Quantum Analysis of Degenerate Three-Level Laser

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Abstract
We have studied the squeezing and statistical properties of the light produced by degenerate three level lasers. In order to carry out the analysis, we have obtained the c-number Langevin equations associated with the normal order using the pertinent master equations. Applying the Q-functions for degenerate three-level laser, we have determined the mean and variance of the photon number as well as the photon number distribution for degenerate three-level laser. It has been found that the mean photon number and quadrature squeezing increase with linear gain coefficient.
Keywords: Degenerate, Laser light, Squeezing, Quantum Noise, Photon

1. Introduction

A three-level laser is a quantum optical system in which light is generated by three level atoms inside a cavity usually coupled to a vacuum reservoir. When a three-level atom in cascade configuration makes a transition from the upper to the intermediate level and then from the intermediate to the bottom level, two photons are emitted. If the two photons have the same frequency, the three-level atom is said to be a degenerate three-level atoms; otherwise it is called a non-degenerate three-level atoms [1]. Squeezing is one of the non-classical features of light that has been studied by several authors [1]-[14]. In squeezed light the fluctuations in one quadrature is below the vacuum level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. Squeezed light has potential applications in low-noise communication and weak signal detection [1].

The squeezing and statistical properties of the light generated by three-level laser have been investigated by several authors [1], [7]-[14]. This study shows that three level lasers can generate a squeezed light under certain conditions. Sintayehu [7] has a detailed analysis of the squeezing properties of the light produced by the degenerate three-level cascade laser coupled to a vacuum reservoir via one of the coupler mirrors and an external resonant coherent radiation in the other. He has employed the stochastic differential equation associated with the normal ordering. The cavity radiation exhibits 98.3% squeezing under certain conditions pertaining to the initial preparation of the superposition and strength of the coherent radiation. Moreover, Eyob [8] has studied the quantum statistical properties of the light produced by a degenerate three-level cascade laser with the cavity mode driven by coherent light. He has seen that better squeezing is obtained when slightly more atoms are initially in the lower level. In addition, the coherent light has no effect on the half width of the three-level cascade laser.

In this paper we seek to study the squeezing and statistical properties of the light produced by a degenerate three-level laser whose cavity is coupled to a vacuum reservoir. We consider a degenerate three-level laser into which degenerate three-level atoms in a cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected into the cavity at a constant rate r_1 and removed from the cavity after certain time τ , as shown in Fig. 1.1. We obtain the c-number Langevin equation associated with the normal ordering for the cavity mode variable. Employing the solution of the resulting

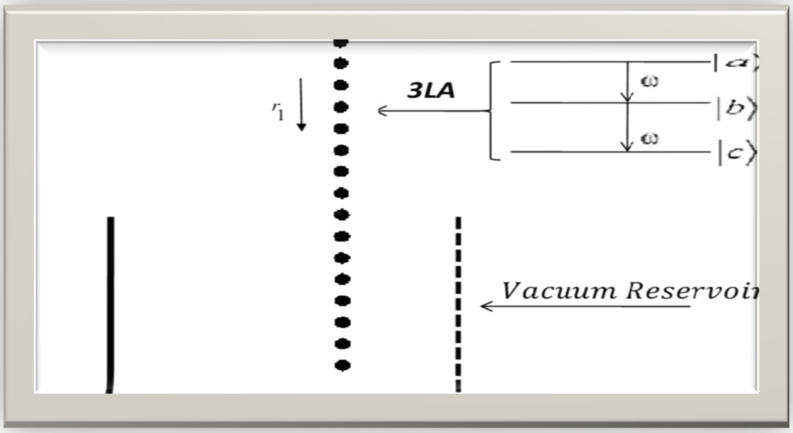

Fig. 1.1: Schematic representation of a degenerate three-level laser coupled to a vacuum reservoir.

