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ON 1D FRACTIONAL SUPERSYMMETRIC THEORY

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Abstract
Following our previous work on fractional supersymmetry (FSUSY) [1,2], we focus here our contribute to the study of the superspace formulation in $1D$ that is invariant under FSUSY where $F=3$ and defined by $Q^3=H$, we extend our formulation in the end of our paper to arbitrary F with $F>3$.
Key-words Fractional superspace - Fractional Supersymmetry of order F - Fractional Supercharge - Covariant Derivative

1. Introduction

Motivating by the results founded in [1,2], the aim of this paper is to develop a superspace formulation in $1D$ QFT that is invariant under fractional supersymmetry (FSS). In such construction, the Hamiltonian H is expressed as the F^{th} power of a conserved fractional supercharge: $Q^F=H$, with $[H,Q]=0$ and $F\geq 3$. Here, we shall reformulate these results in fractional superspace, using generalized Grassmann variable of order F satisfying $\theta^F=0$. Additionally, we construct the Noether fractional supercharges in the case where $F=3$.
The presentation of this paper is as follow: In section 2, we present the Fractional Superspace and Fractional Supersymmetry $F=3$. In section 3, we will give the Fractional supercharges and Euler-Lagrange equations for $F=3$. In section 4, we will generalise the FSUSY in arbitrary order and finally, we give a conclusion.

2. Fractional Superspace and Fractional Supersymmetry $F=3$

The FSUSY of order 3 are generated by the Hamiltonian H generator of the time translation, and Q the generator of FSUSY transformations . they satisfies:

$$[Q,H]=0 \qquad ; \qquad Q^3=H \tag{1}$$

In fields quantum theory, this symmetry can be realized on generalized superspace (t,θ) , where t is the time, and θ is a real generalized Grassmann variable. The latter variable and his derivative $\frac{\partial}{\partial\theta}=\partial$ satisfies:

$$\theta^3=0 \qquad ; \qquad \partial^3=0$$

$$\partial_\theta \theta - q\theta\partial_\theta = \mathrm{I} \tag{2}$$

$$\int d\theta \equiv \partial_\theta^2$$

The introduction of the ε (parameter of the transformation associated to Q) et f (parameter of the transformation associated to H) in the case where $F = 3$ give the following transformations [3]:

$$t' = t - f - q(\varepsilon^2\theta + \varepsilon\theta^2) \tag{3}$$

$$\theta' = \theta + \varepsilon$$

where ε verify:

$$\varepsilon^3 = 0$$

$$\theta\varepsilon = q\varepsilon\theta \tag{4}$$

where $q = e^{\frac{2i\pi}{3}}$. The q-commutation relation between the two variables ε and θ ensures that:

- if $\varepsilon^3 = \theta^3 = 0$ then $(\theta + \varepsilon)^3 = 0$;
- the reality of the time is not affected by the FSUSY transformation;
- the FSUSY transformations q-commute with covariant derivative;
- the FSUSY transformations satisfied the Leibnitz rules.

We now can introduce the scalar superfield Φ of order 3:

$$\Phi(t,\theta) = \varphi_0 + q^{\frac{1}{2}}\theta\varphi_1 + q^2\theta^2\varphi_2 \tag{5}$$

where φ_0, φ_1 et φ_2 are the extension of the bosonic and the fermionic field. These fields verifies:

$$\begin{aligned} \theta\varphi_0(t) &= \varphi_0(t)\theta \\ \theta\varphi_1(t) &= q^2\varphi_1(t)\theta \\ \theta\varphi_2(t) &= q\varphi_2(t)\theta \end{aligned} \tag{6}$$

We now can see that $\Phi = \Phi^*$. Using relations (3), we get easily the FSUSY transformations upon the fields:

$$\begin{aligned} \delta\varphi_0 &= q^{1/2}\varepsilon\varphi_1 \\ \delta\varphi_1 &= -q^{-1/2}\varepsilon\varphi_2 \\ \delta\varphi_2 &= -\varepsilon\partial_{-1}\varphi_0 \end{aligned} \tag{7}$$

