A General Theory of Micro and Macro Mechanics in Complex Vector Space: Applications to Quantum Theory and Gravitation

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Abstract

A new pre-quantum level physical theory underlying the quantum mechanics has been developed and its epistemological features are mainly derived from the fluctuating classical electromagnetic zeropoint field which induces random oscillations on charged particle. In complex vector space, the average random oscillations of the charged particle are treated as complex rotations so that a particle has extended structure. It is found that the quantum nature of particles arises from the superposition of internal oscillations on the center of mass motion. Using Lagrangian mechanics in complex vector space, a generalized Newton's equation is derived and which is shown to be acceptable for both micro and macro mechanics. The generalized Newton's equation contains additional relativistic terms at the micro level. Further, a complex Hamilton-Jacobi equation is derived and which is used to solve quantum mechanical problems in classical approach. The presence of zeropoint field also produces modifications in the central potential and the additional terms are responsible for the effects like advance of planets perihelion and deflection of light near massive objects at macro level. Further, in the case of atoms such terms lead to the estimation of Lamb shift using stochastic electrodynamics. The consideration of extended particle structure in stochastic electrodynamics allows deriving mass correction and charge correction and consequently the estimation of anomalous magnetic moment. The fundamental deeper level theory developed here, may further enhance the efforts for research connecting micro and macro aspects of matter.

Keywords: Foundations of Quantum Mechanics, Geometric Algebra, Stochastic Electrodynamics, Zeropoint Energy, Gravitation.

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1. Introduction

An endeavour to find a common physical theory having both mathematical and epistemological features that properly address the problem of micro and macro aspects of matter advocates the development of classical approach to quantum mechanics. The Newtonian and relativistic mechanics deals with the macro-objects which are considered non-oscillatory or their oscillatory behaviour is too small that can be neglected and will be having fixed position and momentum and the path of these bodies is deterministic in time evolution. The space in which these bodies are moving is considered to be continuous or smooth. However, at micro level what we consider smooth may not be but contains certain structure. The speculation of micro-structure of space in the form of distortions in space was initially proposed by Clifford (1956) and such distortions in space are like waves leading to motion of matter. Thus, consideration of small variations in space leads to fluctuations of micro-particles.

The following enthralling concepts and findings are elucidated lucidly in this article. (a) It is considered that the valid cause for such fluctuations in space or fluctuations of the material particles may be assigned to the randomly fluctuating electromagnetic zeropoint field present all over space. (b) The modification from space representing macro nature of particles to the fluctuating space representing micro nature of particles must contain precise functions representing fluctuations in addition to the functions describing space. (c) A micro-particle like electron is perceived to have intrinsic charge which oscillates in accordance with the random oscillations of the zeropoint field and such oscillations are considered as complex rotations in the complex vector space. (d) The angular momentum of these complex rotations is the zeropoint angular momentum or the spin angular momentum of the particle. (e) The mass is produced due to complex rotations in the zeropoint field and the particle mass and spin angular momentum are related. (f) Thus, the particle may be conceptualized as an extended object in space and cannot be treated as a point particle at micro level. When the particle is in motion, the path of the center of charge is helical in nature and such motion produces the relativistic effects in general. The extended particle motion influences certain change in the surrounding zeropoint field and leads to an additional energy or mass or mass correction. (g) The existence of such extended structure of micro-particle allows considering additional functions to classical mechanics of motion and one can achieve entire micro mechanics or quantum mechanics. A composite of such extended micro-particles forms a macro-particle. Clarification of the above concepts requires proper review of previous developments and those are discussed briefly in this introductory section.

A century old quantum mechanics is a fascinating subject even today. Schrödinger (1928) derived the wave equation from the classical Hamilton-Jacobi equation and considered the action $J = k \log \psi$, where ψ is a complex function which is everywhere real, single valued, finite and continuously differentiable up to second order and k is a constant having dimensions of action. The Schrödinger wave equation is considered as one of the most fundamental equations of motion of the particle in the micro-world. The state of fluctuating nature of particles is described by the wave function which turns out to be the most possible description of quantum reality. In the accepted interpretation, the wave function $\psi(\boldsymbol{x}, t)$ represents the state of the system in position space and the probability density $\rho(\boldsymbol{x}, t) = |\psi(\boldsymbol{x}, t)|^2$ determines the probability of finding the quantum system. The notion of wave function as a force field was initially proposed by de Broglie (1930) and further systematic development of the concept was developed by Bohm (1952a, 1952b). The particle under the influence of force field traverses in a causally determined path continuously in the course of time. In the causal interpretation, the wave function takes the form $\psi(\boldsymbol{x}, t) = \sqrt{\rho} \exp iJ/\hbar$, where $\hbar$ is the reduced Planck constant and J is the classical action. The wave equation turns out to be the Hamilton-Jacobi equation with an additional potential called quantum potential Q which arises from the fluctuations of the particle. The cause of such fluctuations is suggestively attributed to the presence of vacuum fluctuations in the Bohmian theories. The presence of quantum potential in the Schrödinger equation elucidates the existence of certain internal structure of the particles. In the causal interpretation the particle acceleration is determined by the relation $m\ddot{\boldsymbol{x}} = -\boldsymbol{\nabla}[V + Q]$ and the classical force term $-\boldsymbol{\nabla}\mathrm{V}$ is generally local and the quantum force term $-\boldsymbol{\nabla}\mathrm{Q}$ represents the quantum contribution that depends on the form of the fluctuating field and therefore generally non-local (Bohm & Hiley 1993). The concept of a particle moving under its own energy and guided by the quantum force field suggests that it has a complex and subtle inner structure. The spin angular momentum of a particle has been visualised to have certain relation to the circulatory motion in a spherical structure. In the case when the spin is directed along velocity, an element in spherical structure executes rotational motion (Bohm, Schiller & Tionmo 1955). A complete understanding of non-relativistic quantum nature of micro-systems is possible in the Bohmian analysis except for the cause of quantum randomness (Durr & Teufel 2009).

The cause of fluctuating nature of particles may be attributed to the presence of electromagnetic fluctuating random zeropoint field. It is a well-known fact that a quantum mechanical harmonic oscillator has ground state energy equal to $\hbar\omega/2$ per mode and in the second derivation of Planck's blackbody radiation law (Milonni 1994), when the field oscillators are treated as quantum oscillators, the radiation density at absolute zero remains non-zero and the density of normal modes times the energy per mode gives the zeropoint field radiation density

$$\varrho(\omega) = \frac{\omega^2}{\pi^2 c^3}\frac{\hbar\omega}{2}, \quad (1)$$

where c is the velocity of light and ω is the angular frequency. Using Lorentz transformation for electric and magnetic fields, the Lorentz invariance of zeropoint field was established by Boyer (1969). The ubiquitous electromagnetic zeropoint radiation is found to be homogeneous, isotropic, and Lorentz invariant and therefore treated as classical in its nature (Boyer 1985).

The following experimental and theoretical findings clearly establish the presence of zeropoint field. Mulliken (1924) first established the reality of zeropoint energy in the spectroscopic studies of band spectrum of carbon monoxide. Even in the advanced Dirac theory, the energy levels $2S_{1/2}$ and $2P_{1/2}$ have been found to be degenerate. However, the shift between these levels of about 1000 MHz was observed in the microwave studies of hydrogen atom by Lamb and Rutherford (1947). Assuming the interaction between electron and zeropoint field, Welton (1948) explained the Lamb shift of atomic hydrogen. Casimir (1948), showed the existence of a force of attraction between two parallel plates induced by zeropoint field and that force was measured experimentally by Sparnaay (1958) and for detailed discussion of Casimir effect see (Milton 2004). Interestingly, the zeropoint radiation connection to the cosmological constant and dark energy has been speculated by the consideration of the fact that any type of energy that adds to the energy density tensor in the Einstein field equation in general theory of relativity does not change the form of field equation (de La Pena & Cetto 1996). The above and many other theoretical and experimental findings suggest that the zeropoint fields are more general and quite real in nature and not virtual in the sense of quantum vacuum in quantum electrodynamics. Critical arguments about the existence of zeropoint energy was given by Wesson (1991) and argued that the source of zeropoint energy in quantum theories or in cosmological theories is an open question. However, according to Puthoff (1990, 1989), the oscillations of charged particles in the universe produce zeropoint field and simultaneously charged particles fluctuate in the presence of the zeropoint field in a regenerative manner and finally expedite grand ground state.

Classical electrodynamics was developed without incoming radiation field, but as discussed above it is necessary to consider the incoming radiation and introducing zeropoint field in the classical electrodynamics simply yields stochastic electrodynamics (Boyer 1975). The randomness in the theory is attributed to the zeropoint field fluctuations and it is comparable to the randomness considered in the statistical mechanics. The aim of stochastic electrodynamics was to develop a logical classical approach to the quantum foundations. Many quantum phenomena were studied within the scope of stochastic electrodynamics. Due to random impulses from the fluctuating zeropoint field, a charged particle placed in it oscillates about its equilibrium position. Marshall (1963) studied such particle oscillator in the zeropoint field and found a close resemblance between quantum oscillator and classical oscillator in the ground state. Later the harmonic oscillator problem was extensively studied by many others, mainly Boyer (1975a), Santos (1974) and de la Pena and Cetto (1979). Within the scope of stochastic electrodynamics, mean square displacement $\langle x^2 \rangle = \hbar/2m\omega_0$ mean square momentum $\langle p^2 \rangle = \hbar m\omega_0/2$ and average energy $\langle \mathcal{E}_0 \rangle = \hbar\omega_0/2$ of the harmonic oscillator was estimated in a systematic manner. It was found that the product of position and momentum mean square displacements gives the Heisenberg minimum uncertainty relation and further, it was shown that the probability distribution of classical oscillator is same as that of ground state quantum oscillator. Using delicate balance between gain of energy from zeropoint field and loss of energy due to radiation damping of an atomic electron, the Bohr quantum condition has been derived. All these results obtained are in agreement with the quantum oscillator in the ground state. A wide variety of problems were also studied within the purview of stochastic electrodynamics and found to give excellent results. It has been found that the Van der Waals force (Boyer 1972, 1970) derived in stochastic electrodynamics is completely in agreement with the quantum result. For example, the approach of stochastic electrodynamics has satisfactorily given the explanation for diamagnetism (Boyer 1980a), paramagnetism (Barranco, Brunini & Franca 1989), particle spin angular momentum (Moore & Ramirez 1982; de la Pena & Jauregui 1982) and connection between random electrodynamics and quantum electrodynamics (Boyer 1975b). A complete account of stochastic electrodynamics studies was elaborately discussed in several reviews (de La Pena & Cetto1996; Boyer 1975a; Boyer 1980b; Daniel C. Cole 1993). Devoid of explaining many interesting quantum results, the theory of stochastic electrodynamics contains several significant drawbacks. For a non-divergent particle oscillator, it is required to impose an upper cut-off frequency to the spectrum of zeropoint field. If the influence of zeropoint magnetic field is neglected, energy levels of hydrogen atom and explanation of sharp spectral lines would not be possible. Stochastic electrodynamics has been found to be inapplicable for nonlinear forces. Further, the Schrödinger equation can be obtained in some particular cases only. These drawbacks can be eliminated by considering an extended structure of a charged particle and introducing spin into stochastic electrodynamics.

