Interaction of Gravitational and Electromagnetic Fields

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Abstract
In the present paper some results are obtained for Interaction of Gravitational and Electromagnetic Fields.The obtained results are generalized form of well known results in the field of general theory of relativity and Tensor.

Keywords: Gravitational Field, Electromagnetic Field, Tensor

Introduction: The interaction between gravitational field and electromagnetic field by the use of dyadics. Thus for a non-null nonaligned electromagnetic field interacting with a Petrov type N shear free gravitational field we have simple expressions for the 3-vectors $\underline{j}$ and $\underline{k}$ and the 3-dyadics J and K. It is useful in Perfect fluid distribution, related general physics and also of hydrodynamics. In this paper to discussed the interaction between gravitational field and electromagnetic field by the use of dyadics. It is very useful results of general theory of relativity is obtained where we use that the gravitational field may be algebraically special and principal of null direction coincide on some condition. The gravitational and electromagnetic field are non-aligned. It is very useful in perfect fluid.

Main Results

1. The electromagnetic field:

The electromagnetic field is represented by skew-symmetric tensor $F_{\mu r} = F_{[\mu r]}$ satisfying Maxwell's equations.

$$F_{\mu r}^{+;v} = \left(F_{\mu v} + iF_{\mu v}^{*}\right)^{;v} = 0 \quad (1)$$

Where $F_{\mu v}^{*} = \frac{1}{2}\varepsilon_{\mu v}{}^{\sigma\gamma}F_{\sigma\gamma}$ is the dual of $F_{\mu v}$.

$$F_{\mu v} \leftrightarrow \ \varepsilon_{AB}\,\overline{\emptyset}_{\dot{x}\dot{y}} + \emptyset_{AB}\varepsilon_{\dot{x}\dot{y}} \quad (2)$$

The dyad components of $\emptyset_{AB}$ are the 3 complex quantities

$$\underline{\overline{\emptyset}}_0 = S_1^A S_1^B \emptyset_{AB}\,, \qquad \underline{\overline{\emptyset}}_1 = S_1^A S_2^B \emptyset_{AB}\,, \qquad \underline{\overline{\emptyset}}_2 = S_2^A S_2^B \emptyset_{AB}$$

Which correspond to the six real components of $F_{\mu v}$.

The symmetric 2-spinor $\emptyset_{AB}$ can be represented as $\emptyset_{AB} = \alpha_{(A}\beta_{B)}$

And hence $F_{\mu v}^{+}$ may be classified as:

(I) Non-null (general) when $\alpha_A \neq \alpha_B$

(II) Null when $\alpha_A = \alpha_B$ so that $\emptyset_{AB} = \alpha_A \alpha_B$

When the self-dual bi-vector $F_{\mu v}^{+}$ is null , there exists a null vector K^{μ}

Such that $F_{\mu v}^{+} K^{\mu} = 0$ (3)

And ${F_{\mu v}}^{+}$ may be written in the form

$$F_{\mu v}^{+} = 2K_{[\mu}\bar{t}_{v]} = V_{\mu v} \quad (4)$$

Where t^{μ} is a complex null vector satisfying

$$K_{\mu} t^{\mu} = 0\,,\ t^{\mu}\bar{t}_{\mu} = 1 \quad (5)$$

Equation (1) now implies

$$K_{\mu;v}\, K^{v}\bar{t}^{\mu} = 0\,,\ K_{\mu v}\bar{t}^{v}\bar{t}^{\mu} = 0 \quad (6)$$

Einstein’s field equations

$$T_{\mu v} = F_{\mu\sigma}F_{v}^{\sigma} - \frac{1}{4} g_{\mu v} F_{\sigma\gamma}F^{\sigma\gamma} = -R_{\mu v} \quad (7)$$

Give $R_{\mu v} = -\frac{1}{2} K_{\mu} K_{v}$ (8)

While Binachi identities take the form

$$C_{\lambda\mu v p}{}^{;p} = R_{v[\lambda;\mu]} = -\frac{1}{2}\{(K_v K_{[\lambda;\mu]}) + (K_{v;[\mu}K_v)\} \quad (9)$$

According to the theorem by Goldberg , Sachs , Kundt and Thompson

The gravitational field which is a pure gravitational radiation field in the case is algebraically special and its principal null direction l^{μ} conncides with K^{μ} , which from (6) is geodesic and sheer free .

In the non-null case, $F_{\mu v}^{+}$ has the form

$${F_{\mu v}}^{+} = 2A\left(p_{[\mu}q_{v]} + t_{[\mu}\bar{t}_{v]}\right) \quad (10)$$

Where p^{μ} and q^{μ} are the principal null directions of the electromag-netic field and $\{p^{\mu}, q^{\mu}, t^{\mu}, \bar{t}^{\mu}\}$ form a null tetrad . A is the complex field-strength .

Denoting the tetrad components of the optical vectors of the principal directions p^{μ} and q^{μ} by x_1 , σ_1 , p_1, γ_1 and x_2 , σ_2 , p_2 , γ_2 respectively .we may write Maxwell’s equation (1) as

$$\frac{1}{2}(\log A)_{,\mu} = -P_1 q_{\mu} - P_2 p_{\mu} + \gamma_1 \bar{t}_{\mu} + \gamma_2 t_{\mu} \quad (11)$$

Field equations (7) give

$$R_{\mu v} = |A|^2 \left(2p_{(\mu}q_{v)} - \frac{1}{2} g_{\mu v}\right) \quad (12)$$

Two cases may arise for fields:

(1) The gravitational field may be algebraically special and its principal null direct coincides with one of p^{μ} or q^{μ} .

Two fields are said to be aligned.

(2) Gravitational and electromagnetic fields are non-aligned i.e. the principal null vector l^{μ} of the gravitational field does not coincides with either p^{μ} or q^{μ} .

