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Quantum Entanglement of Twin Light Beams

Assaye Gedifew Yismaw
Physics Department, Injibara University, PO box: 040 Injibara, Ethiopia

Abstract
A detailed quantum entanglement of twin light beams with the same and different frequencies produced by the process of parametric oscillation has been presented. We have described the process of the parametric oscillation with first-order Hamiltonian regardless of whether the twin light beams have the same or different frequencies. According to our study we have the quadrature variance decreases with the amplitude of pump mode γ and it approaches to the value of the quadrature variance of a two-mode vacuum state as cavity damping constant k approaches to 1. In addition to this the degree of entanglement of the cavity modes for the system under consideration increase with the amplitude of the pump mode γ and decrease with the cavity damping constant, which implies that the degree of entanglement of twin light beams decreases as quadrature variances approaches to that of a two-mode vacuum state.
Keywords: Quadrature Variance**,** Entanglement

Introduction

Quantum entanglement also called quantum non-local connection, is a properties of a quantum mechanical state of a system of two or more objects in which the quantum states of the constituting objects are linked together so that one object can no longer be adequately described without full mention of its counterpart- even if the individual objects are spatially separated in a spacelike manner.

The main task of this paper is devoted to the quantum entanglement of the twin light beams with the same or different frequencies generated by a parametric down converter coupled to a vacuum reservoir via a single-port mirror which is represented by a first-order Hamiltonian.

Master Equation

The process of parametric oscillation [1] leading to the creation of twin light modes with the same or different frequencies can be described by the Hamiltonian

$$\hat{H} = i\gamma\left(\hat{b}^{\dagger} - \hat{b}\right) + i\varepsilon\left(\hat{b}^{\dagger}\hat{a}_1\hat{a}_2 - \hat{b}\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\right)$$

where $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators for the light modes emitted from the top and intermediate levels of the fundamental mode, $\hat{b}$ is the annihilation operator for the pump mode and part of the pump mode that emerges from non-linear crystal without being down-converted (residue mode), ε is the coupling constant, and γ is proportional to the amplitude of the coherent light deriving the pump mode. We assume that the operators $\hat{a}_1$ and $\hat{a}_2$ commute and satisfy the commutation relation

$$\left[\hat{a}_1, \hat{a}_1^{\dagger}\right] = \left[\hat{a}_2, \hat{a}_2^{\dagger}\right] = 1$$

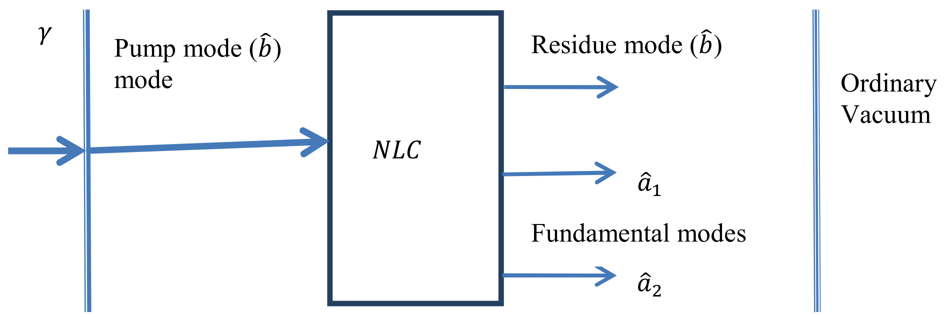

Fig.1: Plot of the external pumping radiation of frequency (ω) is down converted to the fundamental (signal-signal or signal-idler) mode of frequencies (ω_s, ω_i) by the nonlinear crystal.

We may refer to a Hamiltonian of the form described by Eq. (1) as first order Hamiltonian. We next seek to calculate the master equation and operator dynamics for the twin light modes by applying the pertinent Hamiltonian described by Eq. (1). The master equation for the system under consideration turns out to be

