Advances in Physics Theories and Applications
ISSN 2224-719X (Paper) ISSN 2225-0638 (Online)
Vol.17, 2013

Fractional Conformal spin of pseudo differential operators and KP hierarchy

Abderrahman EL Boukili[1*], Abdellah EL Boukili[2], Abderrahim El Hourch[2] and Moulay Brahim Sedra[1]

[1] *Université Ibn Tofail, Faculté des Sciences, Département de Physique, Laboratoire des Hautes énergies, Sciences de l'Ingénierie et Réacteurs (LHESIR), Kénitra 14000, Morocco.*

[2] *Université Mohamed V, Faculté des Sciences Agdal Département de Chimie (LECA), Rabat, Morocco.*

ABSTRACT

Given the importance of pseudo differential operators in physics, we present in this paper the useful background to study the algebraic structure of fractional pseudo differential operators. We present a brief account of basic properties of the space of higher conformal spin differential lax operators in the bosonic case and we give some applications through the fractional KP hierarchy

Keywords: Lax operators, pseudo differential operators, fractional KP hierarchy

1. INTRODUCTION

The pseudo differential operators have been actively used in quantum mechanics, in fact, the procedure of operator quantization is based on the correspondence between classical observables, functions on the phase space and quantum observables through pseudo-differential operators [1].

There are several versions of the theory of pseudo-differential operators, adapted to the solution of various problems in analysis and mathematical physics[2]. However, the algebra of pseudo-differential operator is an extension of the concept of the algebra of differential operator and they are integrated into the construction of the Gelfand-Dickey Poisson bracket of 2d integrable models [3,4].

Let us first consider the algebra A of all local and non local differential operators of arbitrary conformal spins and arbitrary degrees. One may expand A as [5-8]

$$\mathbf{A} = \underset{p\le q}{\oplus} \mathbf{A}^{(p,q)} = \underset{p\le q}{\oplus} \quad \underset{s\in Z}{\oplus} \mathbf{A}_s^{(p,q)}, \; p,q,s \in Z \tag{1}$$

where, $\mathbf{A}_s^{(p,q)}$ is the ring of differential operators type.

$$d_s^{(p,q)} := \sum_{i=p}^{q} u_{s-i}(z)\partial^i \tag{2}$$

This article is devoted to an introduction of the basic properties of the space of fractional pseudo differential operators, we present a brief account of the basic properties of the space of higher conformal spin differential Lax operators in the bosonic case[9-11]. We show in particular that any differential operator is completely specified by a conformal spin s, $s \in Z$, two fractional number integers defining the lowest and the highest degrees respectively and the analytic fields $u_j(z)$.

This paper is organized as follows. In section 2, we present Fractional pseudo differential operators. In section 3 we present the Algebraic structure of Fractional pseudo differential operators. We give some application through the fractional KP hierarchy. Finally we discuss our results in Section5.

2. FRACTIONAL PSEUDO DIFFERENTIAL OPERATORS

This section provides the extensions of the standard pseudo differential operators by fractional order integral operators, which is one of the main topic of this of the present paper.

2.1 Extended Lax operator

Recall that the Leibniz rule

$$\partial^{-j}.f = \sum_{k=0}^{\infty} \binom{-j}{k} f^{(k)} \partial^{-j-k}. \tag{3}$$

is applicable when the order j of "integral" is an arbitrary real (or complex) number. It will be interesting to consider the case when the pseudo-differential operator includes fractional order integrals, and then, to inquire whether the system gives a consistent hierarchy or not. In the following consideration, we accept the axiom that the fractional order integral operators exist and also its exponential law $\partial^{-i}\partial^{-j}=\partial^{-(i+j)}$ holds for fractional i and j, for a while. After the construction of the fractional pseudo differential operators, we will see the Riemann-Liouville integral of fractional order enjoys these requirements for the KP hierarchies.

2.2 Extension by the half-order integrals

For the simplest case of an extension of pseudo differential operators, we consider the pseudo-differential operator including the half-order integrals in addition to the pseudo-differential operator. Accordingly, we define the most general half-order integral operator,

$$d_s^{(\frac{p}{2},\frac{q}{2})} := \sum_{i=p}^{q} u_{s-\frac{i}{2}}(z)\partial^{i/2}, \tag{4}$$

where u_s are the dependent variables of degree $s \,?\, i/2$. With this definition, we can consider the following development

$$d_s^{(\frac{p}{2},\frac{q}{2})} = d_s^{(m,n)} + d_s^{(m+1/2,n+1/2)} \tag{5}$$

where the first operator $d_s^{(m,n)}$ with $\frac{p}{2}\le m\le n\le\frac{q}{2}$ Corresponds to all the even values of i ranging between p and q, such operators coincides with that defines in the first paragraph. The second operator $d_s^{(m+1/2,n+1/2)}$ with $p/2\le m+1/2\le n+1/2\le q/2$ correspond to the odd values of number i, such operators present of the degrees purely half-entirety.