In this case we scale the null vectors so that

$$l_{\mu}p^{\mu} = -l_{\mu}q^{\mu} = -1 \qquad (13)$$

Rotating the space-like vectors $\left(t_{\mu}^{\rho} \rightarrow e^{i\theta}t_{\mu}\right)$ we may write ,

$$l_{\mu} = \frac{1}{\sqrt{2}}\left(p_{\mu} - q_{\mu} + t_{\mu} + \bar{t}_{\mu}\right) \qquad (14)$$

And complete the null tetrad of gravitational field by choosing

$$n_{\mu} = \frac{1}{2}\left(-p_{\mu} + q_{\mu} + t_{\mu} + \bar{t}_{\mu}\right) \qquad (15)$$

$$m_{\mu} = \frac{1}{2}\left(p_{\mu} + q_{\mu} + \bar{t}_{\mu} - t_{\mu}\right) \qquad (16)$$

2. The null field :

Using equation (9) we calculate the current components

$$J_{rst} = C_{\lambda\mu v p}{}^{;p} = r^{l^{\lambda}}s^{l^{\mu}}t^{l^{v}} = J_{[rs]t}$$

In this case we have $R_{\mu v} = -\frac{1}{2}l_{\mu}l_{v}$ so that $R = 0$

$$T_{\mu v} = R_{\mu v}{}' = -\frac{1}{4}(0^{l_{\mu}} + 1^{l_{\mu}})(0^{l_{v}} + 1^{l_{v}})$$

$$= -\frac{1}{4}[0^{l_{\mu}}0^{l_{v}} + 0^{l_{\mu}}1^{l_{v}} + 1^{l_{\mu}}0^{l_{v}} + 1^{l_{\mu}}1^{l_{v}}]$$

Strangling the components of Ricci tensor we have

$R_{00} = -\frac{1}{4}$, $R_{01} = -\frac{1}{4}$, $R_{02} = R_{03} = 0$, $R_{11} = -\frac{1}{4}$, R_{ij}=0

$\therefore\ T_{00} = \frac{1}{2}R_{00} = -\frac{1}{8}$, $t_{1} = T_{01} = \frac{1}{2}R_{01} = \frac{1}{8}$, $T_{11} = \frac{1}{2}R_{11} = -\frac{1}{8}$

i.e.

$P = -\frac{1}{8}$, $\underline{t} = \frac{1}{8}\underline{u}$, $T = -\frac{1}{8}\underline{u}\ \underline{u}$

Where $\underline{u} = 1\underline{l}$, $\underline{v} = 2\underline{l}$, $\underline{\omega} = 3\underline{l}$ are the unit vectors along the space-like congruence .

Hence for J_{rst} , we have

$$j = -\dot{t} + \frac{1}{3}\underline{\nabla}(t_{r}T + 2P) + \underline{a}[T + P_{1}] - \underline{\omega} \times \underline{t} + 2\underline{\Lambda} \times \underline{t} - 2s.\underline{t} \qquad (D_{1})$$

$j_{1} = \frac{1}{4}(a_{1} + S_{11})$,

$j_2 = \frac{1}{8}(a_2 + 2S_{12} - 2\Lambda^3 + \omega^3)$,

$j_3 = \frac{1}{8}(a_3 + 2S_{13} - 2\Lambda^2 - \omega^2)$

$\underline{k} = -\underline{\nabla} \cdot \underline{t} - \left(\underline{\Lambda} \times [T + P_1]\right)_\times + S_\times^{\cdot}[T + P_1]$ $\qquad (D_2)$

$K_1 = \frac{1}{8}(2\Lambda^1 - N_{11})$, $K_2 = \frac{1}{8}(S_{13} + 3\Lambda^2 - N_{12} - L_3)$,

$K_3 = -\frac{1}{8}(S_{12} - 3\Lambda^3 + N_{13} - L_2)$

$J = \underline{\nabla}\,\underline{t} - [\dot{T} + \dot{P}_1] + \frac{1}{3}[t_r\dot{T} + 2\dot{P}]1 + \underline{a}\,\underline{t} + \underline{t}\,\underline{a}$ $\qquad (D_3)$

$-\underline{\omega} \times [T + P_1] + [T + P_1] \times \underline{\omega} + \underline{\Lambda} \times [T + P_1] - S[T + P_1]$

$J_{11} = -\frac{1}{4}(a_1 + S_{11})$, $J_{12} = -\frac{1}{8}(a_2 + S_{12} + \Lambda^3 + \omega^3 - N_{13} + L_2)$,

$J_{13} = -\frac{1}{8}(a_3 + S_{13} - \Lambda^2 - \omega^2 + N_{12} + L_3)$,

$J_{21} = -\frac{1}{8}(a_2 + 2S_{12} - 2\Lambda^3 + \omega^3)$, $J_{22} = -\frac{1}{8}(S_{22} - N_{23} - L_1)$,

$J_{23} = -\frac{1}{8}\left[S_{23} + \Lambda^1 - \frac{1}{2}(N_{11} - N_{22} + N_{33})\right]$,

$J_{31} = -\frac{1}{8}(a_3 + 2S_{13} + 2\Lambda^2 - \omega^2)$, $J_{33} = -\frac{1}{8}(S_{33} + N_{23} - L_1)$,

$J_{32} = -\frac{1}{8}\left[S_{23} - \Lambda^1 + \frac{1}{2}(N_{11} + N_{22} - N_{33})\right]$,

$K = -\underline{\nabla} \times [T + P_1] + \frac{1}{3}\underline{\nabla}(t_rT + 2P) \times 1 - \underline{t} \times S + 3\underline{\Lambda}\,\underline{t} - \left(\underline{\Lambda} \cdot \underline{t}\right)$ $\qquad (D_4)$

