Investigating Muonic Hydrogen Atom Energy Spectrum Using Perturbation Theory in Lowest Order

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Abstract

This work presents a quantum mechanical calculation of corrections in energy levels of muonic hydrogen atom using a potential due to finite size proton. Muonic hydrogen (μ-p) is an exotic atom in which the muon (μ⁻) replaces the electron "orbiting around" the proton in normal atomic hydrogen. Corrections in energy levels of Muonic hydrogen atom are calculated using a potential due to finite size proton. This Thesis analyzes the implications of Muonic hydrogenic atoms compared to standard hydrogenic atoms. These calculations are performed with Schrödinger wave functions with Coulomb potential using perturbation theory. The finite size of proton gives values of Lamb shift higher than that of point charge. The fine structure correction is very small compared to the Lamb shift values of Muonic hydrogen as we can see from the literature review. Therefore, as we have seen through all the above calculation of 1s, 2s and 2p, the application of perturbation theory has shown us that the energy correction is very small at each state. So the perturbation at higher order become small and small compared to the zero order at each state and even it can be ignored for higher orders. From this we can say that the interaction of electron with proton at higher state will be low which justifies that proton is not a large spherical shaped but it has finite size.

Keywords: coulomb potential; Point proton; Finite size proton; muonic hydrogen atom; Energy levels corrections; Lamb shift.

1 Introduction

Quantum mechanics (QM) is a powerful theoretical framework within which it has been found possible to describe, correlate and predict the behaviour of a vast range of physical systems, including the elementary particles, nuclei, atoms and radiation. It is a fundamental theory of matter and energy that explains physical phenomena which found no explanation in classical physics. In particular, facts the energy is absorbed and released in small, discrete quantities (quanta), and that all matter displays both wave and particle like properties, especially when viewed at atomic and subatomic scales. Quantum field theory is currently the most successful frame work for describing interaction between particles at the sub-atomic spectrum.

In 1912, Niels Bohr proposed the first electronic hydrogen atom model which successfully predicted the main energy levels of the hydrogen atom in the framework of a semi-classical theory based on Planck's hypothesis. But one can put into evidence significant failures of the Bohr's model by solving the Schrödinger's equation for hydrogen atom. The main spectral lines of hydrogen atom can be described by the Schrödinger equation, without using any postulates. When we consider the main spectral lines of hydrogen atom, it includes only the largest interaction terms in the Hamiltonian, namely the electron kinetic energy, the electron nuclear attraction, and the electron-electron repulsion (Fallieros and Hadjimichael, 1993). Later, experiments revealed that the main spectral lines have structures, called the fine structure.

2. An atomic nucleus (positively charged) in whose field a negative meson moves is called a mesonic atom. This implies that the properties of such a system are closely related to the properties of an ordinary muonic atom. This is so because in both systems the electromagnetic interaction, i.e. the Coulomb field plays a dominant role. At first sight this might be astonishing, because the mesons are subject to the strong interaction which, by definition, dominates other interactions. However, the range of the strong interacting field is much shorter than that of the Coulomb field, and there is a wide range in which the Coulomb field dominates. The meson itself is captured by the atom in a highly excited state. It can undergo many transitions into lower states before it enters the range of the strong interactions (Sapirstein *et al.,* 1990).

The Lamb Shift in the Muonic Hydrogen

3. Quantum electrodynamics (QED) is employed in this study to explain the Lamb shift of muonic hydrogen atom. First, one calculates the energy of muon bound to the atom and then the muon energy of a free muon. This consists of the Lamb shift (Biswas, 1998; Mandl and Shaw, 2010). In quantum mechanics, an electron can spontaneously emit a photon and subsequently can absorb it. This virtual process happens continuously. In fact, muon is always surrounded by a cloud of virtual photons. Due to this virtual process of the muon-photon interaction, the muon acquires an extra energy which is known as the self- energy of the muon. This self energy of the muon has observable effects. In particular, it can lead to a shift in the

energy of muon in the atom (Landau *et al.,* 1982).

4. Since the muon mass is about 200 times larger than the electron mass, the muon wave function overlaps with the proton Mass of muon per mass of electron is approximately 10^7 stronger than that of the electron (in hydrogen), leading to an increased sensitivity to the hadronic structure of the nucleus (charge and magnetic moments nuclear polarizability). Precision measurements of hydrogen atomic spectra can also be used to determine the proton's electric charge radius. In this case, the hyperfine 1s Lamb shift of atomic hydrogen is sensitive to the proton's finite charge radius since there is a small (but nonzero) probability that the electron's orbit will be inside the proton. Since the effect is small, a careful bound-state QED calculation of the many radiative effects must be performed to yield the proton charge radius (Melnikov and van Ritbergen, 2000).

5. In the experimental investigation (performed at Paul Scherrer Institute (PSI) of the proton structure from the measurement of the Lamb shift (2s−2p transition frequencies) of muonic hydrogen (Rolf, *et, al.,* 2010), ($\mu^- p$ atom a proton orbited by a negative muon), the proton charge radius. As well known, the contribution of the finite-size proton effect Δ_{finite} to the total energy splitting $\Delta E_L = 2P_{1/2} - 2S_{1/2}$ (Lamb shift) is 1.8 percent of the total ΔE_L in a "muonic hydrogen" atom, two orders of magnitude more than in a hydrogen atom H. The atomic energy levels of H or μp are affected by the finite size of the proton charge distribution by:

6. $\Delta E_{finite} = \frac{2\pi Z\alpha}{3}|\psi(0)|^2 r_P^2$, $\qquad$ (2.1)

7. Where $|\psi(0)|$ the atomic wave function at the origin, α is the fine structure constant, $Z = 1$ the proton charge, and r_p is the root mean square proton radius given in femtomers. For the S-states, $|\psi(0)|^2$ is proportional to m_r^3 (m_r is the reduced mass). Since the muon mass (m_μ) is 207 times more than the electron mass (m_e) then the reduced mass of muon $(m_r^\mu \approx 186 m_r^e)$, which leads to the sharp enhancement of the contribution of the finite-size proton effect ΔE_{finite} to the total energy difference ∇E_L in a "muonic hydrogen" atom in comparison with a hydrogen atom. In addition, the Lamb shift in μp differs from H in that the electron vacuum polarization gives the most significant contribution, because the Compton wavelength of the electron (which determines the spatial distribution of the vacuum polarization charge density) is of the order of the muonic hydrogen Bohr radius. This leads to a higher sensitivity to the proton finite size of μp in comparison with a electronic hydrogen atom. When a proton is orbited by a negative muon, its much smaller Bohr radius compared to ordinary atomic hydrogen causes an enhancement of effects related to the finite proton size.

8. The Lamb shift is the frequency of a microwave field that induced transition from one excited state of the hydrogen atom to another. Also results in single spectral lines appearing as two or more closely grouped thinner lines due to relativistic correction with energy shifts typically orders of magnitude smaller than the fine structure, results from the interactions of the nucleus (or nuclei, in molecules) with internally generated electric and magnetic fields (Jauch and Rohrlich, 1980).

9. The Lamb shift as it is obseverable in the spectrum is the difference between the shift of bound electron and free muon which has the same average kinetic energy (Biswas, 1998). Spectroscopic contribution of the relativistic quantum electrodynamics include Lamb's measurement of the shift between the $2S_{1/2}$ and $2P_{1/2}$ states in the hydrogen atom. This experiment is the basis for the initial formulation of the renormalization program of QED and still remains one of the most delicate tests of more sophisticated formulations of the theory. Spectroscopic measurements of the anomalous magnetic moment of the muon have also provided an important test of quantum electrodynamics (Lamb and Rutherford, 1947).

10. According to the Dirac theory for Hydrogen atom, the $2s^{1/2}$ and $2p^{1/2}$ states should have the same energies, which is degenerate. However, the interaction between the muon and the vacuum causes a tiny energy shifts on $2s^{1/2}$ .A careful experimental study by Willis Lamb and Robert Rutherford discovered that this was not in fact the case: state the state resulting in a small shift of the corresponding spectral line; it is called the Lamb Shift (Lamb and Rutherford, 1947). The effect is explained by the theory of quantum electrodynamics (Bethe, 1947; Welton, 1948; Greiner and Reinhardt, 1994); in the electromagnetic interaction itself is quantized. It is assumed that the ground state of the electromagnetic field is not zero, but rather the field undergoes "vacuum fluctuations" that interact with the muon (Karshenboim, *et, al.,* 1989).

11. The proton is the primary building block of the visible Universe, but many of its properties such as its charge radius and its anomalous magnetic moment are not well understood. The root-mean square charge radius,r_p has been determined by electron–proton scattering experiments and (Pohl, R, 2010) the present most accurate value of r_p is given by the CODATA compilation of physical constants.This value is based mainly on precision spectroscopy of atomic hydrogen and calculations of bound-state quantum electrodynamics. The improved the accuracy in the measurement of r_p is provided by muonic hydrogen

(a proton orbited by a negative muon); it is much smaller Bohr radius compared to ordinary atomic hydrogen causes enhancement of effects related to the finite size of the proton. In the Lamb shift experiment (the energy difference between the $2S_{1/2}$ and $2P_{1/2}$ states) is affected by finite size effect of proton radius. They were used pulsed laser spectroscopy to measure a muonic Lamb shift of frequency of 49,881.8867GHz. this frequency corresponds to energy shift value is given below:

12. $\Delta E = 206.294932 \times 10^{-3} \text{eV}$ (2.2)

13. The leading finite size effect in muonic hydrogen contributes to about 2% to the *2S- 2P* Lamb shift, $r_p = 0.8750\ fm$ (P.J.Mohr and B.N. Taylor, 2005) in the 1s Lamb shift in hydrogen. The Lamb shift measurement in muonic hydrogen aims to improve the precision of the proton radius by a factor of 20 compared to the value extrapolated from electron proton scattering data. The Lamb shift cannot be explained by the Schrödinger or Dirac formulations of quantum mechanics. It can, however, be explained by a theory known as quantum electrodynamics a theory whose development was intimately linked to experimental.