C-number Langevin equation, we calculate the quadrature variance. In addition, we determine the mean and the variance of the photon number as well as the photon number distribution for the cavity mode employing the Q-function. The Q-function is obtained with the aid of the antinormally ordered characteristic function defined in the Heisenbeg picture.

2. C-number Langevin Equation

Now we seek to determine the c-number Langevin equation associated with the normal ordering for the cavity mode variables applying the pertinent master equation. The master equation for degenerate three-level laser coupled to vacuum reservoir is found in the linear and large time approximation schemes to be [1].

$$\frac{d}{dt}\hat{\rho}(t)=\frac{1}{2}A_1\rho_{aa}^{(0)}(2\hat{a}^{+}\hat{\rho}\hat{a}-\hat{\rho}\hat{a}\hat{a}^{+}-\hat{a}\hat{a}^{+}\hat{\rho})+\frac{1}{2}(k+A_1\rho_{ac}^{(0)})(2\hat{a}\hat{\rho}\hat{a}^{+}-\hat{\rho}\hat{a}^{+}\hat{a}-\hat{a}^{+}\hat{a}\hat{\rho})+\frac{1}{2}A_1\rho_{ac}^{(0)}(\hat{\rho}\hat{a}^{+2}+\hat{a}^{+2}\hat{\rho}-2\hat{a}^{+}\hat{\rho}\hat{a}^{+})+\frac{1}{2}A_1\rho_{ac}^{(0)}(\hat{\rho}\hat{a}^{2}+\hat{a}^{2}\hat{\rho}-2\hat{a}\hat{\rho}\hat{a})$$

(1). Where $A_1=\frac{2g^2r_1}{\gamma_1^2}$ is linear gain coefficient

Employing the relation

$\frac{d}{dt}<\hat{A}>=Tr(\frac{d}{dt}\hat{\rho}(t)A)$ along with Eq(1), we see that we readily obtain

$$\frac{d}{dt}<\hat{a}(t)>=-\frac{1}{2}\mu_1<a(t)> \qquad (2)$$

$$\frac{d}{dt}<\hat{a}^2(t)>=-\mu_1<a^2(t)>+A_1\rho_{ac}^{(0)} \qquad (3)$$

And

$$\frac{d}{dt}<\hat{a}^{+}(t)a(t)>=-\mu_1<\hat{a}^{+}(t)a(t)>+A_1\rho_{aa}^{(0)} \qquad (4)$$

$$\text{Where}\quad \mu_1=k+A_1(\rho_{ac}^{(0)}-\rho_{aa}^{(0)}) \qquad (5)$$

We see that Eqs. (2), (3) and (4) are in normal order. The C-number Equation corresponding to Equations (2), (3) and (4), associated with the normal ordering. On the basis of corresponding Eq (2), one can write:

$$\frac{d}{dt}\alpha(t)=-\frac{1}{2}\mu_1\alpha(t)+f(t) \qquad (6)$$

Where $f(t)$ is noise force the properties of which remain to be determined, one can readily establish the correlation properties of the noise force

$$<f(t)>=0 \qquad (7)$$

$$<f(t)f(t')>=A_1\rho_{ac}^{(0)}\delta(t-t') \qquad (8)$$

$$< f^*(t')f(t) >= A_1\rho_{aa}^{(0)}\delta(t-t') \tag{9}$$

We note that Eqs. (7), (8) and (9) represent the correlation properties of the noise force associated with the normal ordering.

3. Photon Statistics

In this section, we seek to study the statistical properties of light generated by degenerate three-level laser which can be described in terms of the mean photon number, the variance of photon number as well as the photon number distribution.