In both quantum and stochastic electrodynamics theories, a charged particle means a point particle and the zero size of a particle gives infinite energy. However, to avoid such infinite energy one needs to impose a cut-off procedure which consequently leads to an extended particle structure. In the zeropint field, a charged particle oscillator is assumed to have a center of mass and center of charge and the center of mass cannot follow the motion of center of charge similar to a phenomenon that happens in extended charged bodies. Based on this assumption, Rueda and Cavalleri (1983) proposed a classical model for the particle containing internal structure. The vibrations of charge with respect to center are assumed due to the zeropoint field and a cut-off to the zeropoint field distribution corresponds to the particle extended structure (Rueda 1993a; 1993b). It has been shown recently that the drawbacks of stochastic electrodynamics can be successfully eliminated by considering particle spin angular momentum which arises from the internal circular motion (Cavalleri *et al*. 2010 ; Bosi *et al*. 2008). The natural cut-off frequency for the zeropoint field spectrum simply equals the spin frequency or the frequency of internal circular motion and this is the maximum frequency radiated (absorbed) by the charged particle so that the problem of divergence in stochastic electrodynamics has been addressed. Several interesting phenomena were explained by Cavalleri *et al.,* (2010) which include the stability of elliptical orbits in an atom, the origin of special relativity and explanation for diffraction of electrons. The internal circular motion of charge center appears to be helical when the particle is in motion which is responsible for the observed effects of special relativity (Cavalleri 1997).

Recently in the complex vector approach, the particle spin angular momentum and its emergence from the particle internal structure was explored by the author (Muralidhar 2014). The energy of the charged particle oscillator was derived and found that the presence of spin actually gives quantum results from a classical oscillator system considered in the complex vector space and the particle mass can be visualised as the energy due to a local complex rotation in the zeropoint field. The complex vector formalism has an additional advantage of considering a multivector expansion containing a scalar, a vector and a bivector parts and it provides a better classical approach to quantum phenomena (Muralidhar 2015). For example, a classical approach to the Schrödinger equation leads to the understanding of basic foundations of a quantum system.

The above theoretical studies lead to an understanding that there is a classical fluctuation mechanism that contributes to the quantum behaviour of particles and it must be formulated as a physical theory. This article is organized as follows. The particle structure and spin angular momentum is discussed in section 2 and the subsection 2.2 deals with the correspondence between internal parameters of the particle and the quantum operators and the nature of quantum behaviour of micro-particles. Lagrangian formulation of particle dynamics in the complex vector space and the applications of generalized Newton's equation for both micro and macro-particles are given in the section 3. Stochastic electrodynamics with extended particle structure is presented in the section 4. A short summary and conclusions of the findings are given in the section 5. Detailed mathematical derivations of equations at various places in the main text are presented in the Appendix.

In the following subsection, a review of Schrödinger equation in stochastic theories and other related works, a short account of *zitterbewegung* motion of charged particles, the theory classical spinning particles, and the substructure of electron in the zeropoint field are given. Complex vector approach to particle structure in complex vector space has an added advantage over normal way of treating the particle and its dynamics. A short account of mathematical apparatus, the complex vector algebra is given subsequently.

1.1. Schrödinger equation in stochastic and other theories

The stochastic interpretation of quantum mechanics in the Markov process was initially developed by Fenyes in 1952 and a detailed development of the theory was discussed by Claverie and Diner (1976). Later, considering the diffusion process satisfying the Nelson-Newton's law, the derivation of Schrödinger equation was derived by Nelson (1985; 1966). Nelson considered that every particle of mass m moves in a random environment similar to Brownian motion with diffusion constant $D = \hbar/2m$ and introduced osmotic velocity $\boldsymbol{u}_0$ and current velocity $\boldsymbol{v}_0$ and found that the stochastic acceleration is equal to the derivative of potential function, $\boldsymbol{a}(x,t) = -\boldsymbol{\nabla}V(x,t)$ which is known as Nelson-Newton's law for a particle of unit mass. Further, independent studies of such random motion in quantum mechanics were carried out by Della Riccia and Wiener (1966) and Favella (1967). In a review by Nelson (2012), the status of stochastic mechanics in the Markov process and its successes and failures have been elaborately discussed. A detailed quantum theory from random process was discussed by Pena and Cetto (1975) by introducing zeropoint field in stochastic electrodynamics and a generalization of Newtonian mechanics formulation led to the derivation of Schrödinger equation (de la Pena-Auerback 1969). In stochastic electrodynamics, random oscillations of a charged particle are due to fluctuations of the electromagnetic zeropoint field. The total velocity of the oscillating particle may be considered as a sum of systematic velocity $\boldsymbol{v}_s$ and stochastic velocity $\boldsymbol{u}_s$. From a generalized Fokker-Planck equation in the configuration space, the Schrödinger equation has been derived by assuming a specific value of diffusion constant and the quantum mechanical operators and commutation relations arise as a part of the derivation. Using variational method in stochastic theory, Yasue (1978) presented field theoretic treatment of Schrödinger equation. In the presence of external electromagnetic field, considering the particle as a spinning rigid body, Pauli-Schrodinger equation, Feynman and Gell-Mann type and Dirac equations were derived in the stochastic theory (de la Pena-Auerback 1971). The choice of adopting particular value for diffusion constant was shown to be derived from the stochastic electrodynamics approach (de la Pena & Cetto 2015; Cetto & de la Pena 2015). In the extended density gradient expansion using the non-Markovian stochastic process, Cavalleri *et al.*, (1985; 1990; 1991) derived the Schrödinger equation which contained higher order correction terms resembling quantum electrodynamic radiative terms and the quantum behaviour of particles can be visualised as superposition of *zitterbewegung* motion on classical translational motion.

The classical formulation of quantum mechanics was studied by Wigner (1932) by introducing a probability distribution function of an ensemble of particles in phase space and found a similarity between Schrödinger equation and classical Liouville equation. Dochoum and Franca (2002) showed that the classical probability amplitude is related to the Wigner distribution function and derived the Liouvillian form of time evolution of particles in the presence of zeropoint field. Faria *et al.* (2007) showed that the zeropoint electromagnetic fields are dynamically related to the momentum operator in the Schrodinger equation. Similar considerations show the correspondence between the classical equations of neutral spinning particles and Pauli-Schrödinger equation (Dechoum *et al*. 1998). A new classical approach to quantum formalism was investigated by considering infinitesimal Wigner-Moyel transformation by Olavo (1999a; 1999b) and derived Schrödinger equation in an axiomatic approach and established the connection between stochastic approach in quantum theory and stochastic electrodynamics (Olavo 2000). Considering the random momentum fluctuations of the classical ensemble, Haal and Reginatto (2002) formulated exact form of the uncertainty principle and found the resulting classical equations of motion represent the Schrödinger equation. Schleich *et al*., (2013; 2015) obtained Schrödinger equation from a mathematical identity containing space and time derivatives and showed that the linearity of quantum mechanics is intimately connected to the strong coupling between amplitude and phase of quantum wave.

Based on the non-equilibrium thermodynamics, Grössing (2010; 2008; 2009) argued that the quantum theory really emerges from a deeper sub-quantum theory. In thermal fluctuation background, a particle is treated as a fluctuating quantum system and the total momentum of the system is then expressed as a sum of classical momentum $\boldsymbol{p}$ and momentum due to fluctuations. In the Hamilton-Jacobi classical equation, the gradient of action function $\mathcal{S}\,(\boldsymbol{x}, t)$ is equal to the momentum of the particle and then the momentum fluctuation is expressed as $\delta\boldsymbol{p} = \nabla(\delta\mathcal{S})$ and the corresponding additional kinetic energy is $(\delta\boldsymbol{p})^2/2m = [\nabla(\delta\mathcal{S})]^2/2m$. The Schrödinger equation was derived by formulating an action integral containing an additional kinetic energy corresponding to thermal fluctuations. The probability density was found to satisfy an equation similar to the Maxwell-Boltzmann distribution function. It has been shown that the diffusive velocity is related to $\delta\boldsymbol{p} = m\boldsymbol{u}_s$. In one way or other in all these theories, the Hamilton-Jacobi equation plays an impotent role in bridging the classical and quantum theories.

One of the interesting unifications of classical and quantum theories sprouts from the idea of representing canonical position and momentum coordinates as complex quantities containing real and imaginary parts, $q = q_r + iq_i$, $p = p_r + ip_i$ and these complex variables generate two dimensional systems consisting of canonical variables of real classical position and momentum and fluctuating imaginary parts. As a consequence, the Hamiltonian can be expressed as a sum of real and imaginary parts $H = H_r + iH_i$ and it is supposed to be an integral of particle motion. The particle nature is attributed to the real part and the wave nature of the particle is attributed to the multipath behaviour of complex trajectories due to imaginary part of the Hamiltonian. The complex Hamilton-Jacobi equation contains an additional quantum potential term and such equation has been shown to yield exact eigen values for the potential problems without necessarily solving the corresponding Schrödinger equation (Leacock and Padgett 1983). Several authors studied classical approach to quantum mechanics in complex domain (Yang 2006; Bender *et. al*. 2010) and a detailed study of complex trajectories of particle motion using complex Hamiltonian has been developed by Yang (2007) and the equality between complex Hamilton-Jacobi equation and quantum Schrödinger equation has been established.

Another approach to classical-quantum transformation is through consideration of fractal spacetime in scalar relativity theory developed by Nottale (2011). Based on the Feynman statement that quantum paths are continuous but not differentiable, it has been assumed that there exists a scale dependant inertial coordinate system locally and the physical description at quantum level is described by a fluctuating variable. Then a small increment in the path of a particle contains classical and fluctuating parts $X^{\mu} = dx^{i} + d\xi^{j}$, where the fluctuating non-differentiable element $d\xi$ describes fractal behaviour and depends on fractal dimension, $d\xi \propto dt^{1/D}$. Using the Nelson's concept of forward and backward velocities, a complex velocity is defined in the form $V = v - iu$, where v is the classical velocity and u is the non-differentiable fractal velocity similar to osmotic velocity and this complex velocity field is one of the main tools of scale relativity. The complex time derivative is expressed in terms of complex velocity and diffusion like constant D, $\hat{d}/dt = (\partial/\partial t) + V.\nabla - iD\nabla^2$. The passage from classical to non-differentiable theory is obtained by replacing time derivative with complex derivative. The wavefunction is defined to represent particle paths instead of the velocity of a classical point particle and introduced in the form $\psi = exp(i\mathcal{S}/\mathcal{S}_0)$, where $\mathcal{S}$ is the complex action variable and $\mathcal{S}_0 = imD$. The complex wavefunction is related to complex velocity by the relation $V = i\mathcal{S}_0\nabla(\ln\psi)/m$ and with the use of time derivative, a fundamental equation of dynamics has been derived which can be applicable to both classical and quantum physics as well. Under particular conditions this fundamental dynamical equation can be integrated and the Schrödinger equation has been recovered (Celerier and Nottale 2004).

The story of Schrödinger equation for the past hundred years is not limited to the views presented in this section but much more that cannot be covered by referring few and it is clear that the close connection between classical and quantum phenomena can be achieved by deriving Schrödinger equation on classical grounds and further provides an impetus to the development of a common physical theory covering classical and quantum aspects on the same foundation.