$K_{11} = -\frac{1}{8}(2\Lambda^1 - N_{11})$, $K_{12} = K_{13} = 0$

$K_{21} = -\frac{1}{8}(S_{13} + 3\Lambda^2 N_{12} - L_3)$,

$K_{22} = -\frac{1}{8}\left[S_{23} - \Lambda^1 + \frac{1}{2}(N_{11} + N_{22} - N_{33})\right]$,

$K_{23} = -\frac{1}{8}\ (S_{33} + N_{23} - L_1)$,

$K_{31} = \frac{1}{8}\ (S_{12} - 3\Lambda^3 + N_{13} - L_2)$,

$K_{32} = \frac{1}{8}\ (S_{22} - N_{23} - L_1)$,

$K_{33} = \frac{1}{8}\ \left[S_{23} + \Lambda^1 - \frac{1}{2}(N_{11} - N_{22} + N_{33})\right]$

From the relations

$t_rJ = \underline{\nabla} \cdot \underline{t} - [\dot{P} + (t_rS)P] + 2\underline{a} \cdot \underline{t} - S{:}T = 0$ and

$\underline{\nabla} \cdot T - [\underline{\dot{t}} + \underline{\omega} \times \underline{t} + (t_rS)\underline{t}] = S \cdot \underline{t} + \underline{\Lambda} \times \underline{t} - \underline{a} \cdot [T + P_1]$, We have

$a_1 + S_{11} = -\frac{1}{2}(S_{22} + S_{33} - 2L_1)$; $\varepsilon + \bar{\varepsilon} = P + \bar{P}$

$-2L_1 + t_r S = -2a_1 - S_{11}$; $a_1 + S_{11} = -\frac{1}{2}(S_{22} + S_{33} - 2L_1)$

$a_2 + S_{12} + \Lambda^3 + \omega^3 - N_{13} + L_2 = 0$, $a_3 + S_{13} - \Lambda^2 - \omega^2 + N_{12} + L_3 = 0$

The last two relations give $x = 0$ and $J_{12} = J_{13} = 0$ substituting $F_{\mu\nu} + iF^*_{\mu\nu} = 2l_{[\mu}m_{\nu]}$ is Maxwell's equation , we have $\sigma = l_{\mu;\nu}m^\mu m^\nu = 0$

i.e. $S_{22} - S_{33} = 2N_{23}$, $N_{22} - N_{33} = -2S_{23}$

From these we have

$$K_2 = j_3 , \qquad K_3 = -j_2 ,$$

$$J_{22} = J_{33} = K_{23} = -K_{32} = \frac{1}{16}(S_{22} + S_{33} - 2L_1)$$

$$K_{21} = -j_3 = -K_2 , \qquad K_{31} = j_2 = -K_3 ; J_{21} = -j_2 , J_{31} = -j_3$$

$J_{23} = -\frac{1}{16}(2\Lambda^1 - N_{11}) = -\frac{1}{2}K_1 = \frac{1}{2}K_{11}$

$$J_{32} = \frac{1}{16}(2\Lambda^1 - N_{11}) = \frac{1}{2}K_1 = -\frac{1}{2}K_{11}$$

$$K_{22} = K_{33} = \frac{1}{16}(2\Lambda^1 - N_{11}) = \frac{1}{2}K_1 = -\frac{1}{2}K_{11}.$$

Hence setting

$$f = \frac{1}{4}(a_1 + S_{11}) \ , g = \frac{1}{8}(a_2 + 2S_{12} - 2\Lambda^3 + \omega^3)$$

$$h = \frac{1}{8}\{(a_3 + 2S_{13}) - 2\Lambda^2 - \omega^2\} , \qquad K = \frac{1}{8}(2\Lambda^1 - N_{11})$$

we have

$$\underline{j} = f\underline{u} + g\underline{v} + h\underline{\omega} \quad , \quad \underline{K} = K\underline{u} + h\underline{v} - g\underline{\omega}$$

$$J = -f\underline{u}\,\underline{u} - g\underline{v}\,\underline{v} + \frac{1}{2}f\underline{v}\,\underline{v} - \frac{1}{2}K\underline{v}\,\underline{\omega} - h\underline{\omega}\,\underline{u} + \frac{1}{2}K\underline{\omega}\,\underline{v} + \frac{1}{2}f\underline{\omega}\,\underline{\omega}$$

$$K = -K\underline{u}\,\underline{u} - h\underline{v}\,\underline{v} + \frac{1}{2}K\underline{v}\,\underline{v} + \frac{1}{2}f\underline{v}\,\underline{\omega} + g\underline{\omega u} - \frac{1}{2}f\underline{\omega}\,\underline{v} + \frac{1}{2}K\underline{\omega}\,\underline{\omega}$$

Where $\underline{u}, \underline{v}, \underline{\omega}$ are unit vectors along the space like auxiliary congruences.

If l^μ defines a geodesic null congruence , $\varepsilon + \bar{\varepsilon} = 0$.

Therefore , f=0

This implies $P + \bar{P} = 0$ i.e. e^μ is geodesic shear-free and divergenceless.