3.1 The Mean Photon Number

We now proceed to calculate the mean photon number for the cavity mode employing Q-function. But the Q-function for degenerate three- level laser is found to be [1]

$$Q(\alpha,\alpha^*,t) = \frac{\sqrt{u_1^2 - v_1^2}}{\pi}\exp[-u_1\alpha^*\alpha + v_1(\alpha^2 + \alpha^{*2})/2] \tag{10}$$

in which

$$u_1 = \frac{a_1}{a_1^2 - b_1^2}, v_1 = \frac{b_1}{a_1^2 - b_1^2} \tag{11}$$

$$a_1 = 1 + \frac{A_1(1-\eta)}{2(A_1\eta + k)}(1 - e^{-(A_1\eta+k)t} \tag{12}$$

$$b_1 = \frac{A_1\sqrt{1-\eta^2}}{2(A_1\eta + k)}(1 - e^{-(A_1\eta+k)t} \tag{13}$$

The mean photon number can be expressed in terms of Q-function as

$$\bar{n} = \int d^2\alpha Q(\alpha^*,\alpha,t)\alpha\alpha^* - 1 \tag{14}$$

Using Eq (10), the mean photon number for degenerate three level lasers at steady state is found to be

$$\bar{n}_{ss} = \frac{A_1(1-\eta)}{2(A_1\eta + k)} \tag{15}$$

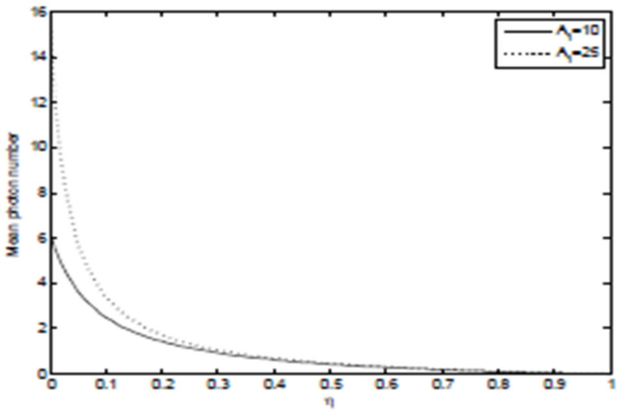

Fig. 2: Plots of the mean photon number [Eq. (15)] versus η for k = 0.8 and for different values of the linear gain coefficient.

The plots in Fig.2, represent that the mean photon number [Eq.15] versus η for k = 0.8, A_1 = 10(solid curve) and A_1 = 25(dotted curve). The figure indicates that the mean photon number decrease as η increase and A_1 decrease.

3.2 The Variance of Photon Number

The variance of the photon number can be expressed by

$$\Delta n^2 = < \hat{a}^2\hat{a}^{+2} > -3\bar{n} - \bar{n}^2 - 2 \tag{16}$$

Using Eq. [10], so that carrying out the integration and upon performing the differention, we easily get

$$< a^{+2} > = \frac{2u_1^{\ 2} + v_1^{\ 2}}{(u_1^{\ 2} - v_1^{\ 2})^2} \tag{17}$$

With the aid of Equations (11), (12) and (13), the variance of photon number at steady state

$$\Delta n_{ss}^{\ 2} = \bar{n}\left(1 + \frac{A_1}{A_{1\eta} + k}\right) \tag{18}$$

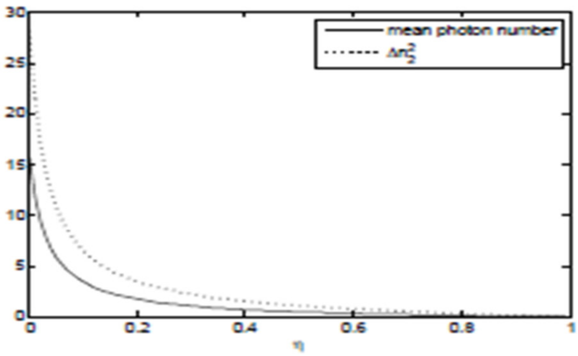

Fig. 3: Plots of the mean photon number [Eq. (18)] and the variance of photon number using [Eq.(2.48)] versus η for k = 0.8 and A_1 = 25.