Muonic Hydrogen Spectral Series

14. Spectroscopic studies of the electronic hydrogen spectrum by Balmer, Paschen and Lyman resulted in the discovery of the several series of lines in the visible, near infrared and ultraviolet regions, the frequencies of which could be expressed in terms of simple integers and a simple empirical frequency known as the Rydberg constant; by arbitrarily introducing the integers as quantum numbers, Bohr was able to set up a simple model of the hydrogen atom in terms of which the Rydberg constant could be calculated with amazing accuracy Some field's extension of the Bohr's ideas to interpret the spectra of more complicated atoms is a familiar story; the resulting "old quantum mechanics" became an increasingly complicated theoretical work that involved numerous arbitrary assumptions but works fairly well in giving an account of various features observed spectra. During this period, investigation of the spectral lines shapes promoted important developments in dispersion theory. With the development of modern quantum mechanics by Heisenberg and Schrodinger, a new era in physics began; the detailed application of quantum mechanics to the vast body of spectroscopic information (Duncan, 2009).

15. Spectroscopic investigation as interpreted in terms of quantum mechanics have provided valuable, basic information regarding the structure of matter and the interaction of electromagnetic radiation with matter observation of so-called multiple fine structure in atomic spectral were interpreted in terms of electron spin and spin-orbit coupling; related interpretations of anomalous Zeeman effect provided information regarding the magnetic moment of the electron. The Spectroscopic hyperfine structure provided evidence of nuclear spin; later applications of Zeeman methods in the radio frequency region have provided highly precise values of magnetic moment of the nuclei by Nuclear Magnetic Resonance techniques. Spectroscopic studies in the γ-ray region have revealed the existence of well-excited energy states in nuclei; quantum mechanics provided an understanding of the nature of radiative transitions between energy levels in atoms, molecules and nuclei (Cagnac et al., 1994).

Result and discussion
First and second order Energy shift and correction to the wave function
First order Energy shift and correction to the wave function

To find the first-order correction to the energy we multiply the first-order equation from the left with $\psi_m^{(0)*}$ from the Eq. (3.13b) and integrate over all space.

$$\left\langle \psi_m^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle + \left\langle \psi_m^{(0)} \middle| \hat{H}^0 \middle| \psi_n^{(1)} \right\rangle = \left\langle \psi_m^{(0)} \middle| E_n^{(1)} \middle| \psi_n^{(0)} \right\rangle + \left\langle \psi_m^{(0)} \middle| E_n^{(0)} \middle| \psi_n^{(1)} \right\rangle \tag{4.1}$$

Using the properties of Hermitian operators we can rewrite the equations as

$$\left\langle \psi_m^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle + \left\langle \psi_n^{(1)} \middle| \hat{H}^0 \middle| \psi_m^{(0)} \right\rangle^* = E_n^{(1)} \left\langle \psi_m^{(0)} \middle| \psi_n^{(0)} \right\rangle + E_n^{(0)} \langle \psi_m^{(0)} | \psi_n^{(1)} \rangle \tag{4.2}$$

Using the zero-order equation we can rewrite the equation as

$$\left\langle \psi_m^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle + E_m^{(0)} \left\langle \psi_n^{(1)} \middle| \psi_m^{(0)} \right\rangle^* = E_n^{(1)} \left\langle \psi_m^{(0)} \middle| \psi_n^{(0)} \right\rangle + E_n^{(0)} \left\langle \psi_m^{(0)} \middle| \psi_n^{(1)} \right\rangle \tag{4.3}$$

Solving for $E_n^{(1)}$ we get

$$E_n^{(1)} \left\langle \psi_m^{(0)} \middle| \psi_n^{(0)} \right\rangle = \left\langle \psi_m^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle + E_m^{(0)} \left\langle \psi_n^{(1)} \middle| \psi_m^{(0)} \right\rangle^* - E_n^{(0)} \left\langle \psi_m^{(0)} \middle| \psi_n^{(1)} \right\rangle \tag{4.4}$$

Using the orthonormality of the zero-order equation we get

$$E_n^{(1)} \delta_{mn} = \left\langle \psi_m^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle + \left(E_m^{(0)} - E_n^{(0)} \right) \left\langle \psi_m^{(0)} \middle| \psi_n^{(1)} \right\rangle \tag{4.5}$$

This finally gives us:

$$E_n^{(1)} = \left\langle \psi_n^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle \tag{4.6}$$

Therefore, the first-order correction to the energy is simply given by the average of the perturbation over the unperturbed wave functions. From the above equation the first-order correction to the wave function is obtained by ensuring $m \neq n (\delta_{mn} = 0)$

$$\left(E_m^{(0)} - E_n^{(0)}\right)\left\langle\psi_m^{(0)}\middle|\psi_n^{(1)}\right\rangle = -\left\langle\psi_m^{(0)}\middle|\hat{H}^1\middle|\psi_n^{(0)}\right\rangle \tag{4.7}$$

To find the first-order correction to the wave function we expand it in terms of a complete orthonormal set of unperturbed eigenfunctions as

$$\left|\psi_n^{(1)}\right\rangle = \sum_m a_\mu\left|\psi_m^{(0)}\right\rangle, \qquad a_\mu = \left\langle\psi_m^{(0)}\middle|\psi_n^{(1)}\right\rangle$$

And using this gives

$$\left(E_m^{(0)} - E_n^{(0)}\right)a_\mu = -\left\langle\psi_m^{(0)}\middle|\hat{H}^1\middle|\psi_n^{(0)}\right\rangle \tag{4.8}$$

Since we are assuming non-degenerate energies and $m \neq n$ we can divide by the energy difference to give

$$a_m = \frac{\left\langle\psi_m^{(0)}\middle|\hat{H}'\middle|\psi_n^{(0)}\right\rangle}{E_n^{(0)} - E_m^{(0)}}$$

Therefore, the first-order correction to the wave function is given by

$$\left|\psi_n^{(1)}\right\rangle = \sum_{m\neq n} \frac{\left\langle\psi_m^{(0)}\middle|\hat{H}^{(1)}\middle|\psi_n^{(0)}\right\rangle}{E_n^{(0)} - E_m^{(0)}}\left|\psi_m^{(0)}\right\rangle \tag{4.9}$$

Where we assume that $\left\langle\psi_n^{(i)}\middle|\psi_n^{(j)}\right\rangle = \delta_{ij}$

Second order Energy shift and correction to the wave function

To find the second-order correction to the energy we multiply the first-order equation from the left with $\psi_m^{(0)*}$ from the Eq. $(3.13c)$ and integrate over all space. We begin with the second-order equation is given by:

$$\hat{H}^{(0)}\psi_n^{(2)} + \hat{H}'\psi_n^{(1)} = E_n^{(2)}\psi_n^{(0)} + E_n^{(1)}\psi_n^{(1)} + E_n^{(0)}\psi_n^{(2)} \tag{4.10}$$

Multiplying from the left with $\psi_m^{(0)*}$ from the left and integrate over all space gives

$$\left\langle\psi_m^{(0)}\middle|\hat{H}^{(0)}\middle|\psi_n^{(2)}\right\rangle + \left\langle\psi_m^{(0)}\middle|\hat{H}'\middle|\psi_n^{(1)}\right\rangle = \left\langle\psi_m^{(0)}\middle|E_n^{(2)}\middle|\psi_n^{(0)}\right\rangle + \left\langle\psi_m^{(0)}\middle|E_n^{(1)}\middle|\psi_n^{(1)}\right\rangle + \left\langle\psi_m^{(0)}\middle|E_n^{(0)}\middle|\psi_n^{(2)}\right\rangle \tag{4.11}$$

If we consider the first term it on the left hand side can be written as

$$\left\langle\psi_m^{(0)}\middle|\hat{H}^{(0)}\middle|\psi_n^{(2)}\right\rangle = \left\langle\psi_m^{(2)}\middle|\hat{H}^{(0)}\middle|\psi_n^{(0)}\right\rangle^* = E_m^{(0)}\left\langle\psi_n^{(2)}\middle|\psi_m^{(0)}\right\rangle = E_m^{(0)}\left\langle\psi_m^{(0)}\middle|\psi_n^{(2)}\right\rangle \tag{4.12}$$

Using this we can write the second order equation as

$$E_m^{(0)}\left\langle\psi_m^{(0)}\middle|\psi_n^{(2)}\right\rangle + \left\langle\psi_m^{(0)}\middle|\hat{H}'\middle|\psi_n^{(1)}\right\rangle = E_n^{(2)}\left\langle\psi_m^{(0)}\middle|\psi_n^{(0)}\right\rangle + E_n^{(1)}\left\langle\psi_m^{(0)}\middle|\psi_n^{(1)}\right\rangle + E_n^{(0)}\left\langle\psi_m^{(0)}\middle|\psi_n^{(2)}\right\rangle$$

Solving for $E_n^{(2)}$ from above equations gives

$$E_n^{(2)}\left\langle\psi_m^{(0)}\middle|\psi_n^{(0)}\right\rangle = \left\langle\psi_m^{(0)}\middle|\hat{H}'\middle|\psi_n^{(1)}\right\rangle + \left(E_m^{(0)} - E_n^{(0)}\right)\left\langle\psi_m^{(0)}\middle|\psi_n^{(2)}\right\rangle - E_n^{(1)}\left\langle\psi_m^{(0)}\middle|\psi_n^{(1)}\right\rangle \tag{4.13}$$

Using that the unperturbed wave functions are orthonormal we can eliminate the second and third term and the right since $\left\langle\psi_m^{(i)}\middle|\psi_n^{(j)}\right\rangle = \delta_{ij}\delta_{mn}$ therefore, we have

$$E_n^{(2)} = \left\langle\psi_n^{(0)}\middle|\hat{H}^{(1)}\middle|\psi_n^{(1)}\right\rangle \tag{4.14}$$

Substitution of the first-order correction to the wave function into the equation for the second-order correction to the energy gives

$$E_n^{(2)} = \left\langle\psi_n^{(0)}\middle|\hat{H}'\middle|\psi_n^{(1)}\right\rangle = \langle\psi_n^{(0)}|\hat{H}'\sum_{m\neq n}\frac{\left\langle\psi_m^{(0)}\middle|\hat{H}'\middle|\psi_n^{(1)}\right\rangle}{E_n^{(0)} - E_m^{(0)}}\left|\psi_m^{(0)}\right\rangle$$

$$= \sum_{m\neq n}\frac{\left\langle\psi_m^{(0)}\middle|\hat{H}'\middle|\psi_n^{(0)}\right\rangle\left\langle\psi_m^{(0)}\middle|\hat{H}'\middle|\psi_n^{(0)}\right\rangle^*}{\left(E_n^{(0)} - E_m^{(0)}\right)}$$

$$\sum_{m\neq n}\frac{\left|\left\langle\psi_m^{(0)}\middle|\hat{H}'\middle|\psi_n^{(0)}\right\rangle\right|^2}{\left(E_n^{(0)} - E_m^{(0)}\right)}\left(E_n^{(0)} - E_m^{(0)}\right)$$

$$E_n^{(2)} = \sum_{m \neq n} |a_m|^2 \left(E_n^{(0)} - E_m^{(0)} \right)$$

If we consider the first-order energy correction to the ground state of the atom due to the perturbation. We choose the ground state for simplicity and since the ground state energy is non-degenerate. The spectrum in the ground state energy is given by:

$$\Delta E = \langle 1,0,0|B|1,0,0\rangle \tag{4.23}$$

We are going to answer after it has been simplified with the assumption that $R \ll a_\mu$, which is needed in any case to ensure that the perturbation is sufficiently small to render our approximation scheme viable.