$$\frac{d\hat{\rho}}{dt} = \gamma\left(\hat{b}^{\dagger}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger} + \hat{\rho}\hat{b} - \hat{b}\hat{\rho}\right) + \varepsilon\left(\hat{b}^{\dagger}\hat{a}_1\hat{a}_2\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{a}_1\hat{a}_2 + \hat{\rho}\hat{b}\hat{a}_1^{\dagger}\hat{a}_2^{\dagger} - \hat{b}\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\hat{\rho}\right)$$
$$+\frac{k_1}{2}\left(2\hat{a}_1\hat{\rho}\hat{a}_1^{\dagger} - \hat{a}_1^{\dagger}\hat{a}_1\hat{\rho} - \hat{\rho}\hat{a}_1^{\dagger}\hat{a}_1\right) + \frac{k_2}{2}\left(2\hat{a}_2\hat{\rho}\hat{a}_2^{\dagger} - \hat{a}_2^{\dagger}\hat{a}_2\hat{\rho} - \hat{\rho}\hat{a}_2^{\dagger}\hat{a}_2\right),$$

where, k_1 and k_2 are the cavity damping constant of the signal-idler mode.

Operator Dynamics

At issue here is to find, the equation of evolution of the expectation values of the cavity mode operators, and their steady state solutions. Using the relation

$$\frac{d\langle\hat{A}\rangle}{dt} = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right) \tag{4}$$

On account of the properties of the trace operator, and Eq.(2) along with the fact that the operators $\hat{a}_1$ and $\hat{a}_2$ commute, we find

$$\frac{d\langle\hat{a}_1\rangle}{dt} = -\frac{k_1}{2}\langle\hat{a}_1\rangle - \varepsilon\langle\hat{b}\hat{a}_2^\dagger\rangle \tag{5}$$

$$\frac{d\langle\hat{a}_2\rangle}{dt} = -\frac{k_2}{2}\langle\hat{a}_2\rangle - \varepsilon\langle\hat{b}\hat{a}_1^\dagger\rangle \tag{6}$$

$$\frac{d\langle\hat{a}_1^\dagger\hat{a}_1\rangle}{dt} = -k_1\langle\hat{a}_1^\dagger\hat{a}_1\rangle - \varepsilon\langle\hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle - \varepsilon\langle\hat{b}^\dagger\hat{a}_1\hat{a}_2\rangle \tag{7}$$

$$\frac{d\langle\hat{a}_2^\dagger\hat{a}_2\rangle}{dt} = -k_2\langle\hat{a}_2^\dagger\hat{a}_2\rangle - \varepsilon\langle\hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle - \varepsilon\langle\hat{b}^\dagger\hat{a}_1\hat{a}_2\rangle \tag{8}$$

$$\frac{d\langle\hat{a}_1\hat{a}_2\rangle}{dt} = -\frac{1}{2}(k_1+k_2)\langle\hat{a}_1\hat{a}_2\rangle - \varepsilon\langle\hat{b}\hat{a}_1^\dagger\hat{a}_1\rangle - \varepsilon\langle\hat{b}\hat{a}_2^\dagger\hat{a}_2\rangle - \varepsilon\langle\hat{b}\rangle. \tag{9}$$

On taking $k_1 = k_2 = k$, the steady-state solutions of the above equations become

$$\langle\hat{a}_1\rangle = -\frac{2\varepsilon}{k}\langle\hat{b}\hat{a}_2^\dagger\rangle, \tag{10}$$

$$\langle\hat{a}_2\rangle = -\frac{2\varepsilon}{k}\langle\hat{b}\hat{a}_1^\dagger\rangle, \tag{11}$$

$$\langle\hat{a}_1^\dagger\hat{a}_1\rangle = -\frac{\varepsilon}{k}\langle\hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle - \frac{\varepsilon}{k}\langle\hat{b}^\dagger\hat{a}_1\hat{a}_2\rangle, \tag{12}$$

$$\langle\hat{a}_2^\dagger\hat{a}_2\rangle = -\frac{\varepsilon}{k}\langle\hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle - \frac{\varepsilon}{k}\langle\hat{b}^\dagger\hat{a}_1\hat{a}_2\rangle, \tag{13}$$

$$\langle\hat{a}_1\hat{a}_2\rangle = -\frac{\varepsilon}{k}\langle\hat{b}\hat{a}_1^\dagger\hat{a}_1\rangle - \frac{\varepsilon}{k}\langle\hat{b}\hat{a}_2^\dagger\hat{a}_2\rangle - \frac{\varepsilon}{k}\langle\hat{b}\rangle. \tag{14}$$