The plots in Fig. 3, represent the mean photon number [Eq. (15)] versus η (dashed curve) and the variance of photon number [Eq. (48)] versus η (solid curve), for k =0.8 and A_1 = 25. We observe that the variance of photon number is greater than the mean of photon number. This shows that a degenerate three level laser has super-Poissonian photon statistics.

3.3 photon number distribution

The photon number distribution for a single-mode light is expressible in terms of the Q-function as

$$P(n,t) = \frac{\pi}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}} Q(\alpha^*,\alpha,t)e^{\alpha^*\alpha}\,|_{\alpha^*=\alpha=0} \tag{19}$$

Now on account of Eq. (10), we see that

$$P(n,t) = \frac{1}{n!}(u_1^{\ 2} - v_1^{\ 2})^{\frac{1}{2}} \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}} \exp\left[(1-u_1)\alpha^*\alpha + v_1(\alpha^2 + \alpha^{*2})/2\right]|_{\alpha^*=\alpha=0} \tag{20}$$

Using exponential function in power series, then upon differentiating with the help of the identity, we readily get

$$P(n,t) = \frac{1}{n!}(u_1^{\ 2} - v_1^{\ 2})^{\frac{1}{2}} \sum_{klm} \frac{(1-u_1)^k v_1^{\ l+m}(2m+k)!(2l+k)!}{2^{l+m}k!l!m!(2m+k-n)!(2l+k-n)!}\delta_{2l+k,n}\delta_{2m+k,n} \tag{21}$$

Applying the properties of the Kronecker delta and the fact that a factorial is defined for nonnegative integers, we finally get

$$P(n,t) = (u_1^{\ 2} - v_1^{\ 2})^{\frac{1}{2}} \sum_{l=0}^{[n]} \frac{n!(1-u_1)^{n-2l} v_1^{\ 2l}}{2^{2l}l!^2(n-2l)!} \tag{22}$$

where [n] = n/2 for even n and [n] = (n − 1)/2 for odd n.

	

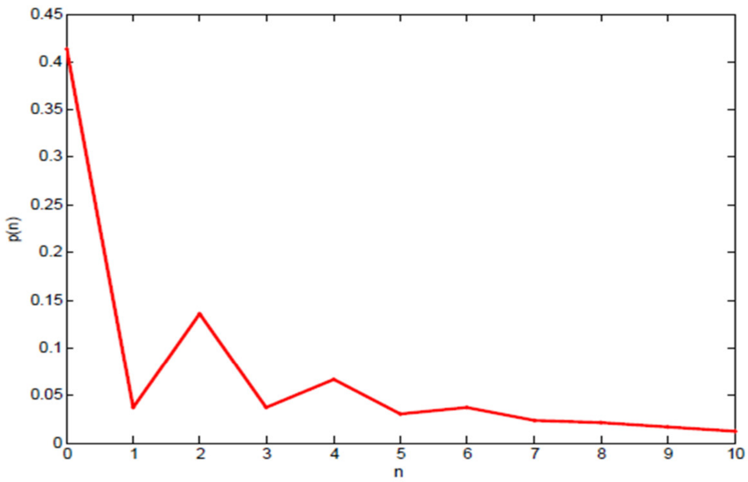

Fig. 4: Plot of the photon number distribution [Eq.(22)] versus photon number for A_1 = 100, k = 0.8 and η = 0.1.

Fig. 4 represents the photon number distribution versus the photon number for A_1 = 100, k = 0.8 and η= 0.1. The figure shows that the probability to observe n numbers of photon in the cavity decreases as n increases. There is a finite probability to observe odd number of photons in the cavity. This is due to the cavity damping.