Thus the effect of the finite size is to shift the energy upward and the scale of the shift is determined by $\left(\frac{R}{a_\mu}\right)^2$. For electronic hydrogen, we note that the charge radius of a proton is on the order of 10^{-15}m and that the Bohr radius is on the order of 10^{-10}m so that the perturbative correction is on the order of one part in 10^{10}.

There are physical systems in which the finite size of the nucleus has a more pronounced effect. If we consider ''muonic hydrogen'', consisting of a proton and a muon. Muons are like electrons only much more massive. For muonic hydrogen the Bohr radius is smaller by a factor 10^{-2} of leading to an energy shift on the order of 10^{-6}.

The difference between actual potential and coulomb from equation (4.19):

$$H^{(1)} = \frac{3e^2}{8\pi\varepsilon_0 r_0}\left(1 - \frac{r^2}{3R}\right) + \frac{Ze^2}{r} \quad,r \, < \, R \tag{4.24}$$

The first-order correction to the ground state energy is calculated by using the first order perturbation theory.

$$2 \quad E_0^{(1)} = \int_0^R \psi(r)\left(V_{finite} - V_{point}\right)\psi(r)\, d^3r$$

$$3 \quad E_0^{(1)} = \int_0^R \psi^*(r)H^{(1)}\psi(r)d^3r \tag{4.25}$$

$$4 \quad \text{But,} \int d^3r = \int \sin\theta d\theta d\varphi r^2 dr = 4\pi \int r^2 dr$$

Then,

$$\Delta E_0^{(1)} = 4\pi \int_0^R \psi(r)\, H^{(1)}\psi(r)r^2 dr$$

Now, from Taylor expansion we can have (Arfken and Weber, 2005):

$$5 \quad e^{-Zr/a_\mu} = 1 - \frac{Zr}{a_\mu} + \frac{Zr^2}{2a_\mu} - \frac{Zr^3}{6a_\mu} + \cdots$$

If,

$$\left|\frac{Zr}{a_\mu}\right| \ll \, 1 \text{ ,then } e^{-Zr/a_\mu} = 1 - \frac{Zr}{a_\mu}$$

Inserting approximate Hydrogenic ground state wave function: $\psi_{1s}^{(0)} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_\mu}\right)^{3/2} e^{\frac{-Zr}{\alpha_\mu}}$

$$\psi_{1s}^{(0)} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_\mu}\right)^{3/2}\left(1 - \frac{Zr}{a_\mu}\right)$$

$$\Delta E_0^{(1)} = 4\pi \int_0^R \left(\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_\mu}\right)^{3/2}\left(1 - \frac{Zr}{a_\mu}\right)\right)^* \left(-\frac{3e^2}{8\pi\varepsilon_0 r_0} + \frac{3r^2 e^2}{24\pi\varepsilon_0 r_0^4}\right)\left(\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_\mu}\right)^{3/2}\left(1 - \frac{Zr}{a_\mu}\right)\right)r^2 dr$$

$$+ 4\pi \int_0^R \left(\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_\mu}\right)^{3/2}\left(1 - \frac{Zr}{a_\mu}\right)\right)^* \frac{e^2}{4\pi\varepsilon_o r}\left(\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_\mu}\right)^{3/2}\left(1 - \frac{Zr}{a_\mu}\right)\right)\frac{e^2}{4\pi\varepsilon_o r}r^2 dr$$

$$\Delta E_0^{(1)} = 4\pi\frac{Z4}{\pi a_\mu^3}\int_0^R \left(1 - \frac{Zr}{a_\mu}\right)^2\left(-\frac{3e^2}{8\pi\varepsilon_0 r_0} + \frac{3r^2 e^2}{24\pi\varepsilon_0 r_0^4}\right)r^2 dr$$

$$+ 4\pi\frac{Z4}{\pi a_\mu^3}\int_0^R \left(1 - \frac{Zr}{a_\mu}\right)^2 \frac{e^2}{4\pi\varepsilon_o r}r^2 dr$$

$$= \frac{16Z}{a_\mu^3} \int_0^R \left(1 - \frac{2Zr}{a_\mu} + \frac{Zr^2}{a_\mu^2}\right)\left(-\frac{3e^2}{8\pi\varepsilon_0 r_0} + \frac{3r^2 e^2}{24\pi\varepsilon_0 r_0^4}\right) r^2 dr + \frac{16Z}{a_\mu^3} \int_0^R \left(1 - \frac{2Zr}{a_\mu} + \frac{Zr^2}{a_\mu^2}\right)\frac{e^2}{4\pi\varepsilon_o r} r^2 dr$$

$$= \frac{16Z}{a_\mu^3} \int_0^R \left(-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 R^3}\right) r^2 dr + \left(\frac{6e^2 r}{4\pi\varepsilon_0 a_\mu R} - \frac{6e^2 r^3}{24\pi\varepsilon_0 a_\mu R^3}\right) r^2 dr$$

$$+ \left(-\frac{3e^2 r^2}{8\pi\varepsilon_0 a_\mu^2 R} + \frac{3e^2 r^4}{24\pi\varepsilon_0 a_\mu^2 R^3}\right) r^2 dr$$

$$+ \frac{16Z}{a_\mu^3} \int_0^R \left(\frac{e^2}{4\pi\varepsilon_o r} r^2 dr - \frac{2e^2 Zr}{4\pi\varepsilon_o r a_\mu} r^2 dr + \frac{Ze^2 r^2}{4\pi\varepsilon_o r a_\mu^2} r^2 dr\right)$$

$$= -\frac{48e^2 Z}{8\varepsilon_0 R a_\mu^3}\int_0^R r^2 dr + \frac{48e^2 Z}{24\varepsilon_0 R^3 a_\mu^3}\int_0^R r^4 dr - \frac{96e^2 Z}{4\varepsilon_0 a_\mu R a_\mu^3}\int_0^R r^3 dr - \frac{96e^2 Z}{24\varepsilon_0 R^3 a_\mu^3}\int_0^R r^5 dr + \frac{48e^2 Z}{8\varepsilon_0 a_\mu^2 R a_\mu^3}\int_0^R r^4 dr$$

$$+ \frac{48e^2 Z}{8\varepsilon_0 R a_\mu^3}\int_0^R r^2 dr + \frac{16e^2 Z}{4r_0\varepsilon_o a_\mu^3}\int_0^R r^2 dr + \frac{32e^2 ZZ}{4\varepsilon_o R a_\mu a_\mu^3}\int_0^R r^3 dr + \frac{16Ze^2 Z}{4\varepsilon_o a_\mu^2 R a_\mu^3}\int_0^R r^2 dr$$

$$= -\frac{6e^2 Z}{\varepsilon_0 R a_\mu^3}\int_0^R r^2 dr + \frac{2e^2 Z}{\varepsilon_0 R^3 a_\mu^3}\int_0^R r^4 dr - \frac{24e^2 Z}{\varepsilon_0 a_\mu R a_\mu^3}\int_0^R r^3 dr - \frac{4e^2 Z}{\varepsilon_0 a_\mu R^3 a_\mu^3}\int_0^R r^5 dr + \frac{6e^2 Z}{\varepsilon_0 a_\mu^2 R a_\mu^3}\int_0^R r^4 dr$$

$$+ \frac{6e^2 Z}{\varepsilon_0 R a_\mu^3}\int_0^R r^2 dr + \frac{4e^2 Z}{R\varepsilon_0 a_\mu^3}\int_0^R r^2 dr + \frac{8e^2 Z}{\varepsilon_o R a_\mu a_\mu^3}\int_0^R r^3 dr + \frac{4Ze^2 Z}{\varepsilon_0 a_\mu^2 R a_\mu^3}\int_0^R r^2 dr$$

$$= -\frac{6e^2 Z}{\varepsilon_0 r_0 a_\mu^3}\left(\frac{r^3}{3}\right)_0^R + \frac{2e^2 Z}{\varepsilon_0 r_0^3 a_\mu^3}\left(\frac{r^5}{5}\right)_0^R - \frac{24e^2 Z}{\varepsilon_0 a_\mu r_0 a_\mu^3}\left(\frac{r^4}{4}\right)_0^R - \frac{4e^2 Z}{\varepsilon_0 a_\mu r_0^3 a_\mu^3}\left(\frac{r^6}{6}\right)_0^R + \frac{6e^2 Z}{\varepsilon_0 a_\mu^2 R a_\mu^3}\left(\frac{r^5}{5}\right)_0^R$$

$$+ \frac{6e^2 Z}{\varepsilon_0 R a_\mu^3}\left(\frac{r^3}{3}\right)_0^R + \frac{4e^2 Z}{R\varepsilon_o a_\mu^3}\left(\frac{r^3}{3}\right)_0^R + \frac{8e^2 Z}{\varepsilon_o R a_\mu a_\mu^3}\left(\frac{r^4}{4}\right)_0^R + \frac{4Ze^2 Z}{\varepsilon_o a_\mu^2 r_0 a_\mu^3}\left(\frac{r^3}{3}\right)_0^R$$