1.2. Zitterbewegung and particle structure

One of the interesting aspects of micro-particles is their fluctuating behaviour and it is also the root cause of statistical interpretation of quantum mechanics. Using Dirac Hamiltonian, Schrödinger showed that an electron contains internal oscillations with amplitude of the order of reduced Compton wavelength and such oscillation is known as *zitterbewegung* (Sakurai 2007). In the Schrödinger *zitterbewegung* derivation of Dirac electron, the position coordinate is given by

$$x = [x(0) + \dot{x}t] + \frac{i\hbar}{2}\eta(0)H^{-1}exp\left(\frac{i2H}{\hbar}\right), \qquad (2)$$

where $\eta(0)$ is a constant, H is the Hamiltonin and over dot denotes differentiation with respect to time. The second term on right represents an additional position coordinate which oscillates with frequency $\omega = 2mc^2\hbar^{-1}$ and it may be represented by oscillating coordinate say $\boldsymbol{\xi}(t)$ and the mean position of oscillation by the first term $\boldsymbol{x}(t)$. The microscopic dynamical variable $\xi(t)$ represents the internal oscillations and the translational motion of the particle is described by the macroscopic variable $\boldsymbol{x}(t)$. The spin angular momentum of the particle arises from the internal motion and the rest mass of the particle appears to be the energy of internal oscillations. It has been further proved by Barut and Bracken (1981) that the phase space of *zitterbewegung* is curved and compact. The above theoretical considerations univocally suggest that a charged particle internal structure contains center of mass and center of charge. The center of charge rotates around the center of mass point in a circular fashion with a frequency of rotation ω_0 and the radius of rotation is equal to the order of Compton wavelength. Huang (1952) made an elegant study of *zitterbewegung* and proposed that the electron executes an internal circular motion about the direction of electron spin with radius of the order of Compton wavelength of electron. The angular momentum of internal circular motion is ascribed to the spin of electron and such circular motion also leads to intrinsic magnetic moment. It has been argued that the particle spin is the angular momentum of *zitterbewegung* oscillations. This theory gives a possibility that the internal structure of an elementary particle like electron or quark, arises from the oscillatory nature of the particle. *Zitterbewegung* motion was investigated by several authors and it would not be possible to refer them all and finally one can say that the electron has certain internal structure and the cause of *zitterbewegung* motion may be attributed to the presence of zeropoint field.

1.3. Classical theories of spinning particles

The classical relativistic theory of a spinning particle moving in a Maxwell's field was developed by Bhabha and Corben (1941). The theory constitutes the equations of motion of a charged particle for both spin and the motion as a whole. The interaction of charged particles with the electromagnetic field automatically includes radiation reaction. The general equation elucidates the deviations in the path of a charged particle and the particle has an internal helical motion which is analogous to the *zitterbewegung* of Dirac electron (Corben 1961; 1968). A classical spinning electron model was developed by Mathisson (2010) and in this model the electron motion contains both internal rotational motion of charge center due to *zitterbewegung* and translational motion of mass center. Mathisson's theory was further extended by Weyssenhoff (1946) and the theory describes two distinct categories differentiated by the assumption that electron spacetime history is time like in one frame and light like in the other. The proper time is attached to the center of mass motion observed in an arbitrary frame. A review of Mathison's model of electron was given by Horvathy (2003) and mentioned that the model reappeared on several occasions.

Using geometric algebra, Barut and Zanghi (1984) explored a classical analogue of *zitterbewegung* wherein the invariant proper time is associated with the center of mass. The orbital angular momentum of internal motion is identified as the particle spin and the internal energy of rotation represents the rest mass of the particle. In the *zitterbewegung* interpretation of Dirac electron, Hestenes (2010) considered electron spin as a zeropoint angular momentum and described as circulation of electron mass and charge. The theory gives a coherent interpretation of the Dirac theory and Pauli-Schrodinger theory of electron. The presence of $i\hbar$ in the Schrödinger equation itself gives the existence of spin. The *zitterbewegung* oscillations of the electron are inherent in the geometric or structural features of electron and give the confirmation of de Broglie hypothesis. In an electron channelling experiment by Gouanere *et al*., (2005) a beam of electrons aligned close to the crystal axis has been found to be trapped spiralling around a single atomic row and the resonance occurred at energy 80.874 MeV. Considering helical motion of electron, a theoretical explanation for the observed energy was given by Hestenes (2010). In the extensions of semi-classical theories, the spin angular momentum was identified with a bivector and the point particle executes circular motion by absorbing energy from zeropoint field (Doran & Lasenby 2003; Doran *et al*. 1996). Considering entirely a different kinematical approach, the extended particle structure was studied by Rivas (2002). Further, the concept of superimposed *zitterbewegung* motion of electron on its translational motion was advocated by Sidharth (2009).

In the Madelung fluid theories (Salesi 1996; Recami & Salesi 1998a), the spinning particle contains internal extended structure and the internal motion of center of charge leads to the so-called quantum potential. The quantum potential term in the Hamiltonian is simply the non-classical energy term. In the absence of spin and the quantum potential term, the particle motion will be converted into purely classical or Newtonian. It appears that the presence of particle spin gives the quantum potential in the Schrödinger equation (Salesi & Recami 1998b).

All the above extended theories of a charged particle elucidate that quantum nature may be a direct consequence of spin angular momentum (Pavsic *et al*. 1993).

1.3.1. Extended particle structure

All the above studies univocally suggests that a charged elementary particle contains a substructure described by circularly rotating point charge around center of mass point with frequency equal to *zitterbewegung* frequency and the average radius of rotation equal to half the Compton wavelength. The deviations of the path of the particle are expected to be due to the extended structure. The internal circular motion is accountable for both spin angular momentum and the magnetic moment. However, in quantum mechanics and in quantum field theories the particles are not having any size and considered as point particles and the high energy scattering experiments reveals that the electron has very small size $\sim 10^{-20}\ m$. The concept of extended structure has been ignored in modern quantum mechanics but singularity syndrome haunts such that certain renormalization or cut-off procedure has been adopted which again leads to finite structure. Since the charge rotation is at the speed of light, the structure of electron cannot be detected in the scattering experiments.

1.4. The language of complex vectors and complex vector algebra

In general, geometric algebra is a superior algebra when compared to vector, matrix and tensor algebras (Hestenes 2003a). The fundamental element of geometric algebra is the geometric product of two vectors. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ are two arbitrary vectors and the geometric product is defined as

$$\boldsymbol{ab} = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b} \quad (3)$$

The symmetric scalar product is defined as

$$\boldsymbol{a}.\boldsymbol{b} = \tfrac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba}) \quad (4)$$

The asymmetric bivector or wdge product is defined as

$$\boldsymbol{a}\wedge\boldsymbol{b} = \tfrac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba}) \quad (5)$$

The bivector $\boldsymbol{a}\wedge\boldsymbol{b}$ represents an oriented plane. When the order of vectors is changed, we call it a reversion operation denoted by an over bar.

$$\overline{\boldsymbol{a}\wedge\boldsymbol{b}} = \boldsymbol{b}\wedge\boldsymbol{a} = -\boldsymbol{a}\wedge\boldsymbol{b} \quad (6)$$

Thus, a bivector product changes its sign under reversion operation. The reversion operation on geometric product changes the sign of a bivector.

$$\overline{\boldsymbol{ab}} = \boldsymbol{a}.\boldsymbol{b} - \boldsymbol{a}\wedge\boldsymbol{b} \quad (7)$$

Fundamentally the angular momentum is the area swept out by the particle as it moves in curved path. In geometric algebra the orbital angular momentum is basically a bivector $L = \boldsymbol{r}\wedge\boldsymbol{p}$ representing an oriented plane rather than a vector perpendicular to the plane. The bivector definition for the quantities like angular momentum, torque, angular velocity etc., is a better choice than using the cross product. A trivector is defined as a wedge product of three vectors and represents an oriented volume.

$$\boldsymbol{abc} = \boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c} \quad (8)$$

However, the product $\boldsymbol{a}.\boldsymbol{b}\wedge\boldsymbol{c} = \boldsymbol{a}\wedge\boldsymbol{b}.\boldsymbol{c}$ represents a vector.

Let a set of unit orthonormal basis vectors $\{\boldsymbol{\sigma}_i; i = 1,2,3\}$ satisfying the relation $\boldsymbol{\sigma}_i.\boldsymbol{\sigma}_j = \delta_{ij}$ span entire Euclidian space. The unit bivectors are defined as $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j = \boldsymbol{\sigma}_i\wedge\boldsymbol{\sigma}_j$ represent oriented unit plane. The product

$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_3 = \boldsymbol{i}$ is called pseudoscalar representing a unit-oriented volume. With this definition, the unit bivectors can be written in the form $\boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j = \boldsymbol{i}\boldsymbol{\sigma}_k$. The pseudoscalar $\boldsymbol{i}$ commutes with all vectors in three-dimensional Euclidean space. The square of a unit bivector and the square of pseudoscalar turn out to be equal to -1 and therefore we have a correspondence between unit imaginary and bivector or pseudoscalar. Throughout this article, we denote the pseudoscalar by $\boldsymbol{i}$ and the unit imaginary by $i = \sqrt{-1}$. A reversion operation on pseudoscalar changes its sign, $\overline{\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3} = -\boldsymbol{i}$. A multivector in geometric algebra is defined as a sum of scalar, vector, bivector and trivector,

$$M = \alpha + \boldsymbol{a} + \boldsymbol{i}\boldsymbol{b} + \boldsymbol{i}\beta, \tag{9}$$

where α and β are scalars,$\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, $\boldsymbol{i}\boldsymbol{b}$ is a bivctor and $\boldsymbol{i}\beta$ is a trivector. Because of the property $\boldsymbol{i}^2 = -1$, a multivector is equal to a sum of complex vector and complex scalar. The complex scalar is defined as

$$z = \alpha + \boldsymbol{i}\beta \tag{10}$$

and it is equivalent to a complex number. The complex vector is defined as a sum of a vector and a bivector.

$$Z = \boldsymbol{a} + \boldsymbol{i}\boldsymbol{b} \tag{11}$$

A complex vector conjugate is obtained by taking reversion operation on Z.

$$\bar{Z} = \boldsymbol{a} - \boldsymbol{i}\boldsymbol{b} \tag{12}$$

Either squaring a complex vector or its conjugate gives a scalar $Z^2 = \bar{Z}^2 = a^2 - b^2$ and the product

$$Z\bar{Z} = a^2 + b^2 - \boldsymbol{i}(\boldsymbol{a} \wedge \boldsymbol{b}) \tag{13}$$

is equal to a sum of a scalar and a vector. It is trivial to find that a sum of two arbitrary complex vectors is equal to a complex vector. Let $Y = \boldsymbol{c} + \boldsymbol{i}\boldsymbol{d}$ and the geometric product

$$ZY = Z.Y + Z \wedge Y, \tag{14}$$

Where $\mathrm{Z.Y} = \boldsymbol{a}.\boldsymbol{c} + \boldsymbol{b}.\boldsymbol{d} + \boldsymbol{i}(\mathbf{b} \wedge \boldsymbol{c} - \boldsymbol{a} \wedge \boldsymbol{d})$ and $\mathrm{Z} \wedge \mathrm{Y} = \mathbf{a} \wedge \boldsymbol{c} + \boldsymbol{b} \wedge \boldsymbol{d} + \boldsymbol{i}(\boldsymbol{b}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{d})$. Thus the geometric product of two complex vectors is equal to a multivector. One of the known examples of complex vector is the electromagnetic field $F = \boldsymbol{E} + \boldsymbol{i}\boldsymbol{B}$, where $\boldsymbol{E}$ is the electric field vector and $\boldsymbol{i}\boldsymbol{B}$ is the magnetic field bivector.