2.3 Extension by the 1/N-th order integrals

Having observed the extension by the half-order integrals is successful, we now consider more generic extensions by the 1/N-th order integrals (N=3,4,…). In these cases, we need to introduce integral operators $\partial^{-1/N}$, $\partial^{-2/N}$, ?,$\partial^{-(N-1)/N}$ simultaneously to give a consistent pseudo differential operators, since we have to close the commutator algebra in the Lax equations under the axiom $\partial^{-i}\partial^{-j}=\partial^{-i-j}$. For example, we give an outline of the N=3 case, in which the pseudo differentiel operateur should be made up of,

$$d_s^{(\frac{p}{3},\frac{q}{3})} = d_s^{(m,n)} + d_s^{(m+1/3,n+1/3)} + d_s^{(m+2/3,n+2/3)}, \tag{6}$$

where the first operator $d_s^{(m,n)}$ with $p/3 \leq m \leq n \leq q/3$ Corresponds to all the values of i ranging between p and q and divisible by 3. Such operators coincides with that defines in the first paragraph. The second operator $d_s^{(m+1/3,n+1/3)}$ with $p/3 \leq m+1/3 \leq n+1/3 \leq q/3$ and $d_s^{(m+2/3,n+2/3)}$ with $p/3 \leq m+2/3 \leq n+2/3 \leq q/3$ are defines to satisfy the condition $\partial^{1/3}\partial^{2/3} = \partial$.

In the case general we have the following decomposition

$$d_s^{(\frac{p}{N},\frac{q}{N})} = d_s^{(m,n)} + d_s^{(m+\frac{1}{N},n+\frac{1}{N})} + \ldots + d_s^{(m+\frac{N-1}{N},n+\frac{N-1}{N})} \tag{7}$$

In this general case, we have the operators which have entirety degrees and others with a fractional degree ($\frac{k}{N}$) like their complementary to degree ($\frac{N-k}{N}$), such operators are defines to satisfy the condition $\partial^{\frac{k}{N}}\partial^{\frac{N-k}{N}} = \partial$ and operators plays a very important part in the studies of the integrables models.

3. ALGEBRIC STRUCTURE OF FRACTIONAL PSEUDO DIFFERENTIAL OPERATEURS

Let us note by $\mathbf{F}_{\frac{a}{b}}^{(\frac{r}{b},\frac{s}{b})}, b \neq 0$ the space of (fractional) pseudo-differential operators of conformal spin a/b and degree $(\frac{r}{b},\frac{s}{b})$ with $r \leq s$. Typical operators of this space are given by

$$\sum_{i=r}^{s} u_{\frac{a}{b}-\frac{i}{b}} \partial^{\frac{i}{b}} \tag{8}$$

Let's consider the huge space F describing the algebra of pseudo-differential operators arbitrary fractionaire conformal spin and arbitrary degrees. It is obtained by summing over all allowed values of spin and degrees in the following way:

$$\begin{aligned} \mathbf{F} &= \oplus_{r\leq s} \oplus_{a\in Z} \oplus_{b\in N^*} \mathbf{F}_{\frac{a}{b}}^{(\frac{r}{b},\frac{s}{b})} \\ &= \oplus_{r\leq s} \oplus_{a\in Z} \oplus_{b\in N^*} \oplus_{k=r}^{s} \mathbf{F}_{\frac{a}{b}}^{(\frac{k}{b},\frac{k}{b})} \end{aligned} \tag{9}$$