$$= -\frac{6e^2 Z}{\varepsilon_0 R a_\mu^3}\left(\frac{R^3}{3}\right) + \frac{2e^2 Z}{\varepsilon_0 R^3 a_\mu^3}\left(\frac{R^5}{5}\right) - \frac{24e^2 Z}{\varepsilon_0 a_\mu R a_\mu^3}\left(\frac{R^4}{4}\right) - \frac{4e^2 Z}{\varepsilon_0 a_\mu R^3 a_\mu^3}\left(\frac{R^6}{6}\right) + \frac{6e^2 Z}{\varepsilon_0 a_\mu^2 R a_\mu^3}\left(\frac{R^5}{5}\right) + \frac{6e^2 Z}{\varepsilon_0 R a_\mu^3}\left(\frac{R^3}{3}\right)$$

$$+ \frac{4e^2 Z}{R\varepsilon_o a_\mu^3}\left(\frac{R^3}{3}\right) + \frac{8e^2 Z}{\varepsilon_0 R a_\mu a_\mu^3}\left(\frac{R^4}{4}\right) + \frac{4Ze^2 Z}{\varepsilon_o a_\mu^2 R a_\mu^3}\left(\frac{R^3}{3}\right)$$

$$= -\frac{2e^2 ZR^2}{\varepsilon_0 a_\mu^3} + \frac{2e^2 ZR^2}{5\varepsilon_0 a_\mu^3} - \frac{6e^2 ZR^3}{\varepsilon_0 a_\mu^4} - \frac{2e^2 ZR^3}{3\varepsilon_0 a_\mu^4} + \frac{6e^2 ZR^4}{5\varepsilon_0 a_\mu^5} + \frac{2e^2 ZR^2}{\varepsilon_0 a_\mu^3} + \frac{4e^2 ZR^2}{3\varepsilon_0 a_\mu^3} + \frac{8e^2 ZR^3}{\varepsilon_o a_\mu^4} + \frac{4Ze^2 ZR^2}{3\varepsilon_o a_\mu^5}$$

$$E_0^{(1)} = \frac{4}{5}\frac{Z^4 e^2}{2a_\mu}\left(\frac{R}{a_\mu}\right)^2 \tag{4.26}$$

This is energy shifty $E_0^{(1)}$ for ground state energy for muonic hydrogen atom and ΔE is proportional to Z^4 and R^2 therefore the most noticeable effect is deep inside large Z nuclei and proportional to the size of proton.

Energy Shifts of the 1s, 2s and 2p Hydrogen Like-atom

Using first-order perturbation theory and hydrogenic eigenfunctions, we will calculate the energy shift to the ground state of a hydrogenic atom (non-relativistic model without spin) due to the finite size of the nucleus. Using the fact that the proton radius is much smaller than the Bohr radius and under the assumption for the nucleus is much smaller than the atomic radius ($r \ll a_\mu$); the energy change is approximately proportional to the nuclear mean square radius. The energy correction for a uniformly charged spherical nucleus of radius R will be evaluated after calculating the energy shifts of the 1s, 2s and 2p states.

In the above Fig. 4.1, the solid lines represent unperturbed energy levels, while the dotted lines represent perturbed energy levels. It is seen that the unperturbed energy level of d state almost overlaps the perturbed energy level.

The shifts of energy levels of s states are larger than these of p states because a muon in s state has a greater probability of staying in the $r\sim0$ region than a muon in p and d states. Besides, the larger the quantum number l the greater is the corresponding orbital angular momentum and the farther is the spread of p cloud from the center, leading to less energy correction.

If the proton where a point particle of charge Ze, the coulomb potential energy of the muon (electron) would be:

$$V_{coulmb} = -\frac{Ze^2}{4\pi\varepsilon_o r} \qquad (4.27)$$

In our zeroth-order Hamiltonian, we used the Coulomb potential everywhere. The perturbation is the difference between the exact potential and the Coulomb potential.

From the Eq. (4.24), we have $H^{(1)} = V - V_0$ and consider it as perturbation. Then the perturbation Hamiltonian of the system is given by:

$$\hat{H}' = \frac{3e^2}{8\pi\varepsilon_0 r_0}\left(1 - \frac{r^2}{3R}\right) + \frac{Ze^2}{r} \ , r \ < \ R \qquad (4.28)$$

When $r \ < \ R, H' \ > \ 0$ and the energy levels shift up on account of the perturbation.

The energy shifty is $\Delta E_{nl} = \int \psi_{nl}^* H^{(1)} \psi_{nl} dv$

This difference is only nonzero inside the nucleus. Only the equation for R(r) depends on the potential energy $V(r)$ and the solution to the radial equation, $R(r)$, depends upon the quantum number l, and also upon principal quantum number or state n for which the wave function be finite. Since $H^{(1)}$ is only a function of r, the angular parts of the integral equals to 1 . Because y_{lm} are normalize $\int\int |y_{lm}(\theta, \varphi)|^2 \sin\theta d\theta d\varphi = 1$ 0r $\int\int d\Omega Y_{lm}^*(\theta, \varphi)Y_{lm}(\theta, \varphi) = \delta_{ll}.\delta_{mm}$. So we only need to do the radial integral part.

$$\Delta E = \int_0^R R_{nl}^* \hat{H}' R_{nl} r^2 dr \qquad (4.29)$$

Now, since we have *from* Eq. (3.4), we have $\psi(r, \theta, \phi) \ = \ R(r)\phi(\theta, \phi)$ (Beiser, 2003), which we re-write as:

$$\psi(r) = \ R_{nl}(r)\phi(\theta, \phi) \qquad (4.30)$$

The energy shift at ground state by first order correction for radial solution of Schrödinger equation is:

$$6 \quad \Delta E_{nl} = \int \psi_{nl}^* \hat{H}'(r)\psi_{nl} d^2 r \qquad (4.31)$$

Since we have $\psi(r) \ = \ R_{nl}(r)\phi(\theta, \phi)$:

$$7 \quad \Delta E_{nl} = \int R_{nl}^* \hat{H}'(r)R_{nl} d^2 r \qquad (4.32)$$

And this can be integrated over the radial path, and then we obtain:

$$\Delta E_{nl} = \int_0^R R_{nl}^* \hat{H}' R_{nl} r^2 dr \qquad (4.33)$$

The energy shift of 1s state to first and second-order perturbation (ΔE_{1s})

For 1s state the zero-order energy for muonic hydrogen is $E_1^{(0)} \ = \ -2815.2 eV$

The energy shifty of $1s$ state to first order perturbation is given by

$$\Delta E_{1s} = \int R_{1s}^* \hat{H}' R_{1s} r^2 dr \text{ and } R_{1s} = 2\left(\frac{1}{a_\mu}\right)^{3/2} e^{-r/a_\mu} \qquad (4.34)$$

Now, from Taylor expansion we can have (Arfken and Weber, 2005):

$$e^{-r/a_\mu} = 1 - \frac{r}{a_\mu} + \frac{r^2}{2a_\mu} - \frac{r^3}{6a_\mu} + \cdots \qquad (4.35)$$

If,

$$\left|\frac{r}{a_\mu}\right| \ll \ 1 \text{ , then } e^{-r/a_\mu} = 1 - \frac{r}{a_\mu} \qquad (4.36)$$

So we get,

$$R_{1,0} = 2\left(\frac{1}{a_\mu}\right)^{3/2}\left(1 - \frac{r}{a_\mu}\right)$$

And, from this approximation the first order energy correction in perturbation over radial will be given by

$$\Delta E_{1s} = \int R_{10}^* \hat{H}'(r)R_{10} r^2 dr \qquad (4.37)$$

We have $\hat{H}'$ from Eq. (4.28)

$$H^{(1)} = -\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 R^3} + \frac{e^2}{4\pi\varepsilon_o r}$$

Inserting the values for each term, we get,

$$\Delta E_{1s} = \int_0^R \left[2 \left(\frac{1}{a_\mu}\right)^{3/2} \left(1 - \frac{r}{a_\mu}\right) \right]^* \left[-\frac{3e^2}{8\pi\varepsilon_0 r_0}\left(1 - \frac{r^2}{3r_0}\right) + \left(\frac{e^2}{4\pi\varepsilon_o r}\right) \right] 2\left(\frac{1}{a_\mu}\right)^{3/2}\left(1\right.$$

$$\left. -\frac{r}{a_\mu}\right) r^2 dr \qquad\qquad \Delta E_{1s}$$

$$= \int_0^R \left[2\left(\frac{1}{a_\mu}\right)^{3/2}\left(1 - \frac{r}{a_\mu}\right)\right]^2 \left[-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 R^3} + \frac{e^2}{4\pi\varepsilon_o r}\right] r^2 dr \qquad (4.38)$$

Taking out constant:

$$\Delta E_{1s}^{(1)} = \frac{4}{a_\mu^3} \int_0^R \left(1 - \frac{r}{a_\mu}\right)^2 \left[-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 r_0^3} + \frac{e^2}{4\pi\varepsilon_o r}\right] r^2 dr$$

$$\boldsymbol{\Delta E_{1s}} = \frac{4}{a_\mu^3} \int_0^R \left(1 - \frac{2r}{a_\mu} + \frac{r^2}{a_\mu^2}\right)\left[-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 R^3} + \frac{e^2}{4\pi\varepsilon_o r}\right] r^2 dr$$

Expanding the terms, we obtain:

$$\Delta E_{1s} = \frac{4}{a_\mu^3} \int_0^R \left(-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 R^3} + \frac{e^2}{4\pi\varepsilon_o r} + \frac{6e^2 r}{4\pi\varepsilon_0 a_\mu R} - \frac{6e^2 r^3}{24\pi\varepsilon_0 a_\mu R^3} - \frac{2e^2}{4\pi\varepsilon_0 a_\mu} - \frac{3e^2 r^2}{8\pi\varepsilon_0 a_\mu^2 R} \right.$$
$$\left. + \frac{3e^2 r^4}{24\pi\varepsilon_0 a_\mu^2 R^3} + \frac{e^2 r}{4\pi\varepsilon_0 a_\mu^2} \right) r^2 dr$$