Then, let us consider the two basic objects Q and D, which represent respectively the FSUSY generator and the covariant derivative [1]

$$Q = -q\theta^2\partial_{-1} - q(\frac{\partial}{\partial\theta})^2\theta + \theta(\frac{\partial}{\partial\theta})^2$$

$$D = -(\theta)^2\partial_{-1} - q^2(\frac{\partial}{\partial\theta})^2\theta + \theta(\frac{\partial}{\partial\theta})^2 \tag{8}$$

Using the equations (2) and (4), we can prove that:

$$D^3 = Q^3 = \partial_z$$
$$QD = q^2 DQ \tag{9}$$
$$\delta_\varepsilon D\Phi = D\delta_\varepsilon \Phi$$

where

$$\delta_\varepsilon \Phi = \varepsilon Q\Phi \tag{10}$$

The invariant action under the FSUSY transformations in equations (7) is:

$$S = \frac{q}{2}\int dt d^2\theta\theta_{-1}\Phi D\Phi = \int dt L$$
$$= \frac{q}{2}\int dt d^2\theta\theta^2[-\dot{\varphi}_0{}^2 + q\dot{\varphi}_1\varphi_2 - q^2\dot{\varphi}_2\varphi_1] \tag{11}$$
$$= \frac{1}{2}\int dt[\dot{\varphi}_0{}^2 - q\dot{\varphi}_1\varphi_2 + q^2\dot{\varphi}_2\varphi_1]$$

3. Fractional supercharges and Euler-Lagrange equations

Following [3] and [6], we can introduce the generalized momenta conjugate to φ_i

$$\pi_0 \equiv \frac{\partial L}{\partial \dot{\varphi}_0} = \dot{\varphi}_0$$
$$\pi_1 \equiv 2\frac{\partial L}{\partial \dot{\varphi}_1} = -q\varphi_2 \tag{12}$$
$$\pi_2 \equiv 2\frac{\partial L}{\partial \dot{\varphi}_2} = q^2\varphi_1$$

If we consider $\dot{\Phi}$ and $D\Phi$ as independent variables, we can prove that the generalized momenta conjugate are the components of the fractional superspace momentum conjugate to $\Phi(t,\theta)$

$$\Pi(t,\theta) \equiv \frac{2}{q}\frac{\partial L}{\partial \dot{\Phi}} = D\Phi \tag{13}$$

which is decomposed as

$$\Pi(t,\theta) = -\theta^2\dot{\varphi}_0 + q^2 q^{\frac{1}{2}}\varphi_1 - q\theta\varphi_2 = \sum_{i=0}^{2} q^{\frac{i^2-1}{2}}\theta^i\pi_{(2-i)} \tag{14}$$

Note that $\Pi^* = \Pi$. If wish to add

$$\int d\theta[\Pi,\Phi] = 0$$
$$\int d\theta[\Pi,\dot{\Phi}] = 0 \tag{15}$$
$$[\dot{\Phi},\Phi] = 0 \tag{16}$$

we must require

$$\varphi_i \dot{\varphi}_j = \dot{\varphi}_j \varphi_i \qquad \text{if} \qquad j \neq 3-i \tag{17}$$

$$\varphi_1 \varphi_2 = q \varphi_2 \varphi_1 \tag{18}$$

Focus on the internal-space part of the Lagrangian $L = -\frac{q}{2}\dot{\varphi}_1 \varphi_2 + \frac{q^2}{2}\dot{\varphi}_2 \varphi_1$. The Lagrangian variation

$$\delta L = -\frac{q}{2}[\delta\dot{\varphi}_1.\varphi_2 + \dot{\varphi}_1.\delta\varphi_2] + \frac{q^2}{2}[\delta\dot{\varphi}_2.\varphi_1 + \dot{\varphi}_2.\delta\varphi_1]$$

$$= -\frac{q}{2}[\delta\dot{\varphi}_1.\varphi_2 + q\delta\varphi_2.\dot{\varphi}_1] + \frac{q^2}{2}[\delta\dot{\varphi}_2.\varphi_1 + q^2\delta\varphi_1.\dot{\varphi}_2] \tag{19}$$

and knowing that

$$\frac{\partial L}{\partial \varphi_1} = \frac{q}{2}\dot{\varphi}_2 \qquad ; \qquad \frac{\partial L}{\partial \varphi_2} = -\frac{q^2}{2}\dot{\varphi}_1$$

$$\frac{\partial L}{\partial \dot{\varphi}_1} = -\frac{q}{2}\varphi_2 \qquad ; \qquad \frac{\partial L}{\partial \dot{\varphi}_2} = \frac{q^2}{2}\varphi_1 \tag{20}$$