Let us consider a special case of complex vector $Z = \boldsymbol{a} + \boldsymbol{i}\boldsymbol{b}$, such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal i.e., $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{0}$ and let $\boldsymbol{c} = \boldsymbol{i}(\boldsymbol{a} \wedge \boldsymbol{b})$ be a vector normal to the bivector plane $\boldsymbol{a} \wedge \boldsymbol{b}$. Choosing the unit orthonormal vector basis $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3)$ along $\boldsymbol{a}$,$\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, thetrivector product $\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} = \boldsymbol{i}|\boldsymbol{a}||\boldsymbol{b}||\boldsymbol{c}|$. Now, the vector basis $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3)$ can be expressed in terms of Z and $\bar{Z}$ by the following relations.

$$\boldsymbol{\sigma}_1 = \frac{Z+\bar{Z}}{2|\boldsymbol{a}|} \;;\quad \boldsymbol{\sigma}_2 = \frac{Z-\bar{Z}}{2|\boldsymbol{b}|} \;;\quad \boldsymbol{\sigma}_3 = \frac{Z\wedge\bar{Z}}{|\boldsymbol{a}||\boldsymbol{b}|} \tag{15}$$

Further, we define a unit scalar $1 = (Z.\bar{Z})/2(a^2 + b^2)$. Thus, the complex vectors Z and $\bar{Z}$ generate a closed complex four-dimensional linear space defined by basis set $\{1, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$. A detailed version of complex vector space is presented previously (Muralidhar 2015). One can express the physical quantities in terms of a product of a complex vector and it's conjugate and therefore, a twofold nature is attributed to the complex vector space.

The four-dimensional spacetime can be obtained by introducing a unit vector $\boldsymbol{\sigma}_0$ along the future light cone and its magnitude $|\boldsymbol{\sigma}_0| = \sqrt{+1}$. Then the spacetime unit covariant vectors are defined in the form $\gamma_0 = \boldsymbol{\sigma}_0$ and $\gamma_k = \boldsymbol{\sigma}_k\gamma_0, (k = 1, 2, 3)$ such that the set of right-handed basis vectors $\{\gamma_\mu;\ \mu = 0,1,2,3\}$ span spacetime. The contravarient unit vectors γ^μ are defined as $\gamma_0 = \gamma^0$ and $\gamma^k = -\gamma_k$. For elaborate details of spacetime and geometric algebra one may refer (Doran & Lasenby 2003).

2. Particle structure and spin angular momentum in the complex vector space

From the introduction, it is clear that a charged particle contains an internal extended structure. Further, the particle structure can be better explained in the complex vector approach adopted here. Let *m* and *e* be the mass and charge of a particle and in the presence of fluctuating zeropoint field it can be visualised as an oscillator, say with frequency ω_0. Due to inertia the center of mass fails to follow the high frequency oscillations of the center of charge and it is identified with the mean deviation of the particle oscillator. Thus, the center of mass and center of charge are separated which is responsible for the mean deviations in the path of the particle. The internal oscillations are better explained by local complex rotations in the complex vector space (Muralidhar 2014; 2015). Such complex rotations are responsible for the average deviation of the path of the particle. Denoting the center of mass position by a vector $\boldsymbol{x}$ and the radius of average rotation by a vector $\boldsymbol{\xi}$, the complex vector representing the particle position can be written in the following form.

$$X = \boldsymbol{x} + \boldsymbol{i\xi} \tag{16}$$

The deviations of the particle are considered to be normal to its center of mass path, the vectors $\boldsymbol{x}$ and $\boldsymbol{\xi}$ satisfy the relation $\boldsymbol{x}.\boldsymbol{\xi} = 0$. In the complex vector Z, the bivector $\boldsymbol{i\xi}$ represents an oriented plane with counter-clockwise rotation and vector $\boldsymbol{x}$ lie in this plane. The conjugate of the complex vector X is obtained by taking a reversion operation on it and we write $\bar{X} = \boldsymbol{x} - \boldsymbol{i\xi}$ and here $-\boldsymbol{i\xi}$ represents an oriented plane with clockwise rotation. Thus, the complex motion of charge contains both clockwise and counter-clockwise rotations. Further, the internal charge rotations may be called complex rotations on a spherical shell with radius ξ in the complex vector space. Another important aspect of complex vector is that it can be decomposed into symmetric and asymmetric parts representing the vector $\boldsymbol{x} = (X + \bar{X})/2$ and bivector $\boldsymbol{i\xi} = (X - \bar{X})/2$ respectively. The average extended particle motion can be expressed by the symmetric scalar product $X.\bar{X} = \bar{X}.X = x^2 + \xi^2$ and as the center mass advances the particle center of charge executes helical motion. The asymmetric product $X \wedge \bar{X} = 2\boldsymbol{i\xi} \wedge \boldsymbol{x}$ is a vector normal to both $\boldsymbol{x}$ and $\boldsymbol{\xi}$ and represents coupling between internal structure and the motion of center of mass. To reveal how the spacetime emerges from the complex vector space, we find the complex element dX by differentiating (16) on both sides.

$$dX = d\boldsymbol{x} + \boldsymbol{i}d\boldsymbol{\xi} \tag{17}$$

In the rest frame of the particle, we visualise the internal charge rotation is at the speed of light, c and the angular frequency of charge rotation $\omega_0 = c/\xi$. Since the vector $\boldsymbol{\xi}$ is an oscillating quantity, one can express it as a rotor on the internal complex plane with frequency of rotation ω_0. Now, the scalar product $dX.d\bar{X}$ becomes

$$dX.d\bar{X} = dx^2 + d\xi^2 = dx^2 - \omega_0^2\xi^2dt^2 \tag{18}$$

Identifying this product with the square of proper distance $ds^2 = -c^2d\tau^2 = dX.d\bar{X} = |dX|^2$, we find the complex vectors, X and $\bar{X}$ represent a four-dimensional physical complex vector space in which the motion of the extended particle is described. Since $\omega_0^2\xi^2 = c^2$, (18) gives the spcetime metric.

$$c^2d\tau^2 = c^2dt^2 - dx^2 \tag{19}$$

Thus, the relativistic effects that we observe in general are the consequence of the particle structure and its complex motion in zeropoint field. It may be seen that in the rest frame of the particle, the charge has a circular motion and when it is observed from an arbitrary frame of reference, the particle motion appears to be helical as the center of mass proceeds with velocity $\boldsymbol{v}$. It has been shown by Cavalleri *et al.* (2010) that the helical path of center of charge is responsible for relativistic effects. When the particle motion is observed from an arbitrary frame of reference, the proper time is attached with the center of mass and in the rest frame of the particle and local time is connected with the center of charge. Differentiating (16) with respect to time gives the velocity complex vector U.

$$U = \boldsymbol{v} + \boldsymbol{iu} \quad (20)$$

Here, the velocity of center of mass $\boldsymbol{v} = d\boldsymbol{x}/dt$ and the instantaneous velocity of internal rotation $\boldsymbol{u} = d\boldsymbol{\xi}/dt$. Now, the product of mass times the internal velocity gives the internal momentum $\boldsymbol{\pi} = m\boldsymbol{u}$. The scalar product of the complex vector U with its complex conjugate $\bar{U}$ is

$$U.\bar{U} = v^2 + u^2. \quad (21)$$

The frequency of internal complex rotation is equal to the ratio between u and ξ, $\omega_0 = u/\xi$ and denoting $\omega_B = v\omega_0/c = v/\xi$ and dividing (21) throughout by ξ^2 gives

$$\omega_c^2 = \omega_0^2 + \omega_B^2, \quad (22)$$

where $\omega_c^2 = |U|^2/\xi^2$ and turns out to be the square of cut-off frequency of the particle complex motion in the zeropoint field. Thus from (22), one can conclude that the extended particle motion in zeropoint field is a superposition of translational motion of center of mass and internal rotational motion of center of charge. The wave length associated with the frequency ω_B is then expressed as

$$\lambda_B = \frac{2\pi c}{\omega_B} = \frac{h}{mv} \ , \quad (23)$$

where the magnitude of ξ is taken to be of the order of Compton wavelength of the particle. The above treatment gives the wave nature of particles in the quantum domain and clarifies the genesis of de Broglie hypothesis. The wave nature of particles in quantum mechanics is thus the manifestation of oscillations of the particle in the zeropoint field. When observed from an arbitrary frame of reference, the particle path is described by complex vectors X and U and their conjugates. In the particle frame, it will be a complex rotation described by internal parameters $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ and these parameters are considered to be the deviations of the particle from its mean path. This point of view univocally suggests that the presence of zeropoint field means the space is not continuous but discrete and the particle may be visualised to travel in different paths in complex vector space. Thus, the Feynman paths of quantum particles simply emerge due to internal oscillations of the particle in zeropoint field and such paths are described by complex vectors in complex vector space. Further, the above analysis shows that time itself is a function of complex vector X and its conjugate $\bar{X}$ in complex vector space, $t = t(X, \bar{X})$ and the time interval dt can be expanded in the form

$$dt = \frac{1}{2}\left(\frac{dX}{U} + \frac{d\bar{X}}{\bar{U}}\right) = \frac{\boldsymbol{u}d\boldsymbol{\xi} - \boldsymbol{v}d\boldsymbol{x}}{(c^2 - v^2)}, \quad (24)$$

where we have used $U^2 = \bar{U}^2 = v^2 - c^2$. Thus, equation (24) reveals the fact that the time derivative of a physical quantity is entirely different from the one in classical form.