In this case, $\underline{j} = g\underline{v} + h\underline{\omega}$, $\underline{K} = K\underline{u} + h\underline{v} - g\underline{\omega}$, $\underline{j}.\underline{K} = 0$

$J = -g\underline{v}\,\underline{u} - \frac{1}{2}K(\underline{v}\,\underline{\omega} - \underline{\omega}\,\underline{v}) - h\underline{\omega}\,\underline{u}$; $K = \underline{u}\,\underline{u} - h\underline{vu} + \frac{1}{2}K\underline{v}\,\underline{v} + g\underline{\omega}\,\underline{u} + \frac{1}{2}K\underline{\omega}\,\underline{\omega}$. t

If the Weyl tensor is of Petrov type N, P = 0, so that

$$f = \frac{1}{4}(a_1 + S_{11}) = -\frac{1}{8}(S_{22} + S_{33} - 2L_1) = 0\,; K = \frac{1}{8}(2\Lambda^1 - N_{11}) = 0$$

$\therefore$ $\underline{j} = g\underline{v} + h\underline{\omega}$, $\underline{K} = h\underline{v} - g\underline{\omega}$ with $\underline{j}\cdot\underline{K} = 0$

$$J = -g\underline{v}\,\underline{u} - h\underline{\omega}\,\underline{u}\,, K = -h\underline{vu} + g\underline{\omega}\,\underline{u}$$

3. Non-null aligned field:

In this case one of the principal null vectors of the electromagnetic field coincides with the principal null direction of the gravitational field and the electromagnetic tetrad frame may be taken to be coincident with the null tetrad frame may be taken to be coincident with the null tetrad frame of the gravitational field. We have

$R_{\mu v} = 2\emptyset\left[p_{(\mu}q_{v)} - \frac{1}{4}g_{\mu v}\right] = \emptyset\left[l_\mu n_v + l_v n_\mu - \frac{1}{2}g_{\mu v}\right]$

Where $\emptyset = |A|^2$, A , the intensity of the electromagnetic field.

$$R = \emptyset g^{\mu v}\left[l_\mu n_v + l_v n_\mu - \frac{1}{2}g_{\mu v}\right] = 0$$

Hence the strangled components of $R_{\mu v}$ are

$$R_{00} = -\frac{1}{2}\emptyset\,, R_{0k} = 0\,, k = 1,2,3\,;\ R_{11} = \frac{1}{2}\emptyset = -R_{22} = -R_{33}, R_{jk} = 0\ j \neq k$$

For the strangled components of the energy-momentum tensor, we have

$$P = T_{00} = \frac{1}{2}R_{00} = -\frac{1}{4}\emptyset\,,\qquad \underline{t} = 0\,, T = \frac{1}{4}\,\emptyset(\underline{u}\,\underline{u} - \underline{v}\,\underline{v} - \underline{\omega}\,\underline{\omega})$$

Where, once again , $\underline{u}$, $\underline{v}$, $\underline{\omega}$ are the unit 3- vectors along the space – like auxiliary congruences.

Therefore, for the components of the 3-vectors $\underline{j}$ and $\underline{k}$ and 3-dydadics J , K , we have

$j_1 = \frac{1}{4}\emptyset_{,1}$, $j_2 = \frac{1}{4}\emptyset_{,2} + \frac{1}{2}\emptyset a_2$, $j_3 = \frac{1}{4}\emptyset_{,3} + \frac{1}{2}\emptyset a_3$

$$K_1 = \emptyset\Lambda^1\,,\qquad K_2 = \frac{1}{2}\emptyset(\Lambda^2 - S_{13})\,,\qquad K_3 = \frac{1}{2}(\Lambda^3 + S_{12})$$

$$J_{11} = \frac{1}{4}\emptyset\,,\qquad J_{12} = -\frac{1}{2}\emptyset(S_{12} + \Lambda^3 - \omega^3)\,,\qquad J_{13} = -\frac{1}{2}\emptyset(S_{13} - \Lambda^2 + \omega^2)$$

$$J_{21} = \frac{1}{2}\emptyset\omega^3\,,\qquad J_{22} = -\frac{1}{4}\emptyset - \frac{1}{2}\emptyset S_{22}\,,\qquad J_{23} = -\frac{1}{2}\emptyset(S_{23} + \Lambda^1)$$

$$J_{31} = -\frac{1}{2}\emptyset\omega^2\,,\qquad J_{32} = \frac{1}{2}\emptyset(S_{23} - \Lambda^1)\,,\qquad J_{33} = -\frac{1}{4}\emptyset - \frac{1}{2}\emptyset S_{33}$$

$$K_{11} = -\frac{1}{2}\emptyset N_{11}\,,\qquad K_{12} = \frac{1}{4}\,\emptyset{,3}\quad, K_{13} = -\frac{1}{4}\emptyset{,2}$$

$$K_{21} = \frac{1}{4}\emptyset,3 - \frac{1}{2}\emptyset(N_{12} + l_3)\,,\ \ K_{22} = \frac{1}{4}\emptyset(N_{11} + N_{22} - N_{33})\,,$$

$$K_{23} = \frac{1}{4}\emptyset,3 + \frac{1}{2}\emptyset(N_{23} - l_1)\,, K_{31} = -\frac{1}{4}\emptyset,2 - \frac{1}{2}\emptyset(N_{13} - l_2),$$

$$K_{32} = -\frac{1}{4}\emptyset,1 + \frac{1}{2}(N_{23} + l_1)\,, \qquad K_{33} = \frac{1}{4}\emptyset(N_{11} - N_{22} - N_{33});$$

$$t_r J = \underline{\nabla}\cdot\underline{t} - \left[\dot{P} + (t_r S)P\right] + 2\underline{a}\cdot\underline{t} - S{:}T = 0$$

Yields $\dot{\emptyset} + 2\emptyset(S_{23} + S_{33}) = 0$ i.e. $\frac{1}{2}\emptyset = -\emptyset(S_{22} + S_{33})$

While $\underline{\nabla}\cdot T - \left[\dot{\underline{t}} + \underline{\omega}\times\underline{t} + (t_r S)\underline{t}\,\right] = S\cdot\underline{t} + \underline{\Lambda}\times\underline{t} - \underline{a}\cdot[T + P_1]$