Then, the Lagrangian variation will be:

$$\delta L = (\delta\varphi_1 \frac{\partial L}{\partial \varphi_1} + \delta\dot{\varphi}_1 \frac{\partial L}{\partial \dot{\varphi}_1}) + (\delta\varphi_2 \frac{\partial L}{\partial \varphi_2} + \delta\dot{\varphi}_2 \frac{\partial L}{\partial \dot{\varphi}_2}) \tag{21}$$

From this equation, it is easy to show that the generalized Euler-Lagrange equations which follow from a least-action principle are:

$$\frac{\partial L}{\partial \varphi_1} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{\varphi}_1}) = 0$$

$$\frac{\partial L}{\partial \varphi_2} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{\varphi}_2}) = 0 \tag{22}$$

Therefore, the quantity

$$C = \sum_{i=1}^{2} \delta\varphi_i \frac{\partial L}{\partial \dot{\varphi}_i} - X \qquad ; \qquad \frac{dC}{dt} = 0 \tag{23}$$

is a constant of motion when the lagrangian varies under a transformation $\delta\varphi_i$ by the total derivative $\delta L = dX/dt$

where $X = \frac{1}{2} q^{\frac{1}{2}} \dot{\varphi}_0 \varphi_1$. The particular case of the Hamiltonian when $\delta\varphi_i = \dot{\varphi}_i$ is:

$$H = \sum_{i=0}^{2} \dot{\varphi}_i \frac{\partial L}{\partial \dot{\varphi}_i} - L = \frac{1}{2}\dot{\varphi}_0^2 \tag{24}$$

For $\delta\varphi_i$ given by (7) and corresponding X, we find the following fractional supercharge associated with the symmetry transformations:

$$Q = \frac{q^{\frac{1}{2}}}{2}(\varphi_1 \dot{\varphi}_0 + \frac{1}{2}\varphi_2^2) \tag{25}$$

Note, like in [6], that $(\varepsilon Q)^* = \varepsilon Q$, i.e., $Q^* = qQ$, using (17), we can prove that $q^{-\frac{1}{2}}Q$ is a real charge.

4. FSUSY of arbitrary order

In this section, we will give a generalisation of the FSUSY of order F where $F \geq 3$. For this, we introduce the expression of the superfield $\Phi(t,\theta)$ of order F .

$$\Phi(t,\theta) = \sum_{i=0}^{F-1} q^{\frac{i^2}{2}} \theta^i \varphi_i \tag{26}$$

where θ is a real generalized Grassmann variable satisfied $\theta^F = 0$ and q is the F-th root of unity ($q = e^{\frac{2\pi i}{F}}$). The superfield components verifies the following commutation relations:

$$\begin{aligned} &\theta\varphi_i = q^{-i}\varphi_i\theta \\ &\varphi_i\varphi_{F-i} = q^i \varphi_{F-i}\varphi_i \end{aligned} \tag{27}$$

the first relation in (27) implies that $\Phi^* = \Phi$. while the second relationship is used to introduce the following commutation relation:

$$\varphi_i\dot{\varphi}_{F-i} = q^i \dot{\varphi}_{F-1}\varphi_i \tag{28}$$

for $F = 2$, the equations (27) and (28) reduces to the usual results of supersymmetry $\theta\varphi_1 = -\varphi_1\theta$, $\varphi_1^2 = 0$ and $\varphi_1\dot{\varphi}_1 = -\dot{\varphi}_1\varphi_1$ while $\varphi_0\dot{\varphi}_0 = \dot{\varphi}_0\varphi_0$.