4. Quadrature Variance

The quadrature variances can be defined by

$$\Delta a^2_{\pm} = < a^2_{\pm} > - < a_{\pm} >^2 \quad (23)$$

can be expressed in terms of c-number variables associated with the normal ordering as

$$\Delta a^2_{\pm} = 1 \pm < \alpha_{\pm}, \alpha_{\pm} > \quad (24)$$

In which

$$< \alpha_{\pm}, \alpha_{\pm} > = < \alpha^2_{\pm} > - < \alpha_{\pm} >^2 \quad (25)$$

$$\alpha_{\pm}(t) = \alpha^*(t) + \alpha(t) \quad (26)$$

Using the solution of Eq. (6) and its complex conjugate, we have

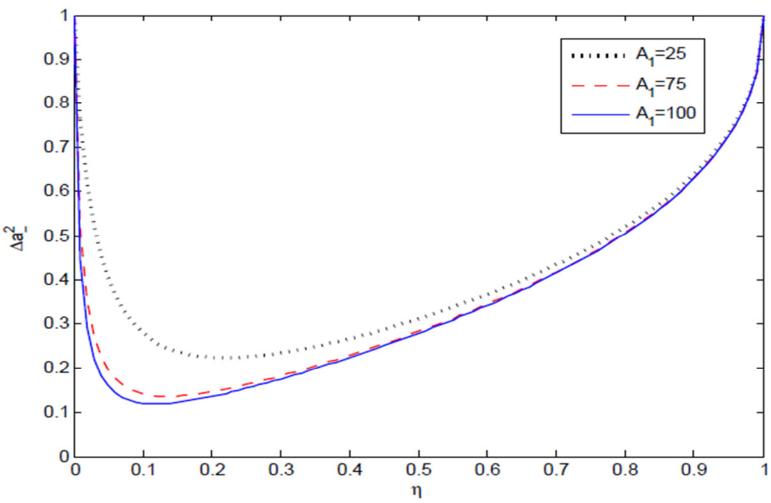

Fig.5: Plots of the quadrature variance Δa^2_- versus η [Eq. (30)] for k = 0.8 and for different values of the linear gain coefficient.

$$\alpha_{\pm}(t) = \alpha_{\pm}(0) e^{-\mu_1 t/2} + \int e^{-\mu_1 (t-t')/2} (f_2^*(t') \pm f_2(t')) dt' \quad (27)$$

Assuming that the cavity mode to be initially a vacuum state, one can easily verify that

$$< \alpha_{\pm}(t) >= 0 \quad (28)$$

$$< \alpha_{\pm}^{2}(t) >= \frac{2A_1[\rho_{ac}^{(0)} \pm \rho_{aa}^{(0)}]}{\mu_1}(1 - e^{-\mu_1 t}) \quad (29)$$

Finally, the quadrature variance of the cavity mode at steady state turns out to be

$$\Delta a^{2}_{\pm} = \frac{k + A_1[1 \pm \sqrt{1-\eta^2}\,]}{A_1\eta + k} \quad (30)$$

.

Fig. 5 represents the variance of the minus quadrature [Eq. (30)] versus η for k = 0.8 and A_1 =25 (dotted curve), A_1 = 75 (dashed curve), and A_1 = 100 (solid curve). The Figure indicates that the quadrature squeezing increase with the linear gain coefficient. Moreover, the maximum value of the quadrature squeezing described by Eq. (30) for A_1 = 100 and k = 0.8, is found to be 88.1% and occurs at η=0.12 below the coherent state level.

5. Conclusion

In this paper we have studied the squeezing and statistical properties of the light produced by degenerate three-level laser. In order to carry out the analysis, we have obtained the C-number Langevin equations associated with the normal order using the pertinent master equations. Employing the Q-function, we have calculated the mean and the variance of the photon number as well as the photon number distribution for degenerate three-level laser. The results showed that the mean photon number increase with linear gain coefficient. The variance of photon number is greater than the mean of photon number and the radiation has super-poissionian photon statistics.
We have also observed that there is a finite probability to observing odd number of photons. This is due to the cavity damping. Degenerate three- level laser produces a light in squeezed state and the squeezing occurs in minus quadrature. Furthermore, for A_1 = 100, k = 0.8, and η = 0.12, the maximum squeezing is found to be 88.1% below the coherent state level.

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