On the other hand, the evolution of the expectation value of the pump mode $\hat{b}$ is given by

$$\frac{d\langle\hat{b}\rangle}{dt} = -iTr\left([\hat{H},\hat{\rho}]\hat{b}\right) + \frac{k}{2}Tr\{(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b})\hat{b}\}, \tag{15}$$

in the absence of parametric oscillation ($\varepsilon = 0$), the $Eq.(1)$ becomes

$$\hat{H} = i\gamma\left(\hat{b}^\dagger - \hat{b}\right), \tag{16}$$

Upon dropping the noise operator, we can write the quantum langevin equation as

$$\frac{d\langle\hat{b}\rangle}{dt} = -\frac{1}{2}k\hat{b} + \gamma, \tag{17}$$

where k is the cavity damping constant. The steady-state solution of this equation is

$$\hat{b} = \frac{2\gamma}{k}. \tag{18}$$

Now substituting the value of $\hat{b}$ in Eqs. $(10), (11), (12), (13)$ and (14), we get

$$\langle\hat{a}_1\rangle = -\frac{2\lambda}{k}\langle\hat{a}_2^\dagger\rangle, \tag{19}$$

$$\langle \hat{a}_2 \rangle = -\frac{2\lambda}{k}\langle \hat{a}_1^\dagger \rangle, \tag{20}$$

$$\langle \hat{a}_1^\dagger \hat{a}_1 \rangle = -\frac{\lambda}{k}\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle - \frac{\lambda}{k}\langle \hat{a}_1 \hat{a}_2 \rangle, \tag{21}$$

$$\langle \hat{a}_2^\dagger \hat{a}_2 \rangle = -\frac{\lambda}{k}\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle - \frac{\lambda}{k}\langle \hat{a}_1 \hat{a}_2 \rangle, \tag{22}$$

$$\langle \hat{a}_1 \hat{a}_2 \rangle = -\frac{\lambda}{k}\langle \hat{a}_1^\dagger \hat{a}_1 \rangle - \frac{\lambda}{k}\langle \hat{a}_2^\dagger \hat{a}_2 \rangle - \frac{\lambda}{k}, \tag{23}$$

in which λ is determined by

$$\lambda = \frac{-2\gamma\varepsilon}{k}. \tag{24}$$

Applying Eq. (19) and (20), we easily find

$$\langle \hat{a}_1 \rangle = \langle \hat{a}_2 \rangle = 0. \tag{25}$$

In addition, by using Eqs. (21), (22) and (23) we get

$$\langle \hat{a}_1^\dagger \hat{a}_1 \rangle = \frac{2\lambda^2}{k^2 - 4\lambda^2}, \tag{26}$$

$$\langle \hat{a}_1^\dagger \hat{a}_1 \rangle = \langle \hat{a}_2^\dagger \hat{a}_2 \rangle, \tag{27}$$

$$\langle \hat{a}_1 \hat{a}_2 \rangle = -\frac{\lambda k}{k^2 - 4\lambda^2}. \tag{28}$$

It can also be readily asserted that

$$\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = \langle \hat{a}_1^\dagger \hat{a}_2 \rangle = 0. \tag{29}$$

We take

$$\hat{a} = \hat{a}_1 + \hat{a}_2, \tag{30}$$

to be the annihilation operator for superposition of light modes $\hat{a}_1$ and $\hat{a}_2$, produced by the parametric oscillator. We can easily assure that

$$[\hat{a}, \hat{a}^\dagger] = 2. \tag{31}$$

We actualize that the superposition of the two light modes, with the same or different frequencies, constitutes a two-mode light. We wish to call superposed light modes with the same frequency and the superposed light modes with different frequencies. It also proves to be convenient to the parametric oscillator which produces the same frequencies as the degenerate parametric oscillator and the one which produces the different as the non-degenerate parametric oscillator. Finally, we would like to mention that the result described by $Eqs. (25) - (29)$ are valid for the signal-signal or signal-idler modes.