ISSN 2224-719X (Paper) ISSN 2225-0638 (Online)
Vol.17, 2013

where $\mathbf{F}_{\frac{a}{b}}^{(\frac{k}{b},\frac{k}{b})}$ is generated by elements type $u_{\frac{a}{b}-\frac{k}{b}}\partial^{\frac{k}{b}}$. If we note by $\mathbf{F}_{a}^{(r,s)} = \oplus_{b\in N^*} \mathbf{F}_{\frac{a}{b}}^{(\frac{r}{b},\frac{s}{b})}$ and $\mathbf{F}^{(r,s)} = \oplus_{a\in Z} \mathbf{F}_{a}^{(r,s)}$ then

$$\mathbf{F} = \oplus_{r\leq s} \mathbf{F}^{(r,s)} \tag{10}$$

The space $\mathbf{F}_{\frac{a}{b}}^{(0,0)} \equiv \mathbf{F}_{\frac{a}{b}}^{(0,0)}$ is nothing but the ring of analytic fields $u_{\frac{a}{b}}$ of conformal spin $a \in Z, b \in N^*$ with respect to this definition, the subspace $\mathbf{F}_{\frac{a}{b}}^{(\frac{k}{b},\frac{k}{b})}$ can be written formally as:

$$\mathbf{F}_{\frac{a}{b}}^{(\frac{k}{b},\frac{k}{b})} \equiv \mathbf{F}_{\frac{a}{b}-\frac{k}{b}}^{(0,0)}\partial^{\frac{k}{b}} \tag{11}$$

Let us return to the equation (4), the operator $d_s^{(\frac{p}{2},\frac{q}{2})}$ is an element of the algebra $\mathbf{F}_s^{(\frac{p}{2},\frac{q}{2})}$, such operator is composed of two operators, the first is $d_s^{(m,n)}$ which has purely integers degrees, so it is a "bosonic" pseudo-differential operator and the second operator $d_s^{(m+\frac{1}{2},n+\frac{1}{2})}$ is an operator of purely half integers degrees. So we can write any pseudo-differential operators of integers spin and fractional degrees as follows

$$\mathbf{F}_s^{(\frac{p}{2},\frac{q}{2})} = \mathbf{F}_s^{(m,n)} \oplus \mathbf{F}_s^{(m'+\frac{1}{2},n'+\frac{1}{2})} \tag{12}$$

where m, n, m' et n' are four integers satisfying the following inequalities $p/2 \leq m \leq n \leq q/2$ and $(p-1)/2 \leq m' \leq n' \leq (q-1)/2$.

In the case of equation (7) the pseudo differential operator $d_s^{(\frac{p}{3},\frac{q}{3})}$ is divided into three operators

$$d_s^{(\frac{p}{3},\frac{q}{3})} = d_s^{(m,n)} + d_s^{(m+1/3,n+1/3)} + d_s^{(m+2/3,n+2/3)} \tag{13}$$

where

$$\begin{aligned} &p/3 \leq m \leq n \leq q/3, \\ &\tfrac{p}{3}+\tfrac{1}{3} \leq m \leq n \leq \tfrac{q}{3}+\tfrac{1}{3}, \\ &\tfrac{p}{3}+\tfrac{2}{3} \leq m \leq n \leq \tfrac{q}{3}+\tfrac{2}{3} \end{aligned} \tag{14}$$

the algebraic structure of this decomposition is given by

$$\mathbf{F}_s^{(\frac{p}{3},\frac{q}{3})} = \mathbf{F}_s^{(m,n)} \oplus \mathbf{F}_s^{(m'+\frac{1}{3},n'+\frac{1}{3})} \oplus \mathbf{F}_s^{(m''+\frac{2}{3},n''+\frac{2}{3})} \tag{15}$$

where m, n, m', n', m'' and n'' are integers satisfying the following inequalities $p/3 \le m \le n \le q/3$, $(p-1)/3 \le m' \le n' \le (q-1)/3$ et $(p-2)/3 \le m'' \le n'' \le (q-2)/3$.

The Non trivial subalgebras $\mathbf{F}_s^{(m'+\frac{1}{3},n'+\frac{1}{3})}$ et $\mathbf{F}_s^{(m''+\frac{2}{3},n''+\frac{2}{3})}$ satisfy the following two equations

$$\mathbf{F}_{s'}^{(m'+\frac{1}{3},n'+\frac{1}{3})} \otimes \mathbf{F}_{s''}^{(m''+\frac{1}{3},n''+\frac{1}{3})} = \mathbf{F}_s^{(m+\frac{2}{3},n+\frac{2}{3})} \tag{16}$$

and

$$\mathbf{F}_{s'}^{(m'+\frac{1}{3},n'+\frac{1}{3})} \otimes \mathbf{F}_{s''}^{(m''+\frac{2}{3},n''+\frac{2}{3})} = \mathbf{F}_s^{(m,n)}. \tag{17}$$