The momentum of center mass will be defined as relativistic momentum and the momentum complex vector can be constructed in the form

$$P = \boldsymbol{p} + \boldsymbol{i\pi}, \quad (25)$$

where $\boldsymbol{p} = \gamma m\boldsymbol{v}$ and γ is the Loretz factor. The complex conjugate of complex momentum is obtained by taking a reversion operation, $\bar{P} = \boldsymbol{p} - \boldsymbol{i\pi}$. Now, the total energy of the particle is expressed as a product of complex momentum and it's conjugate.

$$E^2 = P\bar{P}c^2 = p^2c^2 + m^2c^4 \tag{26}$$

This equation is simply the relativistic energy of the particle obtained from complex vector approach and in the above treatment we have only considered the extended particle structure. However, in the presence of external electromagnetic field, in the minimal coupling prescription, the momentum $\boldsymbol{p} \to \boldsymbol{p} - (e\boldsymbol{A}_{ex}/c)$. In the present approach we consider the external electromagnetic vector potential as a function of complex vector X such that $\boldsymbol{A}_{ex} = \boldsymbol{A}_{ex}(X,t)$ and $\bar{\boldsymbol{A}}_{ex} = \boldsymbol{A}_{ex}(\bar{X},t)$. In general, $\boldsymbol{A}_{ex}$ may contain two parts: an applied electromagnetic vector potential $\boldsymbol{A}_{ap}(X,t)$ and zeropoint electromagnetic vector potential $\boldsymbol{A}_{zp}(X,t)$. Now, the momentum complex vector takes the form $P = \boldsymbol{p} + \boldsymbol{i\pi} - (e\boldsymbol{A}_{ex}/c)$. Considering only the influence of external zeropoint field in the complex momentum function, $P = \boldsymbol{p} + \boldsymbol{i\pi} - [e\boldsymbol{A}_{zp}(X,t)/c]$, the square of total energy can be obtained from the relation $E^2 = P.\bar{P}c^2$. Detailed calculation of finding the product $P.\bar{P}c^2$ is given in the Appendix and (A5) gives

$$E^2 = p^2c^2 + E_0^2. \tag{27}$$

$$E_0^2 = m^2c^4 + e^2A_{zp}^2 + e^2\xi^2\left|\nabla\boldsymbol{A}_{zp}\right|^2 - ec(L+2S).\boldsymbol{iB}_{zp} \tag{28}$$

Where $\boldsymbol{L} = \boldsymbol{x} \wedge \boldsymbol{p}$ is the bivector orbital angular momentum and $S = \boldsymbol{\xi} \wedge \boldsymbol{\pi}$ is the spin angular momentum (see subsection 2.1). To find the meaning of each term in (28) we express E_0 in an approximate form

$$E_0 \sim mc^2 + \frac{e^2A_{zp}^2}{2mc^2} + \frac{e^2\xi^2}{2mc^2}\left|\nabla\boldsymbol{A}_{zp}\right|^2 - \frac{e}{2mc}(L+2S).\boldsymbol{iB}_{zp}, \tag{29}$$

The last terms in the above equation represents interaction energy due to orbital and spin magnetic moments with zeropoint field. The second and third terms on right of (29) actually contribute to particle mass correction in the presence of zeropoint field. Expressing the electromagnetic vector potential in plane wave form, the stochastic average $\langle A_{zp}^2\rangle$ can be calculated. A detailed calculation using the method given by Haisch *et al.* (1994) is given in the appendix and the second term on right of (29) is given by (A6).

$$\frac{e^2}{2mc^2}\langle A_{zp}^2\rangle = \frac{\alpha}{2\pi}\frac{(\hbar\omega_0)^2}{mc^2} \tag{30}$$

Where ω_0 is the cut-off frequency and $\alpha = e^2/4\pi\epsilon_0\hbar c$ is the fine structure constant. The cut-off frequency is the frequency of internal oscillations of the particle and using Einstein-de Broglie formula $\hbar\omega_0 = mc^2$, (30) corresponds to correction to mass and can be expressed as

$$\delta mc^2 = \frac{\alpha}{2\pi}mc^2. \tag{31}$$

Similarly, the third term on right of (29) is expected to give a further very small correction to mass and using

$$\left|\nabla\boldsymbol{A}_{zp}\right|^2 = \left|\frac{\Delta A_{zp}}{\Delta x}\right|^2 \sim \frac{A_{zp}^2}{x^2},$$

it may be expressed as

$$\frac{e^2}{2mc^2}\frac{\xi^2}{x^2}\langle A_{zp}^2\rangle.$$

Since a valid approximation always gives a good result, ξ/x may be approximated as the ratio between Compton wavelength and de Broglie wavelength.

$$\frac{\xi^2}{x^2} = \frac{v^2}{c^2} = \beta^2, \tag{32}$$

Now, the third term on right of (29) can be expressed in the form

$$\frac{e^2\xi^2}{2mc^2}\left|\nabla \boldsymbol{A}_{zp}\right|^2 = -\frac{e^2}{2mc^2}\beta^2 A_{zp}^2. \tag{33}$$

Taking the stochastic average of this expression and using this result in (30), the mass correction due to second and third terms on right of (29) can be obtained as

$$\delta mc^2 = \frac{\alpha}{2\pi}(1+\beta^2)mc^2. \tag{34}$$

Due to local complex rotations in the zeropoint field, the extended particle absorbs zeropoint energy and this appears as the so-called bare mass. When the extended particle is in motion, it interacts with the surrounding zeropoint field and leads to some mass correction. Thus, we have shown here that the consideration of extended particle motion in zeropoint field leads to mass correction term and further, interestingly (34) contains velocity dependent term. The only difference is that we replace the theoretical mass by the observed mass and such correction to mass produces first order corrections what we expect in anomalous magnetic moment, Lamb shift etc. The mass correction calculated in (31) gives the first order correction to the g-factor and it coincides with the value obtained by Schwinger (1948) in quantum electrodynamics. The additional correction terms in (34) gives the higher order corrections in the expansion of anomalous magnetic moment of electron.

The above complex vector approach is also applicable for neutral micro-particle which acquires instantaneous polarity in the zeropoint field over a short period of time and therefore treated as an oscillator. Any neutral particle may be viewed as a composite of charged elementary particles of charge either negative or positive with combined total charge zero and one can expect instantaneous charge center which leads to deviation in the path of the particle. The helical motion of such charge center gives the relativistic effects in general. The spin angular momentum of a neutral particle may be determined from the individual particles contained in it. For example, neutrons are having spin-half and one can consider deviations in the center of mass of such particles in the presence of zeropoint field.

A macroscopic body (a classical object) contains N number of micro-particles either charged or neutral and in the complex vector space, the position of center of mass of such macroscopic body can be defined by considering the position of each individual micro-particle by a complex vector X_a and mass by m_a.

$$X_r = \frac{\sum m_a X_a}{M} = \frac{\sum m_a x_a}{M} + \frac{\sum m_a i\boldsymbol{\xi}_{\boldsymbol{a}}}{M} = \boldsymbol{r} + \boldsymbol{i}\boldsymbol{\xi}_c, \tag{35}$$

where the mass of the macroscopic body $M = \sum m_a$, the average random deviation of the point at the center of mass of the macroscopic body is denoted by $\boldsymbol{\xi}_c = \sum m_a \boldsymbol{\xi}_{\boldsymbol{a}}/M$ and $\boldsymbol{r}$ is the position vector of center of mass. The deviation $\boldsymbol{\xi}_c$ may be conceptualized as the effective deviation that arises considering all the average random deviations of individual particles of the massive body. Further form the above procedure, it is quite clear that the magnitude of average random deviation of individual particle is also equivalent to the magnitude of radius of rotation of individual particles. For instance, when the individual particles have almost same mass m, the effective deviation equals the random deviation of individual particles and therefore the magnitude $|\boldsymbol{\xi}_c| \sim |\boldsymbol{\xi}| = \hbar/2mc$. The complex vector in (35) is similar to the one in (16) and following the similar procedure as above one can arrive at the relativistic effects of a macro-particle. The above treatment reveals the fact the complex vector approach takes care of both internal helical motion of the particle and the center of mass motion and the dynamics reveal the relativistic effects naturally. In the above treatment we have considered the particle as an extended object and in general the average deviations may be considered as path deviations. The complex vector approach considered here is a generalized theory and the following sections are devoted for the elaboration of classical to quantum transition.

2.1. The spin angular momentum

It is possible to develop the foundations of relativistic quantum mechanics from the local fluctuations represented by complex rotation and such complex rotation possesses a bivector angular momentum at the particle level. This angular momentum is known as zeropoint angular momentum or spin angular momentum and it plays an important role in the development of quantum mechanics. Now, the bivector product $X \wedge \bar{P}$ can be expanded in the following form

$$X \wedge \bar{P} = \boldsymbol{x} \wedge \boldsymbol{p} + \boldsymbol{\xi} \wedge \boldsymbol{\pi} + \boldsymbol{i\xi} \wedge \boldsymbol{p} - \boldsymbol{ix} \wedge \boldsymbol{\pi}. \quad (36)$$

Utilising the fact that the reversion operation changes only bivector terms in (36), we obtain the total angular momentum by eliminating vector terms.

$$J = \frac{1}{2}[(X \wedge \bar{P}) + (\bar{X} \wedge P)] = L + S \quad (37)$$

Where $L = \boldsymbol{x} \wedge \boldsymbol{p}$ is the orbital angular momentum and S is the angular momentum of internal complex rotation and represents particle spin angular momentum.

$$S = \boldsymbol{\xi} \wedge \boldsymbol{\pi} \quad (38)$$

This is also known as zeropoint angular momentum because it appears due to charged particle interaction with zeropoint field. The spin angular momentum S lies along the plane normal to the path of center of mass. It implies that the scalar product $S.\boldsymbol{v} = 0$. The conventional spin vector $\boldsymbol{s}$ can be defined by the relation $S = \boldsymbol{is}$ If we choose the center of mass motion along z-axis it represents the component of spin $\boldsymbol{s}_3 = \boldsymbol{\sigma}_3\hbar/2$ and the other two components of spin angular momentum can be defined in a similar way in the form $\boldsymbol{s}_1 = \boldsymbol{\sigma}_1\hbar/2$ and $\boldsymbol{s}_2 = \boldsymbol{\sigma}_2\hbar/2$. These components of spin angular momentum do satisfy the commutation and anti-commutation relations $[\boldsymbol{s}_i, \boldsymbol{s}_j] = \epsilon_{ijk}\hbar\boldsymbol{is}_k$ and $\{\boldsymbol{s}_i, \boldsymbol{s}_j\} = 0$ respectively. Similarly, one can express the bivector spin angular momentum components as $S_k = \boldsymbol{is}_k$ and these components also satisfy the commutation relation $[S_i, S_j] = -\epsilon_{ijk}\boldsymbol{i}\hbar S_k$ and the anti-commutation relation $\{S_i, S_j\} = 0$. One can as well express

$$|S^2| = |S_1^2 + S_2^2 + S_3^2| = \frac{3}{4}\hbar^2 \ . \quad (39)$$

We have seen that the spin bivector is always normal to the path of center of mass. Therefore, it is not necessary that one must consider all the three components of spin. This is because of the fact that the product $S.\boldsymbol{v}$ for other two components turns out to be a vector and it vanishes under complete rotation and if one spin component is measured the other two components become automatically zero. It means all the three components cannot be measured at a time.