Les FSUSY transformations of order F are generated by the generator of the FSUSY Q whose expression is:

$$Q = A[-q\theta^{F-1}\partial_t - q\sum_{i=0}^{F-3}\theta^i(\frac{\partial}{\partial\theta})^{F-1}\theta^{F-2-i} + \theta^{F-2}(\frac{\partial}{\partial\theta})^{F-1}] \tag{29}$$

where $A = (-q)^{-\frac{F-2}{F}}(\{F-1\}!)^{-\frac{F-1}{F}}$. This implies that:

$$Q^F = \partial_t \tag{30}$$

Acting on $\Phi(t,\theta)$, we have

$$\delta\Phi(t,\theta) = \varepsilon Q\Phi(t,\theta) \tag{31}$$

which gives on components:

$$\begin{aligned} &\delta\varphi_i = -Aq(q^{F-1})^i q^{\frac{(i+1)^2}{2} - \frac{i^2}{2}}\{F-1\}!\varepsilon\varphi_{i+1} \\ &\delta\varphi_{F-2} = A(q^{F-1})^{F-2} q^{\frac{(F-1)^2}{2} - \frac{(F-2)^2}{2}}\{F-1\}!\varepsilon\varphi_{F-1} \\ &\delta\varphi_{F-1} = -Aqq^{-\frac{(F-1)^2}{2}}(q^{F-1})^{F-1}\varepsilon\dot{\varphi}_0 \end{aligned} \tag{32}$$

while $i \in \{0,1,...,F-3\}$, $\{F-1\} = \frac{1-q^{F-1}}{1-q}$ and ε is real infinitesimal parameter verify:

$$\theta\varepsilon = q\varepsilon\theta; \varphi_i\varepsilon = q^i\varepsilon\varphi_i \tag{33}$$

To build invariant action under FSUSY transformations need the introduction of the fractional covariant derivative commuting with εQ.

$$D = B[-\theta^{F-1}\partial_t - \sum_{i=0}^{F-3} q^{F-1-i}\theta^i (\frac{\partial}{\partial\theta})^{F-1}\theta^{F-2-i} + \theta^{F-2}(\frac{\partial}{\partial\theta})^{F-1}] \tag{34}$$

where $B = [(-1)^{\frac{F-1}{F}}(q^{F-2}q^{F-3}...q^2)^{-\frac{1}{F}}(\{F-1\}!)^{-\frac{F-1}{F}}]$. Q et D satisfies the following relations:

$$DQ = qQD \quad ; \quad D^F = \partial_t \tag{35}$$
$$\varepsilon Q = qQ\varepsilon \quad ; \quad \varepsilon D = q\varepsilon D$$

After defining the two generators of FSUSY, we can now give the expression of the action S invariant under the FSUSY transformations (32)

$$S = -\frac{1}{2B\{F-1\}!}\int dtd^{F-1}\theta D\Phi\dot{\Phi}$$

$$S = \frac{1}{2\{F-1\}!}\int dtd^{F-1}\theta\theta^{F-1}\{\dot{\varphi}_0^2 + \{F-1\}!\sum_{i=1}^{F-2} q^{\frac{i^2}{2}}q^{F-i}q^{\frac{(F-i)^2}{2}}q^{i(F-i)}\varphi_i\dot{\varphi}_{F-i}$$
$$-\{F-1\}!q^{\frac{1}{2}}q^{F-1}q^{\frac{(F-1)^2}{2}}\varphi_{F-1}\dot{\varphi}_1\} \tag{36}$$

$$S = \frac{1}{2}\int dt\{\dot{\varphi}_0^2 + \{F-1\}!\sum_{i=1}^{F-2} q^{\frac{i^2}{2}}q^{F-i}q^{\frac{(F-i)^2}{2}}q^{i(F-i)}\varphi_i\dot{\varphi}_{F-i}$$
$$-\{F-1\}!q^{\frac{1}{2}}q^{F-1}q^{\frac{(F-1)^2}{2}}\varphi_{F-1}\dot{\varphi}_1\} \tag{37}$$

Conclusion

In this paper, we have extended the results founded in [1] and [2] of fractional symmetry (FSUSY) from $2D$ to $1D$, and following [3] and [6], we are giving the fractional supercharge in $F = 3$. In the last section, we gave the generalized formulation of the generator of the FSUSY Q and fractional covariant derivative D.

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