifth-order KdV equation compare with standard ADM and HPM [5].

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