2.1.1. Spin angular momentum and internal rotations

In depth understanding of spin angular momentum of a particle can be obtained by considering the oscillations of random particle oscillator in the zeropoint field (Muralidhar 2011; 2012). The particle oscillator in general may be visualised as a superposition of two types of oscillation modes corresponding to clockwise and counter clockwise rotations on a complex plane. The vector $\boldsymbol{\xi}$ represents a rotor in the complex plane describing the internal rotation and it can be expressed in the following form.

$$\boldsymbol{\xi} = \boldsymbol{\xi}_0 \exp(\boldsymbol{i\sigma}_s \omega_0 t), \tag{40}$$

where ω_0 is the frequency of resonant oscillations of the particle in the fluctuating zeropoint fields and $\boldsymbol{\xi}_0 = \boldsymbol{\xi}(0)$. The vector $\boldsymbol{\xi}$ in (40) represents counter-clockwise rotations and a reversion operation on $\boldsymbol{\xi}$ gives $\bar{\boldsymbol{\xi}}$ which represents clockwise rotations.

$$\bar{\boldsymbol{\xi}} = \boldsymbol{\xi}_0\, exp(-\boldsymbol{i\sigma}_s \omega_0 t) \tag{41}$$

Superposition of these rotations can be done particularly in two ways and denoted by

$$\boldsymbol{\xi}_+ = \tfrac{1}{2}(\boldsymbol{\xi} + \bar{\boldsymbol{\xi}}) = \boldsymbol{\xi}_0 \cos \omega_0 t \;\; ; \;\; \boldsymbol{\xi}_- = \tfrac{1}{2}(\boldsymbol{\xi} - \bar{\boldsymbol{\xi}}) = \boldsymbol{\xi}_0 \boldsymbol{i\sigma}_s \sin \omega_0 t. \tag{42}$$

The average values of these modes over a complete rotation are zero. However, the average values over half rotation differ.

$$\langle \boldsymbol{\xi}_+ \rangle = 0 \;\; ; \;\; \langle \boldsymbol{\xi}_- \rangle = \tfrac{1}{2}\langle \boldsymbol{\xi} - \bar{\boldsymbol{\xi}} \rangle = \frac{\boldsymbol{\xi}_0 \boldsymbol{i\sigma}_s}{\boldsymbol{\pi}} \tag{43}$$

Physically speaking when particle rotations on complex plane are half way along one direction and half way in the opposite direction means the rotation is considered null and hence the mean position of the oscillator represents any oscillations. This mode $\langle \boldsymbol{\xi}_+ \rangle$ corresponds to symmetric mode. The other mode $\langle \boldsymbol{\xi}_- \rangle$ corresponds to asymmetric mode and its effect is to produce a half rotation in the complex plane and it can be either counter-clockwise or clockwise depending on the mode structure $(\boldsymbol{\xi} - \bar{\boldsymbol{\xi}})$ or $(\bar{\boldsymbol{\xi}} - \boldsymbol{\xi})$ respectively. Assume that the phase angle $\omega_0 t$ is small and differentiating $\boldsymbol{\xi}_+$ and $\boldsymbol{\xi}_-$ with respect to time gives the instantaneous velocities of rotation for symmetric and asymmetric modes denoted by $\boldsymbol{u}_s$ and $\boldsymbol{u}_a$ respectively.

$$\boldsymbol{u}_s(t) = \boldsymbol{\xi}_0 \omega_0 \;\; ; \;\; \boldsymbol{u}_a(t) = \boldsymbol{\xi}_0 \boldsymbol{i\sigma}_s \omega_0 \;\; ; \;\; \boldsymbol{u}_a(-t) = -\boldsymbol{\xi}_0 \boldsymbol{i\sigma}_s \omega_0 \tag{44}$$

The difference of these velocities of asymmetric modes can be written as $\Delta\boldsymbol{u}_+ = \boldsymbol{u}_a(t) - \boldsymbol{u}_a(-t)$ and $\Delta\boldsymbol{u}_- = \boldsymbol{u}_a(-t) - \boldsymbol{u}_a(t)$ respectively. The angular frequency of these two modes is a bivector and written as $\Omega_\pm = \pm\boldsymbol{i\sigma}_s 2\omega_0$. The angular momentum of these internal half rotations can be expressed as a bivector

$$S = \boldsymbol{\xi}_0 m \boldsymbol{u}_a(t) = \boldsymbol{i\sigma}_s m \omega_0 \xi_0^2. \tag{45}$$

Depending on the orientation of bivector $\boldsymbol{i\sigma}_s$, the spin has either positive or negative value and the magnitude of spin frequency is $2\omega_0$. In the classical stochastic electrodynamics approach, the stochastic average deviation from the path has been calculated by Boyer (1975a) and the detailed derivation is given in the appendix. From (A15) the average deviation from the path is given by $\langle \delta x^2 \rangle = \xi_0^2 = \hbar/2m\omega_0$ and substituting this result in (45) gives the bivector spin.

$$S = \frac{\hbar}{2} \boldsymbol{i\sigma}_s \tag{46}$$

Thus, the spin angular momentum arises due to the complex half rotations and it should be noted that the particle spin angular momentum is purely a classical quantity and it is not either attributed to self-rotation of the particle or purely intrinsic nature of the particle as normally addressed in quantum mechanics. Further, the energy associated with the asymmetric modes must be equal to the angular momentum times the frequency of oscillations and the energy of each asymmetric mode i.e., for one half rotation is $\hbar\omega_0/2$ and this is simply the energy per mode of internal oscillator in the zeropoint field. On the other hand, the frequency associated with the full rotation or symmetric mode is equal to ω_0 and the corresponding energy is $\hbar\omega_0$.

In stochastic electrodynamics, the magnitude of spin has been estimated by several authors. The first estimation of spin was due to Marshall (1963) and the ground state angular momentum of particle oscillator in zeropoint field has been found to be $\langle s^2\rangle = 3\hbar^2/4$. Using action angle variables of particle harmonic oscillator, Boyer (Boyer 1978) found average action variable $\langle J\rangle = \hbar/2$. An elegant approach to the internal spin that is connected with the fluctuating zeropoint field has been shown by Sachidanandam (1983) and the internal angular momentum was shown to be equal to $\hbar/2$. In a quasi-Markovian process, the spin has been estimated by Cetto *et al*. (2014).

The rate of change of spin angular momentum in the rest frame of the particle is obtained by differentiating (38) with respect to time.

$$\dot{S} = \dot{\boldsymbol{\xi}} \wedge \boldsymbol{\pi} + \boldsymbol{\xi} \wedge \dot{\boldsymbol{\pi}}. \tag{47}$$

The over dot in the above equation represents differentiation with respect to time. Since $\dot{\boldsymbol{\xi}} = \boldsymbol{i\sigma}_s\omega_{\mathbf{0}}\boldsymbol{\xi}$ and $\dot{\boldsymbol{\pi}} = \boldsymbol{i\sigma}_s\omega_0\boldsymbol{\pi}$, (47) becomes

$$\dot{S} = -\Omega_s \wedge S, \tag{48}$$

where $\Omega_s = -\boldsymbol{i\sigma}_s 2\omega_{\mathbf{0}}$. This can be expressed in an exponential form

$$S(t) = S(0)e^{-\Omega_s t}, \tag{49}$$

where $S(0)$ is the spin at time $t = 0$. Now, we introduce the Lorenz boost with rapidity parameter ϕ in the form

$$L_b = e^{\frac{\hat{v}\phi}{2}}, \tag{50}$$

where $\cos\phi = \gamma$ and $\sin\phi = \beta\gamma$ and $\hat{\boldsymbol{v}}$ is the unit vector along particle velocity $\boldsymbol{v}$. Applying Lorentz boost on the spin bivector S we find

$$L_b^{-1} S L_b = S\gamma + \gamma S.\boldsymbol{\beta}. \tag{51}$$

Since $S.\boldsymbol{\beta} = 0$, one can express proper spin bivector

$$S_0 = \gamma S. \tag{52}$$

Using $dt = \gamma d\tau$ and differentiating the above equation with respect to proper time τ gives

$$\frac{dS_0}{d\tau} = -\Omega_s \wedge S, \tag{53}$$

2.1.2. Quantum states of spin angular momentum

Since the particle spin is identified with half rotations, the state of the particle spin can be expressed by a rotor

$$R_s(t) = R_s(0)\exp\left(\frac{\Omega_s t}{2}\right). \tag{54}$$

The rotor R_s represents a half rotation in $\boldsymbol{i\sigma}_s$ plane and corresponds to the spin up-state. Because of the half angle, the rotor is symmetric with 4π rotation. Denoting $|\Omega_s|t = \theta$, we find the integral

$$\int_0^{4\pi} R_s(\theta)\overline{R_s}(\theta)d\theta = 2\pi\hbar \tag{55}$$

and the rotor can be expressed in normalized form

$$R_s(\theta) = \frac{R_s(0)}{(2\pi\hbar)^{1/2}}\exp\left(-\frac{\boldsymbol{i\sigma}_s\theta}{2}\right). \tag{56}$$

A reversion operation on $R_s(\theta)$gives its complex conjugate $\overline{R_s}(\theta)$ and it corresponds to spin-down state.

$$\bar{R}_s(\theta) = \frac{R_s(0)}{(2\pi\hbar)^{1/2}}\exp\left(\frac{\boldsymbol{i\sigma}_s\theta}{2}\right) \tag{57}$$

With the definition of normalized rotor in (56) at hand, one can define the probability of finding the particle spin in the local space.

$$\mathcal{P} = \int_0^{4\pi} R_s(\theta)\bar{R}_s(\theta)d\theta = 1 \tag{58}$$

Now, we are in a position to define spin operator in the form

$$S_{op} = \boldsymbol{i\sigma}_s\hbar\frac{\partial}{\partial\theta}, \tag{59}$$

and the quantum conditions for the particle in spin-up and spin-down states can be expressed as

$$S_{op}R_s(\theta) = \frac{\hbar}{2}R_s(\theta);\ S_{op}\bar{R}_s(\theta) = -\frac{\hbar}{2}\bar{R}_s(\theta) \tag{60}$$

These relations are simply the consequence of the definition of the operator in (59) and in conventional quantum mechanics we write the expectation value

$$\langle\bar{R}_s(\theta)S_{op}R_s(\theta)\rangle = \frac{\hbar}{2}. \tag{61}$$

In the above treatment, the spin operator is defined for the spin angular momentum in a general case and if we choose $\boldsymbol{\sigma}_s$ along z-direction the operator becomes $\hat{S}_z = \boldsymbol{i\sigma}_3\hbar\,\partial/\partial\theta$. In a similar way one can define the spin operators along y and x directions as $\hat{S}_y = \boldsymbol{i\sigma}_2\hbar\,\partial/\partial\theta$ and $\hat{S}_x = \boldsymbol{i\sigma}_1\hbar\,\partial/\partial\theta$ respectively and it is trivial to find

$$\overline{R_s}(\theta)\hat{S}_y R_s(\theta) = \frac{\hbar}{2}\left(\boldsymbol{i\sigma}_1\cos\frac{\theta}{2} - \boldsymbol{i\sigma}_2\sin\frac{\theta}{2}\right) \tag{62}$$

$$\overline{R_s}(\theta)\hat{S}_x R_s(\theta) = \frac{\hbar}{2}\left(\boldsymbol{i\sigma}_2\cos\frac{\theta}{2} + \boldsymbol{i\sigma}_1\sin\frac{\theta}{2}\right). \tag{63}$$

Thus, we see that x and y components of spin angular momentum are oscillatory in nature and their expectation values vanish as expected.