And integration with respect to r gives us:

$$= \frac{4}{a_\mu^3}\left[-\frac{e^2 r^3}{8\pi\varepsilon_0 R} + \frac{3e^2 r^5}{5(24\pi\varepsilon_0 R^3)} + \frac{e^2 r^2}{8\pi\varepsilon_0} + \frac{6e^2 r^4}{4(8\pi\varepsilon_0 a_\mu R)} - \frac{6e^2 r^6}{6(24\pi\varepsilon_0 a_\mu R^3)} - \frac{2e^2 r^3}{3(4\pi\varepsilon_0 a_\mu)} - \frac{3e^2 r^5}{5(8\pi\varepsilon_0 a_\mu^2 R)} \right.$$
$$\left. + \frac{3e^2 r^7}{7(24\pi\varepsilon_0 a_\mu^2 R)} + \frac{e^2 r^4}{4(4\pi\varepsilon_0 a_\mu^2)}\right]_0^R$$

This will be reduced to:

$$= \frac{4}{a_\mu^3}\left(-\frac{3e^2 R^2}{3(8\pi\varepsilon_0 R)} + \frac{3e^2 R^2}{5(24\pi\varepsilon_0)} + \frac{e^2 r_0^2}{2(4\pi\varepsilon_0)} + \frac{6e^2 R^3}{4(8\pi\varepsilon_0 a_\mu)} - \frac{6e^2 R^3}{6(24\pi\varepsilon_0 a_\mu)} - \frac{2e^2 R^3}{3(4\pi\varepsilon_0 a_\mu)} - \frac{3e^2 R^4}{5(8\pi\varepsilon_0 a_\mu^2)} \right.$$
$$\left. + \frac{3e^2 R^6}{7(24\pi\varepsilon_0 a_\mu^2)} + \frac{e^2 R^4}{4(4\pi\varepsilon_0 a_\mu^2)} \right)$$

$$= \frac{4}{a_\mu^3}\frac{e^2 R^2}{\pi\varepsilon_0}\left(-\frac{1}{8} + \frac{3}{120} + \frac{1}{8} + \frac{6R}{32 a_\mu} - \frac{R}{24 a_\mu} - \frac{R}{6 a_\mu} - \frac{3R^2}{40 a_\mu^2} + \frac{3R^4}{168 a_\mu^2} + \frac{R^2}{16 a_\mu^2} \right)$$

Simplifying and adding like terms together:

$$= \frac{4}{a_\mu^3}\frac{e^2 R^2}{\pi\varepsilon_0}\left(\frac{3}{120} - \frac{16 r_0}{768 a_\mu} - \frac{8 r_0^2}{640 a_\mu^2} + \frac{3 r_0^4}{168 a_\mu^2} \right)$$

$$= \frac{4}{a_\mu^3}\frac{e^2 R^2}{8\pi\varepsilon_0}\left(\frac{1}{5} - \frac{R}{6 a_\mu} \right)$$

Since R is tiny compared to a_μ, and we make the approximation of the energy shift for $1s$ state normal ignoring the second term becomes:

$$\Delta E_{1s} = \frac{2}{5}\frac{e^2}{4\pi\varepsilon_0}\frac{R^2}{a_\mu^3} \qquad\qquad (4.39)$$

Which is first order energy shift for 1s state$(n = 1, l = 0)$. Energy shift for this state is proportional to the size of proton $(\Delta E \simeq r_p^2)$(Yung − Kuo, 2000). we know that the Bohr radius for electronic hydrogen is given by:

$$a_0 = \frac{4\pi\varepsilon_0 \hbar^2}{4\pi^2 m_e e^2} = 5.2977 x 10^{-11} \text{m}$$

For muonic hydrogen we have,

$$a_\mu = \frac{4\pi\varepsilon_0 \hbar^2}{4\pi^2 (207 m_e)e^2} = \frac{5.29 \times 10^{-11}\text{m}}{207} = 2.5 \times 10^{-12} m$$

Substituting constant values to above equation (4.30) yields value:

$$\Delta E_{1s} = \frac{2 \times (-1.60218 x 10^{-19})^2 x (1.05 \times 10^{-15}\text{m})^2}{(5 x 4 \times 3.14) \times (8.85419 x 10^{-12}\,^{F}/_{m}) \times (2.5 \times 10^{-12}\text{m})^3}$$
$$= 0.00059119352 \times 10^{-20}\text{J}$$
$$\Delta E_{1s} = 0.591 \times 10^{-23}\text{J} = 0.36 \times 10^{-3}\text{eV} \qquad\qquad (4.40)$$

Which is first order energy correction for 1s state. The total energy shift for ground state by first order will be calculated as follows. We know that unperturbed Hamiltonian has an eigenvalue for muonic hydrogen is:

$$E_\mu = \frac{m_\mu}{m_e} E_e = 207(-13.6\text{eV}) = -2815.2\text{eV}$$

$$E_1^{(0)} = -2815.2\text{eV}$$

$$E_n = E_n^{(0)} + E_n^{(1)}$$

$$E_1^{(1)} = E_n^{(0)} + \left\langle \psi_0^{(0)} | H' \psi_0^{(0)} \right\rangle$$

$$E_1^{(1)} = -2815.2\text{eV} + 0.36 \times 10^{-3}\text{eV} \cong -2815.20036\text{eV} \quad (4.41)$$

This is approximately equivalent to ground state energy: $\cong -2815.20036\text{eV}$

Since for some systems the first order perturbation vanishes, one can consider the second order perturbation theory that always makes the energy of the ground state, lower (in comparison to the unperturbed one).

The total energy shift for ground state by first order will be calculated as follows. Since for some systems the first order perturbation vanishes, one can consider the second order perturbation theory that always makes the energy of the ground state, lower (in comparison to the unperturbed one). The second order correction done straight forwardly form our estimate for ψ_n^1

$$E_1^{(2)} = \sum_{m \neq 1} \frac{\left| \left\langle \psi_m^{(0)} | \hat{H}' | \psi_1^{(0)} \right\rangle \right|^2}{E_1^{(0)} - E_m^{(0)}}$$

$$= \frac{\left| \left\langle \psi_2^{(0)} | \hat{H}' | \psi_1^{(0)} \right\rangle \right|^2}{E_1^{(0)} - E_2^{(0)}} + \frac{\left| \left\langle \psi_3^{(0)} | \hat{H}' | \psi_1^{(0)} \right\rangle \right|^2}{E_1^{(0)} - E_3^{(0)}} + \frac{\left| \left\langle \psi_4^{(0)} | \hat{H}' | \psi_1^{(0)} \right\rangle \right|^2}{E_1^{(0)} - E_4^{(0)}}$$

$$= \langle \psi_n^0 | H' | \sum_{m \neq n} c_{nm} \psi_n^0 \rangle = \sum_{m \neq n} c_{nm} \langle \psi_n^0 | H' | \psi_n^0 \rangle$$

$$\sum_{m \neq n} \frac{\langle \psi_m^0 | H' \psi_n^0 \rangle}{E_n^0 - E_m^0} \langle \psi_n^0 | H' | \psi_m^0 \rangle$$

But $H^{(1)}$ is also Hermitian (gives us a energy which is always real!) so

$$\langle \psi_n^0 | H' | \psi_m^1 \rangle = \left\langle H' | \psi_n^{(0)} | \psi_m^{(0)} \right\rangle = \left\langle \psi_m^{(0)} | H' | \psi_n^{(1)} \right\rangle^* \text{ so, then we have}$$

$$\langle \psi_m^0 | H' \psi_n^0 \rangle \langle \psi_n^0 | H' | \psi_m^0 \rangle = \langle \psi_m^0 | H' \psi_n^0 \rangle \langle \psi_m^0 | H' \psi_n^0 \rangle^* = |\langle \psi_m^0 | H' \psi_n^0 \rangle|^2 \text{ and finally}$$

$$E_1^{(2)} = \sum_{m \neq n} \frac{|\langle \psi_m^0 | H' \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

$$E_1 = E_n^0 + E_n^1 + E_n^2 + \cdots$$

$$\text{so, } E_1^{(2)} = E_n^0 + E_n^1 + \sum_{m \neq n} \frac{|\langle \psi_m^0 | H' \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

$$E_1^{(2)} = -2815.2\text{eV} - 2815.20036\text{eV} + \frac{|0.36 \times 10^{-3}\text{eV}|^2}{-2815.20036\text{eV} - 0.36 \times 10^{-3}\text{eV}}$$

$$= -5630.40036eV + \left(\frac{0.0000001296\text{eV}}{-2815.200396\text{eV}} \right)$$

$$= -5630.40036\text{eV} - 4.60357991509745 \times 10^{-11}$$

$$E_1^{(2)} = -5630.40036000005\text{eV} \quad (4.42)$$

The energy levels, at least the S-state levels, are affected by the finite proton size and this energy shift for this state is proportional to the size of proton. The size dependent energy shift was worked out non-relativistically and leading order in perturbation theory (in modern notation) given by: $(R = r_p)$ then the radius of proton is given by:

$$r_p^2 = \frac{20\pi\varepsilon_0 a_\mu^3 \Delta E_{1s}}{2e^2}, \quad r_p = \sqrt{\frac{10\pi\varepsilon_0 a_\mu^3 \Delta E_{1s}}{e^2}} \quad (4.43)$$