Quadrature Variances

We have that the variance of the plus and minus quadrature for a two-mode light is given by

$$\left(\Delta \hat{a}_\pm^2\right) = \langle \hat{a}_\pm^2 \rangle - \langle \hat{a}_\pm \rangle^2, \tag{32}$$

where,

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \tag{33}$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}), \tag{34}$$

with $\hat{a}$ being the annihilation operator for the two-mode light. Now on account of $Eq. (25)$ along with Eqs. (27), (33), and, (34) we see that

$$\langle \hat{a}_{\pm} \rangle^2 = 0. \tag{35}$$

Hence in view of this, Eq. (32) is expressible as

$$\left(\Delta \hat{a}_{\pm}^2\right) = \langle \hat{a}_{\pm}^2 \rangle$$

$$= 2(1 + \langle \hat{a}^{\dagger}\hat{a} \rangle) \pm \langle \hat{a}^2 + \hat{a}^{\dagger 2} \rangle$$

$$= 2[1 + \langle (\hat{a}_1^{\dagger}\hat{a}_1 + \hat{a}_2^{\dagger}\hat{a}_2) \rangle \pm \langle (\hat{a}_1\hat{a}_2 + \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}) \rangle]. \tag{36}$$

Therefore, by realizing Eqs. (25), (26), (27), and (2.70) in above, the quadrature variance takes the form

$$\left(\Delta \hat{a}_{\pm}^2\right) = 2 \pm \frac{4\lambda}{k \mp 2\lambda}. \tag{37}$$

Eq.(37) shows that plus and minus quadratures of a two mode light. A two mode-light is said to be in a squeezed state if either $\Delta \hat{a}_+ < 2$ or $\Delta \hat{a}_-$ such that $\Delta \hat{a}_+ \Delta \hat{a}_- \geq 2$ [1]. According to Eq.(37) squeezing occurs in plus quadrature.

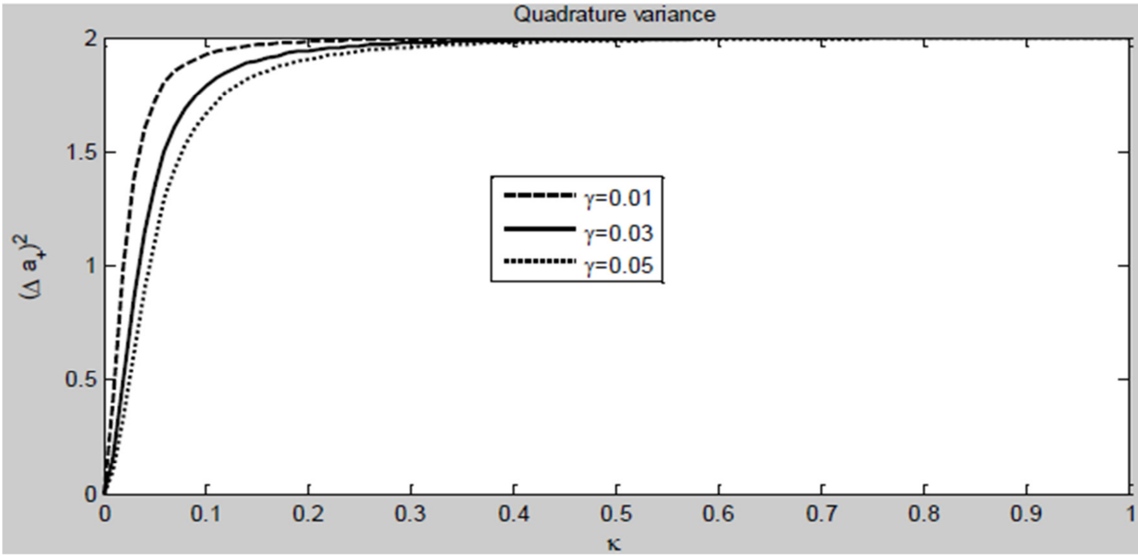

Figure 2: Plots of the quadrature variance [Eq.(37)] versus κ, for, ε = 0.01, and different values of γ, for γ = 0.01 (dashed curve), γ = 0.03 (solid curve), and γ = 0.05 (doted curve).

The plots in Fig.(2) indicate that the quadrature variance decrease with the amplitude of pump mode and it approaches the values of the quadrature variance of a two-mode vacuum state as κ approaches to 1.

We immediately observe from Eq.(37) that the quadrature variance of the signal-signal or signal-idler modes less than the quadrature variance of the two-mode vacuum reservoir. The plots in Fig.(2) indicate that the quadrature variance decrease with the amplitude of pump mode and it approaches the values of the quadrature variance of a two-mode vacuum state as κ approaches to 1.