Yields the three relations

$$\frac{1}{2}\emptyset,1 = 2\emptyset l_1\,, \frac{1}{2}\emptyset,2 = -\emptyset(a_2 + N_{13} - l_2)\,, \frac{1}{2}\emptyset,3 = -\emptyset(a_3 - N_{12} - l_3)$$

Hence

$$j_1 = \emptyset l_1\,, j_2 = -\frac{1}{2}\emptyset(N_{13} - l_2)\,, \qquad j_3 = \frac{1}{2}\emptyset(N_{12} + l_3)$$

$$J_{11} = -\frac{1}{2}\emptyset(S_{22} + S_{33})\,, \qquad J_{22} = \frac{1}{2}\emptyset S_{33}\,, \qquad J_{33} = \frac{1}{2}\emptyset S_{22}$$

$$K_{12} = -\frac{1}{2}\emptyset(a_3 - N_{12} - l_3)\,, \quad K_{13} = \frac{1}{2}\emptyset(a_2 + N_{13} - l_2)\,, \quad K_{21} = -\frac{1}{2}\emptyset a_3$$

$$K_{23} = \frac{1}{2}\emptyset(N_{23} + l_1)\,, \ K_{31} = \frac{1}{2}\emptyset a_2\,, \ K_{32} = \frac{1}{2}\emptyset(N_{23} - l_1)$$

4. Non-null nonaligned field:

If p^μ and q^μ be the principal null directions of the electromagnetic field, we choose the electromagnetic frame as $\{p^\mu, q^\mu, t^\mu, \bar{t}^\mu\}$,

Where t^μ and $\bar{t}^\mu$ are a pair of complex null vectors constructed from two space like vectors with $t^\mu\,\bar{t}_\mu = 1$ and p^μ and q^μ are normalized by $p^\mu q_\mu = 1$.

If l^μ be a principal null direction of the gravitational field, we get

$$l^\mu p_\mu = -l^\mu q_\mu = -1$$

By a rotation $r_\mu \to e^{i\theta} r_\mu$ we may write

$$l_\mu = \frac{1}{2}\left(p_\mu - q_\mu + t_\mu + \bar{t}_\mu\right)$$

And complete the null tetrad by choosing

$$n_\mu = \frac{1}{2}\left(-p_\mu + q_\mu + t_\mu + \bar{t}_\mu\right), \qquad m_\mu = \frac{1}{2}\left(p_\mu + q_\mu + t_\mu - \bar{t}_\mu\right)$$

Consequently ,

$$p_\mu = \frac{1}{2}\left(l_\mu - n_\mu + m_\mu + \overline{m}_\mu\right) = \frac{1}{\sqrt{2}}(2^{l_\mu} + 0^{l_\mu})$$

$$q_\mu = \frac{1}{2}\left(m_\mu + \overline{m}_\mu - l_\mu + n_\mu\right) = \frac{1}{\sqrt{2}}(2^{l_\mu} - 0^{l_\mu})$$

$$t_\mu = \frac{1}{2}\left(l_\mu + n_\mu + m_\mu - \overline{m}_\mu\right) = \frac{1}{\sqrt{2}}(1^{l_\mu} - 3^{l_\mu})$$

Hence the null vectors of electromagnetic frame are expressed in terms of orthonormal tetrad in exactly the same way as those of gravitational null tetrad frame with the difference that the vectors $1^e\mu$ and $2^e\mu$ are interchanged. As a result the 3-vectors $\underline{j}$ and $\underline{k}$ and 3-dyadics J and K have in this case , components given by

$j_1 = -\frac{1}{2}\emptyset(N_{23} - L_1)$, $j_2 = \emptyset L_2$, $j_3 = \frac{1}{2}\emptyset(N_{12} + L_3)$, $K_1 = -\frac{1}{2}\emptyset(S_{23} - \Lambda^1)$, $K_2 = \emptyset\Lambda^2$, $K_3 = \frac{1}{2}\emptyset(S_{12} + \Lambda^3)$, $J_{11} = \frac{1}{2}\emptyset S_{33}$, $J_{12} = \frac{1}{2}\emptyset\omega^3$, $J_{13} = -\frac{1}{2}\emptyset(S_{13} + \Lambda^2)$, $J_{21} = -\frac{1}{2}\emptyset(S_{12} + \Lambda^3 - \omega^3)$, $J_{22} = -\frac{1}{2}\emptyset(S_{11} + S_{33})$,

$$J_{23} = -\frac{1}{2}\emptyset(S_{23} - \Lambda^1 + \omega^1)\,, \qquad J_{31} = \frac{1}{2}\emptyset(S_{13} - \Lambda^2),\ J_{32} = -\frac{1}{2}\emptyset\omega'\,,$$

$$J_{33} = \frac{1}{2}\emptyset S_{11}\,;\ K_{11} = \frac{1}{4}\emptyset(N_{11} + N_{22} - N_{33})\,, K_{12} = -\frac{1}{2}\emptyset a_3\,, K_{13} = \frac{1}{2}\emptyset(N_{13} + L_2)\,, K_{21} = -\frac{1}{2}\emptyset(a_3 - N_{12} - L_3), K_{22} = -\frac{1}{2}\emptyset N_{22}\,,$$

$$K_{23} = \frac{1}{2}\emptyset(a_1 + N_{23} - L_1)\,, K_{31} = \frac{1}{2}\emptyset(N_{13} - L_2)\,, K_{32} = \frac{1}{2}\emptyset a_1\,,$$

$K_{33} = -\frac{1}{4}\emptyset(N_{11} - N_{22} - N_{33})$,

An analysis similar to that carried out in chapter-IV can be done in this case also. For different Petrov types of Weyl tensors , we have