The general case of equations (12) and (15) is given by the following algebraic decomposition

$$\mathbf{F}_s^{(\frac{p}{N},\frac{q}{N})} = \mathbf{F}_s^{(m,n)} \oplus \mathbf{F}_s^{(m_1+\frac{1}{N},n_1+\frac{1}{N})} \oplus \ldots \oplus \mathbf{F}_s^{(m_{N-1}+\frac{N-1}{N},n_{N-1}+\frac{N-1}{N})} \tag{18}$$

where m_k et n_k, $0 \le k \le N-1$ are the integers numbers satisfying

$$\frac{p}{N} - \frac{k}{3} \le m_k \le n_k \le \frac{q}{N} - \frac{k}{N} \tag{19}$$

(here $m = m_0$ and $n = n_0$) and the subalgebra of $\mathbf{F}_s^{(\frac{p}{N},\frac{q}{N})}$ satisfying

$$\mathbf{F}_s^{(m_k+\frac{k}{N},n_k+\frac{k}{N})} \otimes \mathbf{F}_{s'}^{(m_{k'}+\frac{k'}{N},n_{k'}+\frac{k'}{N})} = \mathbf{F}_{s''}^{(m_{k''}+\frac{k''}{N},n_{k''}+\frac{k''}{N})} \tag{20}$$

where $k'' = k + k'$.

4. FRACTIONAL KP HIERARCHY

In this subsection we will study the fractional formulation of the operators with a conformal weight 1 and of degrees $(1,-\infty)$, such operators are called the operators of Kadomtsev-Petviashvili (KP), [12,13,14]

$$\mathbf{L}_{KP} = \partial + \sum_{j=1}^{\infty} u_{j+1}\partial^{-j}, \tag{21}$$

4.1 Extension by the half-order integrals

For the simplest case of an extension of the KP hierarchy, we consider the Lax operator including the half-order integrals in addition to the Lax operator (21). We will define the most general half-order integral operator as follows [12]

$$\mathbf{M}_{1/2} = v_3\,\partial^{-1/2} + v_5\,\partial^{-3/2} + v_7\,\partial^{-5/2} + \cdots, \tag{22}$$

where v_m 's are the dependent variables of degree m/2. We remark that the Lax operator composed only of the half-order integrals (22) itself does not produce any consistent hierarchy, because its products does not close in the half-order integral operators: we need integer order integral/differential operators to close the algebra. With this definition, we consider the following Lax operator,

$$\mathbf{L}_{1/2} = \mathbf{L}_{KP} + \mathbf{M}_{1/2}, \tag{23}$$

and the standard Lax equation for the flows with respect to the time $\mathbf{t} = (t_1, t_2, ?)$,

$$\frac{\partial \mathbf{L}_{1/2}}{\partial t_n} = [\mathbf{B}_n, \mathbf{L}_{1/2}]. \tag{24}$$

If we take the standard definition of the Hamiltonian, $\mathbf{B}_n := (\mathbf{L}^n_{1/2})_+$, whose lower degree sequence is,

$$\begin{aligned} \mathbf{B}_1 &= \partial \\ \mathbf{B}_2 &= \partial^2 + 2v_3\,\partial^{1/2} + 2u_2 \\ \mathbf{B}_3 &= \partial^3 + 3v_3\,\partial^{3/2} + 3u_2\partial + 3(v_5 + v_3')\partial^{1/2} + 3u_3 + 3u_2' + 3v_3^2 \end{aligned} \tag{25}$$

then we find closed coupled nonlinear PDE's, an extended KP hierarchy by the half-order integrals, hereafter $KP_{1/2}$. Each coefficient of the negative powers in ∂ of

$$\frac{\partial \mathbf{L}_{1/2}}{\partial t_2} = [\mathbf{B}_2, \mathbf{L}_{1/2}], \tag{26}$$

is,

$$
\begin{aligned}
\partial^{-1/2} &: \quad \tfrac{\partial v_3}{\partial y} = 2v_5' + v_3'' \\
\partial^{-1} &: \quad \tfrac{\partial u_2}{\partial y} = 2u_3' + u_2'' + 2v_3 v_3' \\
\partial^{-3/2} &: \quad \tfrac{\partial v_5}{\partial y} = 2v_7' + v_5'' + 2(v_3 u_2)' \\
\partial^{-2} &: \quad \tfrac{\partial u_3}{\partial y} = 2u_4' + u_3'' + 2u_2 u_2' + 3v_5 v_3' + v_3 v_5' - v_3 v_3'',
\end{aligned}
\tag{27}
$$