Entanglement of the Cavity Mode

At issue here is to study the entanglement of the fundamental mode (signal-signal or signalidler), the fundamental-residue and the entanglement output modes. It is a well-established fact that a quantum system is said to be entangled if it is not separable. That is, if density operator for the combined state cannot be expressed as a product of the density operators of the individual constituents.

$$\hat{\rho} \neq \sum_j p_j \hat{\rho}_j^{(1)} \otimes \hat{\rho}_j^{(2)}, \tag{38}$$

where $p_j \geq 0$ and $\sum_j p_j = 1$, to ensure the normalization of the combined density operator.

On the other hand, entangled continuous variable state can be expressed as a co-eigenstate of a pair of EPR-type operators such as $\hat{x}_1 + \hat{x}_2$ and $\hat{p}_1 - \hat{p}_2$ [7]. The total variance of these two operators reduce to zero for maximally entangled continuous variable states. Nonetheless according to the criterion set by Duan et al [4] quantum states of the system are claimed to be entangled. From this point of view we need to find the entanglement of the cavity modes. A state of the system is entangled, if the sum of the quantum fluctuations of the two Einstein-Podosky-Rosen (EPR)- like operators[3] $\hat{u}$ and $\hat{v}$ of the two modes satisfies the inequality

$$\Delta u^2 + \Delta v^2 < 2, \tag{39}$$

With $\hat{x}_j = \frac{1}{\sqrt{2}}(\hat{a}_j + \hat{a}_j^\dagger)$, and $\hat{p}_j = \frac{i}{\sqrt{2}}(\hat{a}_j^\dagger - \hat{a}_j)$, (with j=1,2) are the quadratures for the two-modes of the cavity. But

$$(\Delta u^2) = \langle \hat{u}^2 \rangle - \langle \hat{u} \rangle^2$$

$$= \frac{1}{2}(\langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_1 \hat{a}_1^\dagger \rangle + \langle \hat{a}_2^\dagger \hat{a}_2 \rangle + \langle \hat{a}_2 \hat{a}_2^\dagger \rangle + \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle + \langle \hat{a}_1 \hat{a}_2 \rangle + \langle \hat{a}_2 \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_1^\dagger \rangle). \tag{40}$$

Following the same step and substituting, we can readily find

$$(\Delta v^2) = \langle \hat{v}^2 \rangle - \langle \hat{v} \rangle^2$$

$$= \frac{1}{2}(\langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_1 \hat{a}_1^\dagger \rangle + \langle \hat{a}_2^\dagger \hat{a}_2 \rangle + \langle \hat{a}_2 \hat{a}_2^\dagger \rangle + \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle + \langle \hat{a}_1 \hat{a}_2 \rangle + \langle \hat{a}_2 \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_1^\dagger \rangle). \tag{41}$$

On the basis of the boson commutation relation that are mentioned in Eq. (2), we can find that

$$\Delta u^2 + \Delta v^2 = 2(1 + \langle(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_1 \hat{a}_2)\rangle) \tag{42}$$

Taking Eq. (27) and Eq. (36) under consideration, we can simplify and arrive at

$$\Delta u^2 + \Delta v^2 = (\Delta a_+)^2 \tag{43}$$

which implies in the system under consideration, the entanglement and quadrature variance exactly equal. Now the various correlations involved in Eq. (42) would be determined by using the steady-state Eqs. (2.67), (2.68), (2.69) and (2.70), we reach