(i) Weyl tensor is of type N

$$U^{rs}\,V^{t0}\,J_{rst} = -\frac{c_1\sigma}{\sqrt{2}} = U^{rs}\,V^{t0}J_{rst}$$

$$U^{rs}V^{t2}J_{rst} = \frac{c_1 x}{\sqrt{2}} = iU^{rs}V^{t3}J_{rst}$$

(ii) Weyl tensor is of type III

$$V^{rs}J_{rs0} = \sqrt{2}\,c_2\sigma = -V^{rs}J_{rs1}$$

$$V^{rs}J_{rs2} = \sqrt{2}c_2 x = iV^{rs}J_{rs3}$$

(iii) Weyl tensor is of type-II or D

$$V^{rs}V^{t0}J_{rst} = -\,3c_3/\sqrt{2}\ \sigma = V^{rs}V^{t1}J_{rst}$$

$$V^{rs}V^{t2}J_{rst} = 3c_3/\sqrt{2}\,x = iV^{rs}V^{t3}J_{rst}$$

Hence for Petrov N , Weyl tensor

$$\frac{-C_1\sigma}{\sqrt{2}} = \frac{1}{2}[J_{022} - iJ_{023} + iJ_{032} + J_{033} + iJ_{312} + J_{313} - J_{122} + iJ_{123}]$$

$$= \frac{1}{2}\Big[-\frac{1}{2}\emptyset(S_{11} + S_{33}) + \frac{i}{2}\emptyset(S_{23} - \Lambda^1 + \omega^1) - \frac{i}{2}\emptyset\omega^1 + \frac{1}{2}\emptyset S_{11} - \frac{i}{2}\emptyset N_{22}$$

$$+ \frac{1}{2}\emptyset(a_1 + N_{23} - L_1) - \frac{1}{2}\emptyset a_1 - \frac{i}{4}\emptyset(N_{11} - N_{22} - N_{33})\Big]$$

$$= \frac{1}{4}\emptyset[-S_{33} + N_{23} - L_1] + i\emptyset(2S_{23} - 2\Lambda^1 - N_{11} - N_{22} + N_{33})$$

$$\frac{C_1 x}{\sqrt{2}} = \frac{1}{2}[-J_{200} + iJ_{300} + iJ_{310} + J_{120} + J_{021} - iJ_{03} + iJ_{311} + J_{122}]$$

$$= \frac{1}{4}\emptyset(\omega^3 + N_{13} + L_2) - \frac{i}{4}\emptyset(a_3 - S_{13} - \Lambda^2)$$

For type , III Weyl tensor

$$\sqrt{2}\, C_2\, \sigma = 2[V^{20}J_{200} + V^{30}J_{300} + V^{31}J_{310} + V^{12}J_{120}]$$

$$= -J_{200} + iJ_{300} + iJ_{310} + J_{120} = \emptyset L_2 - \frac{i}{2}\emptyset(N_{12} + L_3) + i\emptyset\Lambda^2 + \frac{1}{2}\emptyset(S_{12} + \Lambda^3)$$

$$= -2[V^{20}J_{201} + V^{30}J_{301} + V^{12}J_{121}]$$

$$= J_{201} - iJ_{301} - iJ_{311} - J_{121}$$

$$= -J_{021} + iJ_{031} - iJ_{311} - J_{121}$$

$$= -J_{21} + iJ_{31} - iK_{21} - K_{31}$$

$$= \frac{1}{2}\emptyset(S_{12} + \Lambda^3 - \omega^3) - \frac{i}{2}\emptyset(S_{13} - \Lambda^2) + \frac{i}{2}\emptyset(a_3 - N_{12} - L_3) - \frac{\emptyset}{2}(N_{13} - L_2)$$

$$\therefore \sqrt{2}\, C_2\sigma = \frac{1}{2}\emptyset(S_{12} + \Lambda^3) + \emptyset L_2 + i\emptyset\Lambda^2 - \frac{i}{2}\emptyset(N_{12} + L_3)$$

$$= \frac{1}{2}\emptyset(S_{12} + \Lambda^3) - \frac{1}{2}\emptyset(\omega^3 + N_{13} - L_2) + \frac{i}{2}\emptyset(a_3 - S_{13} + \Lambda^2)$$

$$- \frac{i}{2}\emptyset(N_{12} + L_3) \qquad (A_1)$$

$$\therefore L_2 = -\frac{1}{2}(\omega^3 + N_{13} - L_2)\ ; \quad (N_{13} + L_2) = -\omega^3 \qquad (A_2)$$

$$\Lambda^2 = \frac{1}{2}(a_3 - S_{13} + \Lambda^2)\ ; \quad S_{13} + \Lambda^2 = a_3 \qquad (A_3)$$

Also , $\sqrt{2}\, C_2 x = 2[V^{02}J_{022} + V^{03}J_{032} + V^{31}J_{312} + V^{12}J_{122}]$

$$= J_{022} - iJ_{032} + iJ_{312} + J_{122}$$

$$= -\frac{1}{2}\emptyset(S_{11} + S_{33}) + \frac{i}{2}\emptyset\omega^1 - \frac{i}{2}\emptyset N_{22} + \frac{1}{2}\emptyset a_1$$

$$= i(J_{023} - iJ_{033} + iJ_{313} + J_{123})$$

$$= i\left[-\frac{\emptyset}{2}(S_{23} - \Lambda^1 + \omega^1) - \frac{i}{2}\emptyset S_{11} + \frac{i}{2}\emptyset(a_1 + N_{23} - L_1) - \frac{\emptyset}{4}(N_{11} - N_{22} - N_{33})\right]$$

$$= \frac{\emptyset}{2}S_{11} - \frac{\emptyset}{2}(a_1 + N_{23} - L_1) - \frac{i\emptyset}{2}(S_{23} - \Lambda^1 + \omega^1) - \frac{i\emptyset}{4}(N_{11} - N_{22} - N_{33})$$