And energy shift for this state is proportional to the size of proton ($\Delta E \simeq R_P$) and given by:

$$r_p = \sqrt{\frac{10\pi\varepsilon_0 a_\mu^3 \Delta E_{shift}}{e^2}}$$

We have energy shifty value from the Literature review from the Lamb shift experiment (the energy difference between the $2S_{1/2}$ and $2P_{1/2}$ states) is from Eq. (2.2)

$$(\Delta E = 330.4944298 \times 10^{-22}\text{J} = 206.294932 \times 10^{-3}\text{eV})$$

$$= \sqrt{\frac{10 \times 3.1428571428571 \times 8.85 \times 10^{-12}\,\text{F}/\text{m} \times 0.015625 \times 10^{-30}\text{m}^3(330.4944298 \times 10^{-22}\text{J})}{2.5669807524x10^{-38}\,\text{C}^2}}$$

$$\sqrt{0.559001226391681 \times 10^{-30}\text{m}^2\left(\frac{\text{F}}{\text{m}} \times \text{m}^3\right)}$$

$$r_P = 0.7476714781026306386\text{fm} \tag{4.44}$$

Lamb shifty in (μp) for transition $\left(2p_{3/_2} - 2s_{1/_2}\right)$ was measured in PSI (Paul Scherrer Institute) with results of frequency of 49881.88 GHz or energy of 206.2949meV and from this experiment the value of proton charge radius $r_p = 0.8408739$ Fermi;.and from the electronic hydrogen CODATA value $r_p = 0.8768$Fermi

The energy shift of 2s state to first and second-order perturbation (ΔE_{2s})

For 2s state we have zero-order energy shift $E_2^{(0)} = -703.8$eV

Now from Eq. (4.12), if $\left|\dfrac{r}{a_\mu}\right| \ll 1$, then, $e^{-r/2a_\mu} = 1 - \dfrac{r}{2\alpha_\mu}$

so, $\qquad R_{2,0} = \dfrac{1}{2\sqrt{2}}\left(\dfrac{1}{a_\mu}\right)^{3/2}\left(2 - \dfrac{r}{a_\mu}\right)\left(1 - \dfrac{r}{2a_\mu}\right)$

Then the first order energy shift for this state will be (Phillips, 2003):

$$\Delta E_{2s} = \int R_{nl}^* H^1 R_{nl} r^2 dr \tag{4.45}$$

$$\Delta E_{2s} = \int R_{2s}^* H^1 R_{2s} r^2 dr$$

$$\Delta E_{2s} = \int_0^R \left[\frac{1}{2\sqrt{2}}\left(\frac{1}{a_\mu}\right)^{3/2}\left(2 - \frac{r}{a_\mu}\right)\left(1 - \frac{r}{2a_\mu}\right)\right]^* \left[-\frac{3e^2}{8\pi\varepsilon_0 r_0}\left(1\text{-}\frac{r^2}{3R^2}\right)\right.$$

$$\left. + \left(\frac{e^2}{4\pi\varepsilon_0 r}\right)\right]\left(\frac{1}{2\sqrt{2}}\left(\frac{1}{a_\mu}\right)^{3/2}\left(2 - \frac{r}{a_\mu}\right)\left(1 - \frac{r}{2a_\mu}\right)\right) r^2 dr$$

$$\Delta E_{2s} = \int_0^R \left[\frac{1}{2\sqrt{2}}\left(\frac{1}{a_\mu}\right)^{3/2}\left(2 - \frac{r}{a_\mu}\right)\left(1 - \frac{r}{2a_\mu}\right)\right]^2 \left[-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 R^3} + \frac{e^2}{4\pi\varepsilon_0 r}\right] r^2 dr$$

$$\Delta E_{2s} = \frac{1}{8a_\mu^3}\int_0^R \left[\left(2 - \frac{r}{a_\mu}\right)\left(1 - \frac{r}{2a_\mu}\right)\right]^2 \left[-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 R^3} + \frac{e^2}{4\pi\varepsilon_0 r}\right] r^2 dr$$

Taking out constant term:

$$= \frac{1}{8a_\mu^3}\int_0^R \left[\left(4 - \frac{8r}{a_\mu} + \frac{4r^2}{a_\mu^2} - \frac{r^3}{a_\mu^2} - \frac{r^2}{a_\mu^3} + \frac{r^4}{a_\mu^4}\right)\left(-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 R^3} + \frac{e^2}{4\pi\varepsilon_0 r}\right)\right] r^2 dr$$

Expanding the terms, we obtain:

$$= \frac{1}{8a_\mu^3}\int_0^R \left[4\left(-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 R^3} + \frac{e^2}{4\pi\varepsilon_0 r}\right) - \frac{8r}{a_\mu}\left(-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 r_0^3} + \frac{e^2}{4\pi\varepsilon_0 r}\right)\right.$$

$$+ \frac{4r^2}{a_\mu^2}\left(-\frac{3e^2}{8\pi\varepsilon_0 r_0} + \frac{3e^2 r^2}{24\pi\varepsilon_0 r_0^3} + \frac{e^2}{4\pi\varepsilon_0 r}\right) - \frac{r^3}{a_\mu^2}\left(-\frac{3e^2}{8\pi\varepsilon_0 r_0} + \frac{3e^2 r^2}{24\pi\varepsilon_0 r_0^3} + \frac{e^2}{4\pi\varepsilon_0 r}\right)$$

$$\left. - \frac{r^2}{a_\mu^3}\left(-\frac{3e^2}{8\pi\varepsilon_0 r_0} + \frac{3e^2 r^2}{24\pi\varepsilon_0 r_0^3} + \frac{e^2}{4\pi\varepsilon_0 r}\right) + \frac{r^4}{a_\mu^4}\left(-\frac{3e^2}{8\pi\varepsilon_0 r_0} + \frac{3e^2 r^2}{24\pi\varepsilon_0 r_0^3} + \frac{e^2}{4\pi\varepsilon_0 r}\right)\right] r^2 dr$$

And integration with respect to r gives us:

$$= \frac{1}{8a_\mu^3}\left[-\frac{e^2 r^3}{2\pi\varepsilon_0 r_0} + \frac{e^2 r^5}{10\pi\varepsilon_0 r_0^3} + \frac{e^2 r^2}{2\pi\varepsilon_0} + \frac{3e^2 r^4}{4\pi\varepsilon_0 a_\mu r_0} - \frac{e^2 r^6}{6\pi\varepsilon_0 a_\mu r_0^3} - \frac{2e^2 r^3}{3\pi\varepsilon_0} - \frac{3e^2 r^5}{10\pi\varepsilon_0 a_\mu^2 r_0} + \frac{e^2 r^7}{14\pi\varepsilon_0 a_\mu^2 r_0^3}\right.$$

$$+ \frac{e^2 r^4}{4\pi\varepsilon_0 a_\mu^2} + \frac{e^2 r^6}{16\pi\varepsilon_0 a_\mu^2 r_0} - \frac{e^2 r^8}{64\pi\varepsilon_0 a_\mu^2 r_0^3} - \frac{e^2 r^5}{20\pi\varepsilon_0 a_\mu^2} - \frac{3e^2 r^5}{40\pi\varepsilon_0 a_\mu^3 r_0} + \frac{e^2 r^7}{56\pi\varepsilon_0 a_\mu^3 r_0^3} - \frac{e^2 r^4}{16\pi\varepsilon_0 a_\mu^3}$$

$$\left. - \frac{3e^2 r^7}{32\pi\varepsilon_0 a_\mu^4 r_0} + \frac{e^2 r^9}{32\pi\varepsilon_0 a_\mu^4 r_0^3} + \frac{e^2 r^6}{24\pi\varepsilon_0 a_\mu^4}\right]_0^R$$

This is reduced to:

$$= \frac{1}{8a_\mu^3}\left[-\frac{e^2 r_0^2}{2\pi\varepsilon_0} + \frac{e^2 r_0^2}{10\pi\varepsilon_0} + \frac{e^2 r_0^2}{2\pi\varepsilon_0} + \frac{3e^2 r_0^2}{4\pi a_\mu\varepsilon_0} - \frac{e^2 r_0^2}{6\pi a_\mu\varepsilon_0} - \frac{2e^2 r_0^3}{3\pi\varepsilon_0} - \frac{3e^2 r_0^4}{10\pi a_\mu^2\varepsilon_0} + \frac{e^2 r_0^4}{14\pi a_\mu^2\varepsilon_0} + \frac{e^2 r_0^4}{4\pi a_\mu^2\varepsilon_0} + \frac{e^2 r_0^4}{56\pi a_\mu^3\varepsilon_0}\right.$$
$$\left.- \frac{e^2 r_0^4}{16\pi a_\mu^3\varepsilon_0} - \frac{3e^2 r_0^6}{32\pi a_\mu^4\varepsilon_0} + \frac{e^2 r_0^6}{72\pi a_\mu^4\varepsilon_0} + \frac{e^2 r_0^6}{24\pi a_\mu^4\varepsilon_0}\right]$$

Taking out common terms, we obtain:

$$= \frac{1e^2 r_0^2}{8\pi\varepsilon_0 a_\mu^3}\left[-\frac{1}{2} + \frac{1}{10} + \frac{1}{2} + \frac{3r_0}{4a_\mu} - \frac{r_0}{6a_\mu} - \frac{2r_0}{3} - \frac{3r_0^2}{10a_\mu^2} + \frac{r_0^2}{14a_\mu^2} + \frac{r_0^2}{4a_\mu^2} + \frac{r_0^2}{56a_\mu^3} - \frac{r_0^2}{16a_\mu^3} - \frac{3r_0^4}{32a_\mu^4} + \frac{r_0^4}{72a_\mu^4} + \frac{R^4}{24a_\mu^4}\right]$$

The energy shifty in the 2s state is therefore given by

$$\Delta E_{2s} = \frac{1}{20}\frac{e^2}{4\pi\varepsilon_0}\frac{R^2}{a_\mu^3} \tag{4.46}$$

Substituting the value of constants we have,

$$= \left(\frac{(-1.602 \times 10^{-19})^2 \times (1 \times 10^{-15}m)^2}{20 \times 4 \times \pi \times 8.85 \times 10^{-12}\,{}^F/_m\,(0.254807692308 \times 10^{-10}m\quad)^3}\right)$$
$$\left(\frac{2.5669807524}{34.73625}\right) \times 10^{-38-30+42}J$$
$$\Delta E_{2s} = 0.0731 0^{-18}J = 0.04611707eV \tag{4.47}$$

Which is first order energy correction for 2s state. The total energy shift for ground state by first order will be calculated as follows. We know that unperturbed Hamiltonian has an eigenvalue for muonic hydrogen is:

$$E_\mu = \frac{m_\mu}{m_e}E_e = 207(-13.6eV) = -2815.2eV$$
$$E_n^{(o)} = -2815.2eV$$
$$E_n = E_n^{(o)} + E_n^{(1)}$$
$$E_1^{(1)} = E_n^{(o)} + \left\langle\psi_0^{(0)}|H'|\psi_0^{(0)}\right\rangle$$
$$= -2815.2eV + 0.04611707eV \cong -2814.7388293eV$$

This is approximately equivalent to ground state energy: $\cong -2814.7388293eV$ and This is first order energy shift of 2s state.