whereas of

$$
\frac{\partial \mathbf{L}_{1/2}}{\partial t_3} = [\mathbf{B}_3, \mathbf{L}_{1/2}],
\tag{28}
$$

is,

$$
\begin{aligned}
\partial^{-1/2} &: \quad \tfrac{\partial v_3}{\partial t} = 3v_7' + 3v_5'' + v_3''' + 6(v_3 u_2)' \\
\partial^{-1} &: \quad \tfrac{\partial u_2}{\partial t} = 3u_4' + 3u_3'' + u_2''' + 6u_2 u_2' + 6(v_3 v_5)' \\
&\quad + \tfrac{3}{2}(v_3'^2 + v_3 v_3'').
\end{aligned}
\tag{29}
$$

Noting that the derivative in superspace can be seen as a square root of the derivative, which fact is formally equivalent to the feature of the half order derivative $\partial^{1/2}$. In contrast to the supersymmetric extensions, the extension considered in this section works without using Grassmann numbers.

4.2 Extension by the 1/N-th order integrals

Having observed the extension by the half-order integrals is successful, we now consider more generic extensions by the 1/N-th order integrals (N=3,4,…), $KP_{1/N}$ hierarchies. In these cases, we need to introduce integral operators $\partial^{-1/N}$, $\partial^{-2/N}$, ?, $\partial^{-(N-1)/N}$ simultaneously to give a consistent Lax equation, since we have to close the commutator algebra in the Lax equations under the axiom $\partial^{-i}\partial^{-j} = \partial^{-i-j}$. For N=p, a prime number, there appears a new system coupled to the KP hierarchy. For example, we give an outline of the N=3 case, in which the Lax operator should be made up of,

$$
\mathbf{L}_{1/3} = \mathbf{L}_{KP} + \mathbf{M}_{1/3} + \mathbf{M}_{2/3},
\tag{30}
$$

where,

$$\begin{aligned} \mathbf{M}_{1/3} &= w_4\,\partial^{-1/3} + w_7\,\partial^{-4/3} + w_{10}\,\partial^{-7/3} + \cdots, \\ \mathbf{M}_{2/3} &= w_5\,\partial^{-2/3} + w_8\,\partial^{-5/3} + w_{11}\,\partial^{-8/3} + \cdots, \end{aligned} \tag{31}$$

and $\deg[w_m] = m/3$. We observe that the standard Lax equation and the definition of the Hamiltonian similar to the former case give a consistent hierarchy of coupled PDE's. One can see the lowest coupled PDE arises from the first two non-trivial Lax equations. Each coefficient of ∂ in

$$\frac{\partial \mathbf{L}_{1/3}}{\partial t_2} = [\mathbf{B}_2, \mathbf{L}_{1/3}], \tag{32}$$

is,

$$\begin{aligned} \partial^{-1/3} &: \quad \tfrac{\partial w_4}{\partial y} = 2w_7' + w_4'' \\ \partial^{-2/3} &: \quad \tfrac{\partial w_5}{\partial y} = 2w_8' + w_5'' + (w_4^2)' \\ \partial^{-1} &: \quad \tfrac{\partial u_2}{\partial y} = 2u_3' + u_2'' + 2(w_4 w_5)' \\ \partial^{-4/3} &: \quad \tfrac{\partial w_7}{\partial y} = 2w_{10}' + w_7'' + (w_5^2)' + 2(w_4 u_2)' \\ \partial^{-5/3} &: \quad \tfrac{\partial w_8}{\partial y} = 2w_{11}' + w_8'' + \tfrac{8}{3} w_7 w_4' + \tfrac{4}{3} w_4 w_7' + 2w_5 u_2' - \tfrac{2}{3} w_4 w_4'' \\ \partial^{-2} &: \quad \tfrac{\partial u_3}{\partial y} = 2u_4' + u_3'' + 2u_2 u_2' + \tfrac{10}{3} w_8 w_4' + \tfrac{4}{3} w_4 w_8' + \tfrac{8}{3} w_7 w_5' \\ &\qquad + \tfrac{2}{3} w_5 w_7' - \tfrac{4}{3} w_5 w_4'' - \tfrac{2}{3} w_4 w_5'', \end{aligned} \tag{33}$$