$\therefore 2\sqrt{2}C_2 x = \frac{\emptyset}{2}(-S_{33} - N_{23} + L_1) - \frac{i}{4}\emptyset(2S_{23} - 2\Lambda^1 + N_{11} + N_{22} - N_{33})(A_4)$

For the type II or D , Weyl tensor

$$-\frac{3C_3\sigma}{\sqrt{2}} = V^{rs}V^{t0}J_{rst} = V^{rs}V^{t1}J_{rst}$$

$$= \frac{1}{2}[-J_{022} + iJ_{023} + iJ_{032} + J_{033} - iJ_{312} - J_{313} - J_{122} + iJ_{123}]$$

$= \frac{1}{2}\left[\left[\frac{\emptyset}{2}(S_{11} + S_{33}) - \frac{i\emptyset}{2}(S_{23} - \Lambda^1 + \omega^1) - \frac{i}{2}\emptyset\omega^1 + \frac{1}{2}\emptyset S_{11}\right] + \frac{i}{2}\emptyset N_{22} - \frac{1}{2}\emptyset(a_1 + N_{23} - L_1) - \frac{1}{2}\emptyset a_1 - \frac{i\emptyset}{4}(N_{11} - N_{22} - N_{33})\right] = \frac{1}{4}\emptyset[2S_{11} + S_{33} - 2a_1 - N_{23} + L_1] + \frac{i}{4}\emptyset\left[-S_{23} + \Lambda^1 - 2\omega^1 - \frac{1}{2}(N_{11} - 3N_{22} - N_{33})\right]$ (A_5)

$\frac{3c_3x}{\sqrt{2}} = V^{rs}V^{t2}J_{rst} = -iV^{rs}V^{t3}J_{rst} = \frac{1}{2}[-J_{200} + iJ_{300} + iJ_{310} + J_{120} + J_{021} - iJ_{031} + iJ_{311} + J_{121}]$

$= \frac{1}{2}\left[\emptyset L_2 - \frac{i}{2}\emptyset(N_{12} + L_3)\right] + i\emptyset\Lambda^2 + \frac{\emptyset}{2}(S_{12} + \Lambda^3) - \frac{1}{2}\emptyset(S_{12} + \Lambda^3 - \omega^3) - \frac{i}{2}\emptyset(S_{13} - \Lambda^2) - \frac{i}{2}\emptyset(a_3 - N_{12} - L_3) + \frac{1}{2}\emptyset(N_{13} - L_2) = \; = \frac{1}{4}\emptyset(\omega^3 + N_{13} + L_2) - \frac{i}{4}\emptyset(a_3 - S_{13} - \Lambda^2)$ (A_6)

$$\therefore x = 0$$

For Petrov type N, Weyl tensor we have from (A_2) and (A_3)

Also , in this case

$$C_3 = C_2 = 0$$

Hence , from (A_5)

$$2S_{11} + S_{33} - 2a_1 - N_{23} + L_1 = 0 \quad (A_8)$$

And from (A_1) , (A_4)

$$S_{33} = -N_{23} + L_1 \quad (A_9)$$

$$S_{23} - \Lambda^1 = -\frac{1}{2}(N_{11} + N_{22} - N_{33}) \quad (A_{10})$$

$$S_{12} + \Lambda^3 - \omega^3 - N_{13} + L_2 = 0 \quad (A_{11})$$

$$a_3 - S_{13} + \Lambda^2 - N_{12} - L_3 = 0 \quad (A_{12})$$

From (A_7) and (A_9)

$$S_{11} + S_{33} = a_1$$

From (A_8) and (A_{10}) , $N_{22} = \omega^1$

From (A_2) and (A_{11})

$$2L_2 = -(S_{12} + \Lambda^3)$$

From (A_3) and (A_{12})

$$2\Lambda^2 = N_{12} + L_3$$

$$\therefore \frac{c_1\sigma}{\sqrt{2}} = -\frac{1}{2}\emptyset S_{33} + \frac{i}{2}\emptyset(S_{23} - \Lambda^1)\ ;\ \sqrt{2}\, c_1\sigma = \emptyset[-S_{33} + i(S_{23} - \Lambda^1)]$$

From these , we have for components of 3-vectors $\underline{j}$ and $\underline{k}$ and 3-dyadics

J and K .

$$K_2 = j_3\ , \qquad K_3 = -j_2\ ,\ \ J_{11} = j_1\ ,\ \ J_{13} = K_{12}\ , J_{22} = -K_{32}\ ,$$

$K_{11} = K_1\ , K_{22} = J_{32},\ \ K_{13} = -J_{12}\ ,\ J_{21} = -K_{31}\ ,\ J_{31} = K_{21}$

Also, since

$$S_{23} - \Lambda^1 + \omega^1 = -\frac{1}{2}(N_{11} + N_{22} - N_{33}) + N_{22}$$

$$= -\frac{1}{2}(N_{11} - N_{22} - N_{33})$$

And $a_1 + N_{23} - L_1 = 2S_{11} + S_{33} - S_{11} - S_{33} = S_{11}$

$$\therefore\ J_{23} = -K_{33}$$

And $\qquad K_{23} = -J_{33}$

Thus $\underline{j}\ , \underline{k}\ , \underline{J}\ , \underline{K}$ can be expressed in terms of a_1 only $\underline{a}\ , \underline{\Lambda}\ , \underline{\omega}$ and S , as

$$j_1 = \frac{1}{2}\emptyset S_{33} = J_{11}\ , \qquad j_2 = -\frac{1}{2}\emptyset(S_{12} + \Lambda^3) = -k_3\ ,$$

$$j_3 = \emptyset\Lambda^2 = k_2\ ,\ \ k_1 = -\frac{1}{2}\emptyset(S_{23} - \Lambda^1) = K_{11}\ ,\ \ J_{12} = \frac{1}{2}\emptyset\omega^3 = -K_{13}$$

$$J_{13} = -\frac{1}{2}\emptyset a_3 = K_{12}\ , \qquad J_{21} = -\frac{1}{2}\emptyset(S_{12} + \Lambda^3 - \omega^3) = -K_{31}\ ,$$

$$J_{22} = -\frac{1}{2}\emptyset a_1 = -K_{32}\ , \qquad J_{23} = -\frac{1}{2}\emptyset(S_{23} - \Lambda^1 + \omega^1) = -K_{33}\ ,$$