Since for some systems the first order perturbation vanishes, one can consider the second order perturbation theory that always makes the energy of the ground state, lower (in comparison to the unperturbed one). The second order correction done straight forwardly form our estimate for ψ_n^1

$$E_1^{(2)} = \langle\psi_n^0|H'\sum_{m\neq n}c_{nm}\psi_n^0\rangle = \sum_{m\neq n}c_{nm}\left\langle\psi_n^{(0)}|H'|\psi_n^{(0)}\right\rangle$$

$$= \sum_{m\neq n}\frac{\left\langle\psi_m^{(0)}|H'|\psi_n^{(0)}\right\rangle}{E_n^{(0)} - E_m^{(0)}}\left\langle\psi_n^{(0)}|H'|\psi_m^{(0)}\right\rangle$$

But H' is also Hermitian (gives us a energy which is always real!) so

$$\langle\psi_n^0|H'|\psi_m^0\rangle = \left\langle H'|\psi_n^{(0)}|\psi_m^{(0)}\right\rangle = \left\langle\psi_m^{(0)}|H'|\psi_n^{(1)}\right\rangle^* \text{ so, then we have}$$

$$\left\langle\psi_m^{(0)}|H'|\psi_n^{(0)}\right\rangle\left\langle\psi_n^{(0)}|H'|\psi_m^{(0)}\right\rangle = \left\langle\psi_m^{(0)}|H'|\psi_n^{(0)}\right\rangle\left\langle\psi_m^{(0)}|H'|\psi_n^{(0)}\right\rangle^* = |\langle\psi_m^0|H'\psi_n^0\rangle|^2$$

$$E_1^{(2)} = \sum_{m\neq n}\frac{\left|\left\langle\psi_m^{(0)}|H'|\psi_n^{(0)}\right\rangle\right|^2}{E_n^0 - E_m^0}$$

But

$$E_n = E_n^0 + E_n^1 + E_n^2 + \cdots$$
$$E_1^{(2)} = E_n^0 + E_n^1 + \sum_{m\neq n}\frac{|\langle\psi_m^0|H'\psi_n^0\rangle|^2}{E_n^0 - E_m^0}$$
$$E_1^{(2)} = -2815.2eV - 2814.7388293eV + \frac{|0.04611707eV|^2}{-2814.7388293eV - 0.04611707eV}$$
$$= -5629.9388293 + \frac{0.00212678415}{-8444.72377567}$$
$$E_1^{(2)} = -5629.93882904815eV \tag{4.48}$$

This energy shift for this state is proportional to the size of proton ($\Delta E \simeq r_p^2$).

$$r_p = \frac{204\pi\varepsilon_0 a_\mu^3 \Delta E}{e^2} = \sqrt{\frac{20 \times 4\pi\varepsilon_0 a_\mu^3 \Delta E_{shift}}{e^2}} \qquad (4.49)$$

$$= \sqrt{\frac{20 \times 4 \times 3.1428571428571 \times 8.85 \times 10^{-12}\,^F/_m \times 0.015625 \times 10^{-30}\text{m}^3(330.4944298 \times 10^{-22}\text{J})}{2.5669807524x10^{-38}\,\text{C}^2}}$$

$$\sqrt{0.44763027969933 \times 10^{-30}}$$
$$r_p = 0.6690515675193 \times 10^{-15} \qquad (4.50)$$

Lamb shifty in (μp) for transition $\left(2p_{3/_2} - 2s_{1/_2}\right)$ was measured in PSI (Paul Scherrer Institute) with results of frequency of 49881.88 GHz or energy of 206.2949meV and from this experiment the value of proton charge radius $r_p = 0.8408739$ Fermi;.and from the electronic hydrogen CODATA value $r_p = 0.8768$Fermi

4.3.3. The energy shift of 2p state to first and second-order perturbation (ΔE_{2p})

For 2s state the zero-order muonic hydrogen energy is $E_1^{(0)} = -2815.2\text{eV}$(Schwab, 2008).

$$R_{2,1} = \frac{1}{2\sqrt{6}}\left(\frac{1}{a_\mu}\right)^{3/_2} 2\left(\frac{r}{a_\mu}\right) e^{-r/_{2a_\mu}}$$

And from Taylor expansion we will have, if $\left|\frac{r}{a_\mu}\right| \ll 1$, then:

$$e^{-r/_{2a_\mu}} = 1 - \frac{r}{2a_\mu}$$

$$R_{2,1} = \frac{1}{2\sqrt{6}}\left(\frac{1}{a_\mu}\right)^{3/_2} 2\left(\frac{r}{a_\mu}\right)\left(1 - \frac{r}{2a_\mu}\right)$$

And, from this approximation the first order energy perturbation over radial will be:

$$\Delta E_{2p} = \int R_{nl}^* H^1 R_{nl} r^2 dr$$

Inserting the values for each term, we get,

$$\Delta E_{2p} = \int_0^R \left[\frac{1}{2\sqrt{6}}\left(\frac{1}{a_\mu}\right)^{3/_2} 2\left(\frac{r}{a_\mu}\right)\left(1 - \frac{r}{2a_\mu}\right)\right]^* \left[-\frac{3e^2}{8\pi\varepsilon_0 r_0}\left(1 - \frac{r^2}{3r_0^2}\right)\right.$$
$$\left. + \left(\frac{e^2}{4\pi\varepsilon_o r}\right)\right]\left[\frac{1}{2\sqrt{6}}\left(\frac{1}{a_\mu}\right)^{3/_2} 2\left(\frac{r}{a_\mu}\right)\left(1 - \frac{r}{2a_\mu}\right)\right] r^2 dr$$

$$\Delta E_{2p} = \int_0^R \left[\frac{1}{2\sqrt{6}}\left(\frac{1}{a_\mu}\right)^{3/_2} 2\left(\frac{r}{a_\mu}\right)\left(1 - \frac{r}{2a_\mu}\right)\right]^2 \left[-\frac{3e^2}{8\pi\varepsilon_0 r_0} + \frac{3e^2 r^2}{24\pi\varepsilon_0 r_0^3} + \frac{e^2}{4\pi\varepsilon_o r}\right] r^2 dr$$

Taking out constant term:

$$\Delta E_{2p} = \frac{1}{24a_\mu^3}\int_0^R \left[\left(\frac{4r^2}{a_\mu^2}\right)\left(1 - \frac{r}{a_\mu} + \frac{r^2}{4a_\mu^2}\right)\right]\left[-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 R^3} + \frac{e^2}{4\pi\varepsilon_o r}\right] r^2 dr$$

$$\Delta E_{2p} = \frac{1}{24a_\mu^3}\int_0^R \left[\left(\frac{4r^2}{a_\mu^2} - \frac{4r^3}{a_\mu^3} + \frac{r^4}{a_0^2}\right)\right]\left[-\frac{3e^2}{8\pi\varepsilon_0 r_0} + \frac{3e^2 r^2}{24\pi\varepsilon_0 r_0^3} + \frac{e^2}{4\pi\varepsilon_o r}\right] r^2 dr$$

Now, expanding the terms, we obtain:

$$\Delta E_{2p} = \frac{1}{24a_\mu^3}\int_0^R \left[\frac{4r^2}{a_\mu^2}\left(-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 r_0^2} + \frac{e^2}{4\pi\varepsilon_o r}\right) - \frac{4r^3}{a_\mu^3}\left(-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 R^3} + \frac{e^2}{4\pi\varepsilon_o r}\right)\right.$$
$$\left. + \frac{r^4}{a_\mu^2}\left(-\frac{3e^2}{8\pi\varepsilon_0 R} + \frac{3e^2 r^2}{24\pi\varepsilon_0 R^3} + \frac{e^2}{4\pi\varepsilon_o r}\right)\right] r^2 dr$$

This gives us:

$$= \frac{1}{24a_\mu^3}\int_0^R \left(\frac{-12e^2 r^2}{8\pi\varepsilon_0 a_\mu^2 R} + \frac{12e^2 r^4}{24\pi\varepsilon_0 a_\mu^2 R^3} + \frac{e^2 r^2}{\pi\varepsilon_0 a_\mu^2} + \frac{12e^2 r^3}{8\pi\varepsilon_0 a_\mu^3 R} - \frac{e^2 r^5}{2\pi\varepsilon_0 a_\mu^3 R^3} - \frac{e^2 r^2}{\pi\varepsilon_0 a_\mu^3} - \frac{3e^2 r^4}{8\pi\varepsilon_0 a_\mu^2 R} + \frac{e^2 r^6}{8\pi\varepsilon_0 a_\mu^2 R^2}\right.$$
$$\left. + \frac{e^2 r^3}{4\pi\varepsilon_0 a_\mu}\right) r^2 dr$$

And integration with respect to r gives us:

$$= \frac{1}{24a_\mu^3}\left(-\frac{3e^2r^5}{10\pi\varepsilon_0 a_\mu^2 R} + \frac{e^2r^7}{14\pi\varepsilon_0 a_\mu^2 R^3} + \frac{e^2r^4}{4\pi\varepsilon_0 a_\mu^2} + \frac{3e^2r^6}{12\pi\varepsilon_0 a_\mu^3 R} - \frac{e^2r^8}{16\pi\varepsilon_0 a_\mu^3 R^3} - \frac{e^2r^5}{5\pi\varepsilon_0 a_\mu^3} - \frac{3e^2r^7}{56\pi\varepsilon_0 a_\mu^2 R}\right.$$
$$\left. + \frac{e^2r^9}{72\pi\varepsilon_0 a_\mu^2 R^3} + \frac{e^2r^6}{24\pi\varepsilon_0 a_\mu}\right)\Big|_0^R$$