whereas of

$$\frac{\partial \mathbf{L}_{1/3}}{\partial t_3} = [\mathbf{B}_3, \mathbf{L}_{1/3}], \tag{34}$$

is,

$$\begin{aligned} \partial^{-1/3} &: \quad \tfrac{\partial w_4}{\partial t} = 3w_{10}' + 3w_7'' + w_4''' + 6(w_4 u_2)' + 3(w_5^2)' \\ \partial^{-2/3} &: \quad \tfrac{\partial w_5}{\partial t} = 3w_{11}' + 3w_8'' + w_5''' + 6(w_5 u_2)' + 6(w_7 w_4)' \\ &\qquad + 2w_4'^2 + 2w_4 w_4'' \\ \partial^{-1} &: \quad \tfrac{\partial u_2}{\partial t} = 3u_4' + 3u_3'' + u_2''' + 6u_2 u_2' + 6(w_8 w_4)' + 6(w_7 w_5)' \\ &\qquad + w_5 w_4'' + 3w_5' w_4' + 3w_4^2 w_4'. \end{aligned} \tag{35}$$

These are nine equations for the nine dependent variables so that we can combine them into the coupled PDE of u_2, w_4 and w_5.

For N being a composite number, we observe that the new system is coupled to the system coming from the prime factors of N. For example, the $KP_{1/4}$ system is a new system coupling to the $KP_{1/2}$ system given above.

5. CONCLUSION

In our paper, we have presented an extension of the standard algebra of pseudo differential operators, we have introduced the useful background to study the algebraic structure of fractional pseudo differential operators. we give some applications through the fractional KP hierarchy.

We should also remark that the introduction of pseudo-derivative of irrational order does not make a finite closed system: we need uncountable number of additional M 's like ($M_{1/3}$) and ($M_{2/3}$).

REFERENCES

[1] H. Weyl, The Theory of Groups and Quantum Mechanics, Dover, New-York, 1931.

[2] Andrei Khrennikov, Distributions and Pseudo-Differential Operators on Infinite-Dimensional Spaces with Applications in Quantum Physics, Operator Theory: Advances and Applications, Vol. 164, 161–172

[3] K. Huitu and D. Nemeschansky, Super symmetric Gelfand-Dickey Algebra, Mod. Phys. Lett. A6 (1991) 3179-3190;

[4] A. El Boukili, M.B. Sedra and A. Zemate, Super Gelfand-Dickey Algebra and Integrable Models, CJP, VOL. 47, NO. 6 DECEMBER 2009;

[5] A. El Boukili, M. B. Sedra and A. Zemate, Integrable Systems: Bosonic and Supersymmetric Cases, AJMP, Volume 6 (2008) 123 – 143.

[6] M.B. Sedra, A. El Boukili, H. Erguig, J. Zerouaoui, Some Aspects of 2D Integrability, Adv. Studies Theor. Phys Vol. 2, 2008, no. 2, 87 – 97.

[7] O. Dafounansou, A. El Boukili, M.B. Sedra, Some Aspects of Moyal Deformed Integrable Systems, CHINESE JOURNAL OF PHYSICS, VOL. 44, NO. 4 AUGUST 2006.

[8] A. El Boukili, and M. B. Sedra, Some Physical Aspects of Moyal Noncommutativity, CHINESE JOURNAL OF PHYSICS, VOL. 47, NO. 3 JUNE 2009

[9] K. Bilal, A. El Boukili and M. B. Sedra, KdV Hierarchies on the 2d-torus, The African Review of Physics (2012) 7:0026 243.

[10] A. EL Boukili and M.B. Sedra, On the Algebra of q-Deformed Pseudo Differential Operators, ISRN High Energy Physics, Volume 2012, Article ID 503621, 12 pages.

[11] A. El Boukili, and M. B. Sedra, Integrability of q-Deformed Lax Equations, CHINESE JOURNAL OF PHYSICS, VOL. 47, NO. 3 JUNE 2009.

[12] Masaru Kamata and Atsushi Nakamula, Riemann-Liouville integrals of fractional order and extended KP hierarchy, J. Phys. A35, (2002), 9657-9670.

[13] Samko S G, Kilbas A A and Marichev O I 1993 Fractional Integrals and Derivatives . Theory and Applications (Amsterdam: Gordon and Breach)

[14] Butzer P L and Westphal U 2000, Applications of Fractional Calculus in Physics, E d. Hilfer R (Singapore: World Scientific) p 1