$$\Delta u^2 + \Delta v^2 = \frac{2k}{k+2\lambda} \tag{44}$$

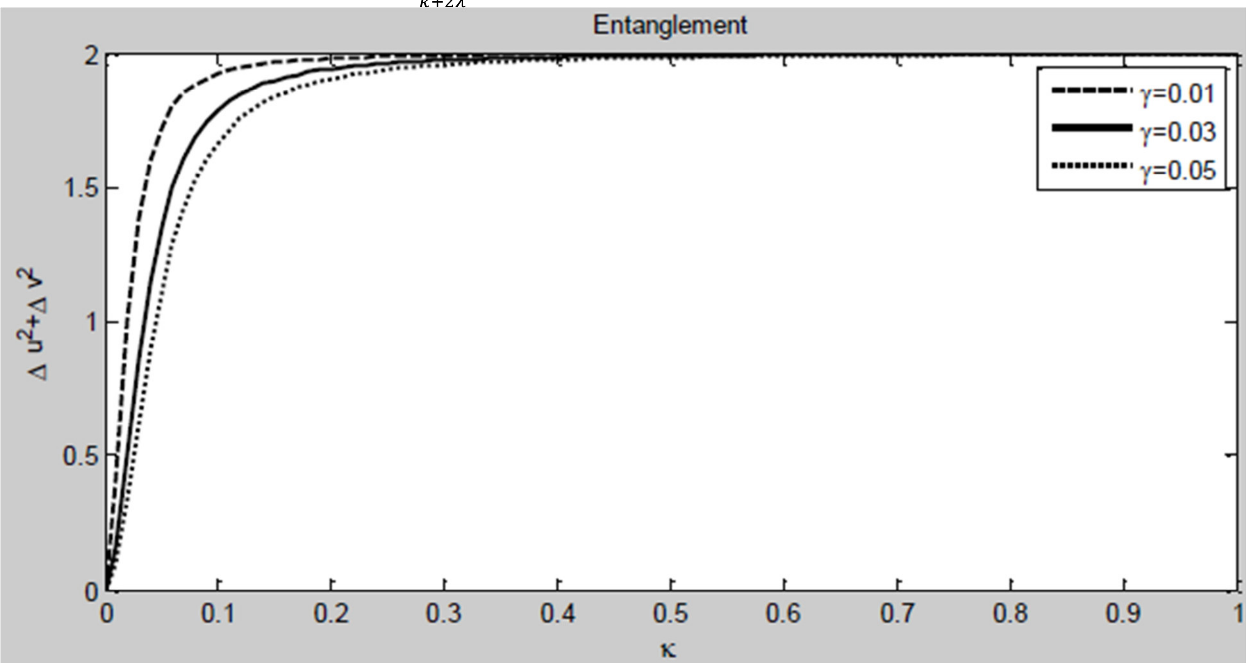

Figure 3: Plots of the sum of the variance cavity modes for EPR-like operators [Eq.(42)] at steady state versus κ, for ε = 0.01,and different value of γ, for γ = 0.01 (dashed curve), γ = 0.03 (solid curve), γ = 0.05 (dotted curve).

The plots in Fig(3) indicate that the degree of entanglement of the cavity modes for the system under consideration increase with the amplitude of the pump mode γ, and decrease with the cavity damping constant, which means that the lesser the cavity damping the more probable for the radiation to stay in the cavity which in turn strengths the correlation that leads to entangled. The entanglement criterion given by Eq.(39) is also satisfied.

Conclusion

In this paper we seen that the quadrature variance and the degree of the quantum entanglement of twin light beams generated by parametric oscillator whose coupled to vacuum reservoir by the aid of steady state solution. Applying the steady state solution we have calculated the quadrature variance and the entanglement of twin light beam. Our result shows that, we have the quadrature variance decreases with the amplitude of pump mode γ and it approaches to the value of the quadrature variance of a two-mode vacuum state as cavity damping constant κ approaches to 1. In addition to this the degree of entanglement of the cavity modes for the system under consideration increase with the amplitude of the pump mode γ and decrease with the cavity damping constant, which implies that the degree of entanglement of twin light beams decreases as quadrature variances approaches to that of a two-mode vacuum state. Eventually we conclude that the degree of the entanglement of twin light beams generated by parametric oscillator exactly equal to the quadrature variance of the cavity mode.

Reference

[1]. K. Fesseha, Fundamentals of Quantum Optics,(Lulu, USA, 2008).
[2]. K. Fesseha, Opt. Commun.**156**,145(1998) .
[3]. A. Einstein, B. Podolsky and R. Rosen, Phys. Rev. **47**, 777 (1935).
[4]. L. M. Duna, G. Giedke, J. I. Cirac and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).
[5]. Hilaery and Zubairy, Phys. Rev. Lett. **96**, 050503 (2006).
[6]. E. Schrodinger, "Discussion of Probability Relations Between Separated Systems,"*Proceedings of the Cambridge Philosophical Society* **31**, 555-563(1935): **32**,446-451(1936).
[7]. Su-Yong Lee, Shahid Qammar, Hai-Woong Lee and M. Suhail Zubairy, J. Phys. B: At. Mol. Opt.Phys.**41**, 145504(2008).