This is reduced to:

$$= \frac{1}{24a_\mu^3}\left(-\frac{3e^2R^4}{10\pi\varepsilon_0 a_\mu^2} + \frac{e^2R^5}{14\pi\varepsilon_0 a_\mu^2} + \frac{e^2R^4}{4\pi\varepsilon_0 a_\mu^2} + \frac{3e^2R^5}{12\pi\varepsilon_0 a_\mu^3} - \frac{e^2R^5}{16\pi\varepsilon_0 a_\mu^3} - \frac{e^2R^5}{5\pi\varepsilon_0 a_\mu^3} - \frac{3e^2R^6}{56\pi\varepsilon_0 a_\mu^2} + \frac{e^2R^6}{72\pi\varepsilon_0 a_\mu^2}\right.$$
$$\left. + \frac{e^2R^6}{24\pi\varepsilon_0 a_\mu}\right)$$

Now, taking out common terms, we obtain:

$$= \frac{1e^2R^2}{24a_\mu^4\pi\varepsilon_0}\left(-\frac{3}{10a_\mu} + \frac{1}{14a_\mu} + \frac{1}{4a_\mu} + \frac{R}{4a_\mu^2} - \frac{R}{16a_\mu^2} - \frac{2R}{5a_\mu^2} - \frac{3e^2R^2}{56a_\mu} + \frac{R^2}{72a_\mu} + \frac{R^2}{24\pi}\right)$$
$$= \frac{1}{24}\frac{e^2R^4}{\pi\varepsilon_0 a_\mu^4}\left(\frac{-42+10+35}{140a_\mu} + \frac{80R-20R-64R}{320a_\mu^2} - \frac{3e^2R^2}{56a_\mu} + \frac{R^2}{72a_\mu} + \frac{R^2}{24\pi}\right)$$
$$= \frac{1}{24}\frac{e^2R^2}{\pi\varepsilon_0 a_\mu^4}\left(\frac{3R^2}{140a_\mu} - \frac{4R^3}{320a_\mu^2} - \frac{3e^2R^4}{56a_\mu} + \frac{R^2}{72a_\mu} + \frac{R^4}{24\pi}\right)$$
$$\Delta E_{2p} = \frac{1}{24}\frac{e^2R^2}{\pi\varepsilon_0 a_\mu^3} \tag{4.51}$$

Substituting constant values from Appendix gives us a numerical value (Yung-Kuo, 2000):

$$\Delta E_{2p} = \frac{1}{24}\left(\frac{(1.602\times10^{-19}C)^2(1.05\times10^{-15}m)^2}{3.14\times8.85\times10^{-12}(2.5\times10^{-12}m)^3}\right)$$
$$= \frac{1}{24}\left(\frac{2.82946041}{434.203125}\times10^{-68+48}\right)$$
$$0.27151851\times10^{-20} = 0.16\times10^{-1}eV$$

The energy difference between

$$\Delta E_{2S} - \Delta E_{2P} = \frac{1}{20}\frac{e^2}{4\pi\varepsilon_0}\frac{R^2}{a_\mu^3} - \frac{1}{24}\frac{e^2R^2}{\pi\varepsilon_0 a_\mu^3}$$
$$\Delta E_{2S} - \Delta E_{2P} = \frac{1}{8}\frac{e^2}{\pi\varepsilon_0}\frac{R^2}{a_\mu^3}\left(\frac{1}{10} - \frac{1}{3}\right)$$
$$\Delta E_{2S} - \Delta E_{2P} = \frac{-7}{240}\frac{e^2}{\pi\varepsilon_0}\frac{R^2}{a_\mu^3} \tag{4.52}$$

Where a_μ is the Bohr radius of muonic hydrogen. Thus by measuring the energy shift, we can deduce the value of R.

The total energy shift for ground state by first order will be calculated as follows. We know that unperturbed Hamiltonian has an eigenvalue for muonic hydrogen is:

$$E_\mu = \frac{m_\mu}{m_e}E_e = 207(-13.6eV) = -2815.2eV$$
$$E_n^{(o)} = -2815.2eV$$

If we add this value to the energy at ground state, there is almost no change.

$$E_n = E_n^{(o)} + E_n^{(1)}$$
$$E_n^{(1)} = E_n^{(o)} + \left\langle\psi_0^{(0)}\middle|H'\middle|\psi_0^{(0)}\right\rangle$$
$$= -2815.2eV + 0.016eV \cong -2815.168eV$$

This is approximately equivalent to ground state energy: $\cong -2815.168eV$

Since for some systems the first order perturbation vanishes, one can consider the second order perturbation theory that always makes the energy of the ground state, lower (in comparison to the unperturbed one).

The second order correction done straight forwardly form our estimate for ψ_n^1

$$E_1^{(2)} = \langle\psi_n^0|H'\psi_n^1\rangle$$

Or we can expand this out the full sum

$$E_1^{(2)} = \left\langle\psi_n^0\middle|H^{H'}\sum_{m\neq n}c_{nm}\psi_n^{(0)}\right\rangle = \sum_{m\neq n}c_{nm}\left\langle\psi_n^{(0)}\middle|H^{H'}\middle|\psi_0^{(0)}\right\rangle$$

$$\sum_{m \neq \neq n} \frac{\langle \psi_m^{(0)} | H' \psi_n^0 \rangle}{E_n^0 - E_m^0} \langle \psi_n^{(0)} | H^{(1)} | \psi_m^{(0)} \rangle$$

But $H^{(1)}$ is also Hermitian (gives us a energy which is always real!) so

$$\langle \psi_n^{(0)} | H^{H'} | \psi_m^{(1)} \rangle = \langle H' | \psi_n^{(0)} | \psi_m^{(0)} \rangle = \langle \psi_m^{(0)} | H^{(1)} | \psi_n^{(1)} \rangle^* \text{ so, then we have}$$

$$\langle \psi_m^{(0)} | H^{(1)} | \psi_n^{(0)} \rangle \langle \psi_n^{(0)} | H^{(1)} | \psi_m^{(0)} \rangle = \langle \psi_m^{(0)} | H^{(1)} | \psi_n^{(0)} \rangle \langle \psi_m^{(0)} | H^{(1)} | \psi_n^{(0)} \rangle^* = |\langle \psi_m^0 | H' \psi_n^0 \rangle|^2$$

$$E_1^{(2)} = \sum_{m \neq n} \frac{|\langle \psi_m^0 | H' \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

$$E_n = E_n^0 + E_n^1 + E_n^2 + \cdots$$

$$\text{so, } E_1^{(2)} = E_n^0 + E_n^1 + \sum_{m \neq n} \frac{\left| \langle \psi_m^{(0)} | H' | \psi_n^{(0)} \rangle \right|^2}{E_n^0 - E_m^0}$$

$$E_1^{(2)} = -2815.2 \text{eV} - 2815.168 \text{eV} \quad + \frac{|0.016 \text{eV}|^2}{-2815.168 \text{eV} \quad - 0.016 \text{eV}}$$

$$= -5630.368 + \left(\frac{0.000256}{-2815.183744} \right)$$

$$= -5630.3679999181 \text{eV}$$

And energy shift for this state is proportional to the size of proton ($\Delta E \simeq r_p^2$).) and given by:

$$r_p = \sqrt{\frac{24\pi\varepsilon_0 a_\mu^3 \Delta E_{shift}}{e^2}} \qquad (4.53)$$

We have energy shifty value from the literature review from the Lamb shift experiment (the energy difference between the $2S_{1/2}$ and $2P_{1/2}$ states) is from Eq. (2.2)

$$(\Delta E = 330.4944298 \times 10^{-22} \text{J} \quad = 206.294932 \times 10^{-3} \text{eV} \quad)$$

$$r_p = \sqrt{\frac{24 \times 3.14 \times 8.85 \times 10^{-12} \times (2.5 \times 10^{-12})^3 \times 330.4944298 \times 10^{-22} \text{J}}{(1.602 \times 10^{-19} C)^2}}$$

$$\sqrt{0.447223474165761 \times 10^{-30}}$$

$$r_p = 0.699 fm \qquad (4.54)$$

Lamb shifty in (μp) for transition $\left(2p_{3/2} - 2s_{1/2} \right)$ was measured in PSI (Paul Scherrer Institute) with results of frequency of 49881.88 GHz or energy of 206.2949MeV and from this experiment the value of proton charge radius $r_p = 0.8408739$ Fermi; and from the electronic hydrogen CODATA value $r_p = 0.8768$ Fermi

Percent errors between experimental and theoretical values of finite sized proton in table below

No	Items or description	Value proton size (r_P)
1	Theoretical values from the energy shift of 1s state	$0.7476714 fm$
2	Theoretical values from the energy shift of 2s state	$0.669 fm$
3	Theoretical values from the energy shift of 2p state	$0.699 fm$
4	Experimental value from the muonic hydrogen CODATA	$0.8408739 fm$
5	Experimental value from the electronic hydrogen CODATA	$0.8768 fm$

Conclusion

In this thesis, we have presented the muonic hydrogen energy level in quantum mechanics using the time-independent perturbation theory, and calculating ground state energy shift and estimation of proton finite size assuming finite size nucleus. The main purpose of this study was to investigate the energy level shift of muonic hydrogen atom with finite-size of proton using perturbation theory. The lamb shift, the interaction between the electron and the vacuum causes a tiny energy shift between *2s to 2p*. The calculations justify that the proton has finite size in which we have calculate, energy shift over a volume of two extreme points ranges from negative infinity to positive infinity and we got some constant value of zero which implies the finite size of the proton. As we can see from the first and second order perturbation at 1s,2s and 2p excited state, there is small amount energy shift from ground state because the wave function goes to zero at nucleus at these state. Therefore, as we have seen through all the calculations of 1s, 2s and 2p, the application of perturbation theory have shown us that the energy correction is very small at each state. So the perturbation at higher order becomes smaller and smaller compared

to the zero order at each state and such that it can be ignored for higher orders. From this we can say that the interaction of electron with proton at higher state will be low which justifies that proton is not a large spherical shaped but it has finite size. The wave equation of particles produces eigen value equation, which can energy operator where Hamiltonian's particle operated on wave function

.

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