Without loss of generality, taking the normalisation constant in (57) equal to unity, the rotor representing spin rotation can also be written in the form

$$R(\theta) = \exp\left(-\frac{i\sigma_s\theta}{2}\right). \tag{64}$$

Such rotors are also known as spinors. Since it is equivalent to a scalar and a bivector, it forms an even sub-algebra of geometric algebra and satisfies the unitary property $R(\theta)\bar{R}(\theta) = 1$. Consider a small rotation $\delta\theta$ in the $\boldsymbol{i\sigma}_s$ plane. Then the rotor can be written as

$$R(\delta\theta) = \exp\left(-\frac{i\sigma_s\delta\theta}{2}\right) = 1 - \frac{i\sigma_s}{\mathbf{2}}\delta\theta \tag{65}$$

Differentiating this expression with $\delta\theta$ shows that $\boldsymbol{i\sigma}_s/2$ is the generator of rotations. If we choose $\boldsymbol{\sigma}_s$ along z-axis then the unit bivector $\hat{S}_3 = \boldsymbol{i\sigma}_3/2$ is the generator of rotations along the plane of spin component S_3. Similarly, $\hat{S}_2 = \boldsymbol{i\sigma}_2/2$ and $\hat{S}_1 = \boldsymbol{i\sigma}_1/2$ are the generators of rotations along the planes of spin components S_2 and S_1 respectively. These three generators satisfy the commutation relations $[\hat{S}_i, \hat{S}_j] = -\epsilon_{ijk}\hat{S}_k$ and hence form a Lie group. This group is also known as spin group.

2.1.3. The spin magnetic moment and anomalous g-factor

In both Pauli and Dirac theories of electron, the value of gyromagnetic ratio was found to be $g = 2$. However, it has been observed experimentally that it is slightly greater than 2 and such slight deviation was addressed in quantum electrodynamics by considering interaction of radiation with electron. Here, we shall pursue the anomalous gyromagnetic ratio by considering the interaction of zeropoint field with the extended particle. The intrinsic bivector magnetic moment due to spin angular momentum can be expressed as

$$\mu_s = -\frac{e}{mc}S. \tag{66}$$

The orientation of magnetic moment bivector is opposite to the orientation of spin bivector. In the present extended particle structure, there appears to be an instantaneous intrinsic dipole moment $e\boldsymbol{\xi}$. The total electric and magnetic moments can be written as a complex vector in the rest frame of the particle.

$$\mu = \frac{e}{mc}(mc\boldsymbol{\xi} - S) \tag{67}$$

The electromagnetic zeropoint field can be expressed as a complex vector $F_{zp} = \boldsymbol{E}_{zp} + \boldsymbol{iB}_{zp}$. The interaction of F_{zp} with μ gives the energy

$$\mu.F_{zp} = \frac{e}{mc}(mc\boldsymbol{\xi}.\boldsymbol{E}_{zp} - S.\boldsymbol{iB}_{zp}). \tag{68}$$

The average of dipole interaction energy over a complete rotation is zero and therefore the magnetic interaction energy due to zeropoint field is

$$E_m = -\frac{e}{mc}S.\boldsymbol{iB}_{zp}. \tag{69}$$

Note that this energy term is already obtained in (29). However, in the presence of applied magnetic field $\boldsymbol{B}_{ap}$, the total energy of interaction may be expressed as sum of interaction energy due to $\boldsymbol{B}_{ap}$ and the interaction energy due to zeropoint field $\boldsymbol{B}_{zp}$.

$$E_m = -\frac{e}{mc} S.\boldsymbol{iB}_{ap} - \frac{e}{mc} S.\boldsymbol{iB}_{zp}. \tag{70}$$

The frequency of rotation associated with the zeropoint magnetic field is the cyclotron frequency $\omega_0 = eB_{zp}/mc$ and the additional frequency due to the presence of external magnetic field may be expressed in the form $\delta\omega = eB_{ap}/mc$. Using the relation $mc^2 = \hbar\omega_0$, one can express $\delta\omega/\omega_0 = \delta m/m$ and we find

$$E_m = -\left(1 + \frac{\delta m}{m}\right) \frac{e}{mc} S.\boldsymbol{iB}_{zp}. \tag{71}$$

When $g \neq 2$ we have $\frac{g}{2} = 1 + \frac{\delta m}{m}$ and the anomalous magnetic moment of electron is

$$a_e = \frac{g}{2} - 1 = \frac{\delta m}{m}. \tag{72}$$

The method of calculation does not include any infinite terms and subtraction process of mass correction term as in quantum electrodynamics and it clearly gives how the ratio $\delta m/m$ is equal to anomalous g-factor. From (31), the ratio $\delta m/m = \alpha/2\pi$ and it coincides with the quantum electrodynamics calculation to the first order in fine structure constant.

2.1.4. Classical approach to quantum oscillator: The roll of spin angular momentum

Harmonic oscillator is one of the fundamental problems in quantum mechanics and it gives the foundation to understand several aspects of quantum phenomena. In this section we shall pursue how spin angular momentum converts a classical oscillator into quantum oscillator. In the non-relativistic case, kinetic energy of the extended particle in complex vector space can be expressed as

$$E_k = \frac{P.\bar{P}}{2m} = \frac{p^2}{2m} + \frac{\pi^2}{2m}. \tag{73}$$

The potential energy connected with the particle oscillations can be expressed by considering the potential as a function of both X and $\bar{X}$.

$$V(X,\bar{X}) = \frac{1}{2} m\omega_0^2 X.\bar{X} = \frac{1}{2} m\omega_0^2 x^2 + \frac{1}{2} m\omega_0^2 \xi^2 \tag{74}$$

The total energy of the extended particle oscillator is a sum of two oscillator systems: one connected with center of mass and the other connected with the center of charge.

$$H = H_c + H_{zp} \tag{75}$$

Where, $H_c = \left(\frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2\right)$ and $H_{zp} = \left(\frac{\pi^2}{2m} + \frac{1}{2} m\omega_0^2 \xi^2\right)$.

The oscillations of the center of charge are attributed to H_{zp} and it must be equal to the zeropoint energy of the oscillator. However, in more general considerations the internal oscillator may not be at resonance with the fluctuations of the zeropoint spectrum and as a consequence the interaction of zeropoint field may produce random displacements of the center of mass position and which leads to the energy H_c. It may be noted that the energy shift from the average minimum takes place without any change in the particle spin angular momentum. Further, the additional shift in the energy may be assumed to produce higher modes of center of charge oscillations. We have seen that the particle internal oscillations correspond to half rotations connected with asymmetric modes with energy equal to zeropoint energy of the oscillator $\hbar\omega_0/2$. The modes connected with center of mass are symmetric modes corresponding to full rotations and the energy shift in the internal oscillator system must be always in discrete energy shifts with energy in multiples of $\hbar\omega_0$. Thus, the total energy of the oscillator is obtained without considering H_c and considering only H_{zp} along with additional condition that $H \geq \hbar\omega_0/2$. Detailed version of classical approach to quantum oscillator has been studied previously (Muralidhar 2014) and a short account of complex vector approach to particle oscillator is given here. Taking $\boldsymbol{a} = (m\omega_0)^{1/2}\boldsymbol{\xi}$ and $\boldsymbol{b} = (m\omega_0)^{-1/2}\boldsymbol{\pi}$, the bivector product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to the particle spin angular momentum.

$$\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{\xi} \wedge \boldsymbol{\pi} = S \tag{76}$$

The Hamiltonian of the oscillator is now expressed in terms of products of complex vector $Z = k(\boldsymbol{a} + \boldsymbol{i}\boldsymbol{b})$ and its conjugate $\bar{Z} = k(\boldsymbol{a} - \boldsymbol{i}\boldsymbol{b})$, where $k = 1/2\hbar$. Now, the product of complex vector Z and its conjugate $\bar{Z}$ is obtained as

$$Z\bar{Z} = \frac{H}{\hbar\omega_0} - \frac{iS}{\hbar}. \tag{77}$$

Similarly,

$$\bar{Z}Z = \frac{H}{\hbar\omega_0} + \frac{iS}{\hbar}. \tag{78}$$

Adding these two equations gives the Hamiltonian in terms of Z and $\bar{Z}$.

$$H = \frac{1}{2}(Z\bar{Z} + \bar{Z}Z)\hbar\omega_0 \tag{79}$$

and subtracting (78) from (77) gives the following commutation relation

$$[Z, \bar{Z}] = Z\bar{Z} - \bar{Z}Z = \boldsymbol{\sigma}_s, \tag{80}$$

where $\boldsymbol{\sigma}_s$ is the unit vector normal to the bivector spin plane. Without loss of generality one can choose the direction of quantization axis along $\boldsymbol{\sigma}_s$. If we define the idempotent in the form $\mathcal{I}_{\pm} = (1 \pm \boldsymbol{\sigma}_s)/2$ then the action ofthe unit vector $\boldsymbol{\sigma}_s$ on $\mathcal{I}_{\pm}$ gives the eigen values ± 1.

$$\boldsymbol{\sigma}_s \mathcal{I}_{\pm} = \pm \mathcal{I}_{\pm} \tag{81}$$

This equation reveals the fact that the $\mathcal{I}_+$ and $\mathcal{I}_-$ represent the spin-up and spin-down states of the particle oscillator. Multiplying (80) from right by $\mathcal{I}_+$ gives the condition

$$[Z, \bar{Z}]\mathcal{I}_+ = 1\mathcal{I}_+. \tag{82}$$

The internal oscillations are in general at random in the fluctuating zeropoint field. The stochastic average of all such random oscillations may be considered as circular motion in the complex plane and which represents the ground state of the harmonic oscillator. In the steady state condition of ground state, the potential energy is equal to kinetic energy and therefore for the ground state $a = b$. Choosing unit vectors $\boldsymbol{\sigma}_a$ and $\boldsymbol{\sigma}_b$ along the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, it is easy to redefine the complex vector $Z = ka(\boldsymbol{\sigma}_a + \boldsymbol{i}\boldsymbol{\sigma}_b) = 2ka\boldsymbol{\sigma}_a\mathcal{I}_-$ and $\bar{Z} = ka(\boldsymbol{\sigma}_a - \boldsymbol{i}\boldsymbol{\sigma}_b) = 2ka\boldsymbol{\sigma}_a\mathcal{I}_+$. It is trivial to find the products $Z\bar{Z} = 4k^2a^2\mathcal{I}_+$ and $\bar{Z}Z = 4k^2a^2\mathcal{I}_-$. For the ground state of the oscillator $4k^2a^2 = 1$ and using the identities $\mathcal{I}_+\mathcal{I}_+ = \mathcal{I}_+$ and $\mathcal{I}_-\mathcal{I}_+ = 0$ we find $Z\bar{Z}\mathcal{I}_+ = 1\mathcal{I}_+$ and $\bar{Z}Z\mathcal{I}_+ = 0$. Combining these results with (79) gives the ground state energy $E_0 = \hbar\omega_0/2$ which coincides with the ground state energy per mode of the particle oscillator in quantum description and in stochastic electrodynamics approach. In general, there may be some higher energy states of the oscillator and in such cases $a \neq b$. To account for higher energy states, we write $H = H_{n-1} + \hbar\omega_0$, $H = H_{n-2} + 2\hbar\omega_0$ etc. and finally $H = H_0 + n\hbar\omega_0$. It is trivial to show that $H = H_{n-l} + l\hbar\omega_0(, l = 0, 1, 2, \dots n)$ satisfy the condition (82). In conclusion, the products $Z\bar{Z}$ and $\bar{Z}Z$ are proportional to integers.

$$Z\bar{Z}\mathcal{I}_+ = (n+1)\mathcal{I}_+ \;\; ; \quad \bar{Z}Z\mathcal{I}_+ = n\mathcal{I}_+ \tag{83}$$

Using the condition $H\mathcal{I}_+ = E_n\mathcal{I}_+$, the relations in (77) and (78) can be written in the form

$$(n+1)\hbar\omega_0\mathcal{I}_+ = (E_n - \boldsymbol{i}\omega_0 S)\mathcal{I}_+ \tag{84}$$

$$n\hbar\omega_0\mathcal{I}_+ = (E_n + \boldsymbol{i}\omega_0 S)\mathcal{I}_+. \tag{85}$$

Similar relations can be obtained by employing $\mathcal{I}_-$ in place of $\mathcal{I}_+$. Adding the above equations, we arrive at the energy of the quantum harmonic oscillator.