$J_{31} = -\frac{1}{2}\emptyset(S_{13} - \Lambda^2) = K_{21}\ ,\ \ J_{32} = -\frac{1}{2}\emptyset\ \ \omega' = K_{22}\ ,\ \ J_{33} = \frac{1}{2}\emptyset S_{11} = -K_{23}$

Also , $C_1\sigma = -\sqrt{2}[j_1 + ik_1]$

Since we have $a_2 = -2\omega^3$ and $a_3 = \frac{1}{2}\omega^2$, the expressions of $\underline{j}\ , \underline{k}\ , J$ and K can be written in terms of corotating axes $\left(\underline{\Lambda} - \underline{\omega} = 0\right)$ as

$$\underline{j} = \frac{1}{2}\emptyset\left[S_{33}\underline{u} - (S_{12} + \Lambda^3)\underline{v} + 2\Lambda^2\underline{\omega}\right]$$

$$\underline{k} = \frac{1}{2}\emptyset\left[-(S_{23} - \Lambda^1)\underline{u} + 2\Lambda^2\underline{v} + (S_{12} + \Lambda^3)\underline{\omega}\right]$$

$$J = \frac{1}{2}\emptyset\left[S_{33}\underline{u}\,\underline{u} + \Lambda^3\underline{u}\,\underline{v} - \frac{1}{2}\Lambda^2\underline{u}\,\underline{\omega} - S_{12}\underline{v}\,\underline{u} - (S_{11} + S_{33})\underline{v}\,\underline{v} - S_{23}\,\underline{v}\,\underline{\omega} - (S_{13} - \Lambda^2)\underline{\omega}\,\underline{u} - \Lambda'\underline{\omega}\,\underline{v} + S_{11}\underline{\omega}\,\underline{\omega}\right]$$

$$K = \frac{1}{2}\emptyset\left[-(S_{23} - \Lambda')\underline{u}\,\underline{u} - \frac{1}{2}\Lambda^2\underline{u}\,\underline{v} - \Lambda^3\underline{u}\,\underline{\omega} - (S_{13} - \Lambda^2)\underline{v}\,\underline{u} - \Lambda'\underline{v}\,\underline{v} + S_{11}\,\underline{v}\,\underline{\omega} + S_{12}\,\underline{\omega}\,\underline{u} + (S_{11} + S_{33})\underline{\omega}\,\underline{v} + S_{23}\,\underline{\omega}\,\underline{\omega}\right]$$

If the gravitational field is shear free , $\sigma = 0$ i.e.

$$S_{33} = 0\ , \qquad S_{23} = \Lambda'$$

Hence,

$$j_1 = 0 = K_1 = J_{11} = K_{11}$$

$$\underline{j} = -(c+h)\underline{v} - uf\underline{\omega}\,, \qquad \underline{k} = -uf\underline{v} \qquad + (c+h)\underline{\omega}$$

$$\underline{j}\,.\underline{k} = 0$$

$$J = h(\underline{u}\,\underline{v} + \underline{v}\,\underline{u}) + f(\underline{u}\,\underline{\omega} + \underline{\omega}\,\underline{u}) - j(\underline{v}\,\underline{\omega} + \underline{\omega}\,\underline{v}) - b(\underline{v}\,\underline{v} - \underline{\omega}\,\underline{\omega}) - (c+h)\underline{v}\,\underline{u} - uf\underline{\omega}\,\underline{u} = J_0 + \underline{j}\underline{u}$$

$$K = f(\underline{u}\,\underline{v} + \underline{v}\,\underline{u}) - h(\underline{u}\,\underline{\omega} + \underline{\omega}\,\underline{u}) + b(\underline{v}\,\underline{\omega} + \underline{\omega}\,\underline{v}) - j(\underline{u}\,\underline{v} - \underline{\omega}\,\underline{\omega}) - uf\underline{v}\,\underline{u} + (c+h)\underline{\omega}\,\underline{u} = K_0 + \underline{k}\,\underline{u}$$

Where

$$2b = \emptyset S_{11}\,, 2c = \emptyset S_{12}\,, 2f = \emptyset S_{13} = -\frac{1}{2}\emptyset\Lambda^2$$

$$2g = \emptyset S_{23} = \emptyset\Lambda^1, 2h = \emptyset\Lambda^3$$

And

$$J_0 = h(\underline{u}\,\underline{v} + \underline{v}\,\underline{u}) + f(\underline{u}\,\underline{\omega} + \underline{\omega}\,\underline{u}) - g(\underline{v}\,\underline{\omega} + \underline{\omega}\,\underline{v})$$

$$K_0 = f(\underline{u}\,\underline{v} + \underline{v}\,\underline{u}) - h(\underline{u}\,\underline{\omega} + \underline{\omega}\,\underline{u}) + b(\underline{v}\,\underline{\omega} + \underline{\omega}\,\underline{v})$$

Two symmetric traceless dyadics.

Thus for a non-null nonaligned electromagnetic field interacting with a Petrov type N shear free gravitational field we have simple expressions for the 3-vectors $\underline{j}$ and $\underline{k}$ and the 3-dyadics J and K .

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