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0. \tag{86}$$

The complex vector approach to the harmonic oscillator elucidates the spin connection to the zeropoint field and the existence of spin itself converts a classical oscillator into quantum oscillator. A detailed discussion on the particle oscillator univocally suggests that the action of complex vectors $\bar{Z}$ and Z resemble creation and annihilation operators in quantum mechanics.

2.1.5. Spin-mass relation

Subtracting (85) from (84) gives

$$\hbar\omega_0\mathcal{I}_+ = \boldsymbol{\sigma}_s\Omega_s.S\mathcal{I}_+. \tag{87}$$

Using Einstein-de Broglie relation $mc^2 = \hbar\omega_0$ and since $\mathcal{I}_+$ absorbs $\boldsymbol{\sigma}_s$, we find the relation between spin and mass.

$$mc^2 = \Omega_s.\mathrm{S} \tag{88}$$

The mass of the particle is then attributed to the local internal rotational energy which is a consequence of the presence of zeropoint field. Thus the particle mass is produced due to local complex internal rotation in the space filled with fluctuating zeropoint field and equivalent to the internal rotational energy. In other words, the particle mass appears due to spin rotation on itself in complex vector space. Thus, mass is generated from the absorption of energy from the zeropoint fields and it seems the mass is purely electromagnetic in nature. Suppose the particles are hydrogen atoms having total half spin angular momentum (nuclear spin may be neglected), we treat particle as an oscillator and there will be no difficulty in deriving the spin mass relation (88). Similarly, one can extend this logical approach to the macro-particles considering it as a composite of micro-particles and as such the total mass in general may be the absorbed zeropoint energy due to local internal complex rotations.

2.1.6. Quantization of total angular momentum

Electron rotational motion around the nucleus in hydrogen atom can be visualised as an oscillator and the radius vector can be expressed as a rotor $\boldsymbol{r}(t) = \boldsymbol{r}_0 e^{i\omega_B t}$, with de Broglie frequency ω_B. In addition to this due to the presence of zeropoint field the electron oscillates with high frequency ω_0. Thus, the electron motion is visualised as a modulated wave with internal frequency ω_0 and envelop frequency ω_B. As we have seen in the previous section, the oscillator has higher energy states representing higher energy modes of the fundamental internal oscillator system. In classical mechanics, in terms of action and angle variables the oscillator energy $E = J\omega$. This general formula is applicable for any periodic system. The action variable is defined as $J = (1/2\pi)\oint pdq$, where integration must be carried out over a complete rotation or oscillation and the action variable represents the angular momentum. Using the formula $J = E/\omega_0$ and the energy in (86), the total angular momentum of the electron can be obtained.

$$|J| = |L + S| = \frac{E_n}{\omega_0} = \left(n + \frac{1}{2}\right)\hbar. \tag{89}$$

This quantum condition along with stationary states of electron is one of the fundamental properties in quantum mechanics.

2.1.7. Lorentz rotation, four vectors and spin tensor

We have seen that the half rotations of spin are represented by general rotor which is given by (64). The extended particle motion is described by charge rotation followed by the center of mass motion and such helical motion is represented by rotation followed by boost and it is called Lorentz rotation.

$$\Lambda = L_b^{-1} R_0 \tag{90}$$

It may be assessed that when the particle is in motion, the direction of internal velocity $\boldsymbol{u}$ changes from its original direction and let this new direction be represented by a unit vector $\boldsymbol{\sigma}_0$. The Lorentz rotation of this vector gives a complex vector

$$\Lambda^{-1}\boldsymbol{\sigma}_0\Lambda = \mathrm{L}_b^{-1}\boldsymbol{\sigma}_0\mathrm{L}_b = \gamma(\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_0 \wedge \boldsymbol{\beta}) \tag{91}$$

The unit bivector $\boldsymbol{\sigma}_0 \wedge \hat{\boldsymbol{v}}$ is called relative bivector and multiplying (96) with the velocity of light gives the proper velocity vector.

$$u_0 = c\gamma(\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_0 \wedge \boldsymbol{\beta}) \tag{92}$$

To find the four vectors, let us consider the basis vectors $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3)$ in Euclidean space and the following identification gives the spacetime basis vectors.

$$\gamma_0 = \boldsymbol{\sigma}_0 \ ; \ \gamma_k = \boldsymbol{\sigma}_0\boldsymbol{\sigma}_k, \ (k = 1, 2, 3) \tag{93}$$

Using the relation $\gamma d\tau = dt$, (92) can be expressed in the form

$$d\tau u_0 = \gamma_0 cdt + \gamma_k dx. \tag{94}$$

In the four-vector notation we have

$$dx_\mu = \boldsymbol{\sigma}_0 d\tau u_0 = cdt + \boldsymbol{dx} \ ; \ dx^\mu = d\tau u_0 \boldsymbol{\sigma}_0 = cdt - \boldsymbol{dx}. \tag{95}$$

Now, the four-velocity vector can be expressed as

$$u_\mu = \boldsymbol{\sigma}_0 u_0 = c\gamma(1 + \boldsymbol{\beta}) \ ; \ u^\mu = u_0\boldsymbol{\sigma}_0 = c\gamma(1 - \boldsymbol{\beta}). \tag{96}$$

The covariant and contravariant velocity vectors represent the so called spacetime split introduced by Hestenes (2003b).

We have $u_\mu u^\mu = c^2$ and the proper momentum vector can be written as $p_\mu = mu_\mu$ and the product

$$c^2 p_\mu p^\mu = E^2 - p^2 c^2 = m^2 c^4.$$

The tensor representation of spin angular momentum can be obtained by assigning $S^{00} = S^{ii} = 0$ and

$$S^{ij} = S_{ij} = \frac{\hbar}{2}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j = \frac{\hbar}{2}\gamma_j\gamma_i. \tag{97}$$

It can be seen from the definition of bivector spin angular momentum (38) that the product $S.\boldsymbol{v} = 0$ and therefore the relation

$$p_\alpha S^{\alpha\beta} = 0 \tag{98}$$

is satisfied. The non-relativistic spin precession in external electromagnetic field can be expressed in the form

$$dS/dt = \frac{e}{mc}F.S, \tag{99}$$

where $F = \boldsymbol{E} + i\boldsymbol{B}$. The electromagnetic field experienced by the moving particle is obtained by taking Lorenz boost.

$$L_b^{-1} F L_b = \gamma\boldsymbol{E} + \gamma[\boldsymbol{iB} + \boldsymbol{i}(\boldsymbol{B}.\boldsymbol{\beta})] = \gamma\boldsymbol{F} \tag{100}$$

Where we have used the identity $\boldsymbol{iB} \wedge \boldsymbol{\beta} = \boldsymbol{i}(\boldsymbol{B}.\boldsymbol{\beta})$ and in the case of transverse magnetic field $\boldsymbol{B}.\boldsymbol{\beta} = 0$. Using the relations $\gamma d\tau = dt$ and $S_0 = \gamma S$, (99) can be expressed in the form

$$\frac{dS_0}{d\tau} = \frac{e}{mc}F.S_0. \tag{101}$$

This equation is simply a form of BMT equation derived in electrodynamics

2.1.8. The wavefunction and its relation to spin angular momentum

The state of a micro-system like an electron can be expressed by a complex function $\psi = \psi(\boldsymbol{x}, t)$. Further, the spin is an essential feature of fermion wavefunction. In the rest frame of the particle, the particle spin eigen states are also the energy eigen states and we write from (54),

$$R_0(t) = R_0(0)\exp\left(\frac{\Omega_S.\mathrm{St}}{2S}\right). \tag{102}$$

Using spin-mass relation (88), we find the rotor in the rest frame of the particle.

$$R_0(t) = R_0(0)\exp\left(\frac{\mathrm{mc^2\,t}}{2S}\right) \tag{103}$$

The wavefunction obtained here is real in the sense it is not presumed but emerges from a more fundamental local complex rotation in complex vector space. It should be noted that the wavefunction reverts back to its original state after 2π rotation unlike 4π rotation in spin states. Let the particle be moving along $\boldsymbol{x}$ direction with velocity $\boldsymbol{v}$ and the corresponding wavefunction will be obtained by applying Lorentz boost to (103).

$$R(\boldsymbol{x}, t) = L_b^{-1} R_0(t) L_b = R_0(0)\, exp\left(\frac{\mathrm{E}t - \boldsymbol{p}.\boldsymbol{x}}{2S}\right) \tag{104}$$

Where, $E = \gamma mc^2$and $\boldsymbol{p} = \gamma m\boldsymbol{v}$. The rotor R_0 is a consequence of complex rotation in a local space and it is actually obtained from the stochastic average of many numbers of entities that represent local rotations and reveals the statistical correspondence of the wavefunction. To account for such statistical nature, we replace R_0 in (104) by $\sqrt{\rho} = \sqrt{\rho(\boldsymbol{x},t)}$ and the particle wavefunction can be written in the form

$$\psi(\boldsymbol{x},t) = \sqrt{\rho}(\boldsymbol{x},t)\, exp\left(\frac{\mathrm{E}t - \boldsymbol{p}.\boldsymbol{x}}{2S}\right), \tag{105}$$

In general, in quantum mechanics we come across with $i\hbar$ in the wavefunction and here, it is replaced by 2S. It shows that the unit imaginary in usual quantum mechanics implies the presence of spin angular momentum. Since $\psi\bar{\psi} = \rho$, the quantity $\rho(\boldsymbol{x},t)$ is identified as the probability density. The above treatment suggests that the wavefunction emerges from the average internal complex rotation and further, the probability interpretation gives the complete meaning of wavefunction representing an individual micro-system.

The momentum observable can be obtained by taking partial position derivative of wavefunction and rearranging gives

$$\boldsymbol{p} = -2S\boldsymbol{\nabla} \ln \psi + S\boldsymbol{\nabla} \ln \rho, \tag{106}$$

where $\boldsymbol{\nabla} = \partial / \partial \boldsymbol{x}$. Thus $-2S\boldsymbol{\nabla}$ is the momentum operator which acts on wavefunction to yield the observable momentum in the case of constant probability density. In Bohmian mechanics, the wavefunction has been treated as a force field that guides the particle motion. The quantum nature of particles will be revealed by the replacement of classical momentum by the quantum momentum function $p(x,E)$ which depends on the energy of the particle and substitution of $p(x,E)$ in the Schrödinger equation gives an additional potential called quantum potential. The ontological and epistemological features of wavefunction have been studied over several decades and the explanation for causal, real, nonlocal and global behaviour of wavefunction basically involves the quantum potential (Durr Teufel 2009; Holand 1993).

2.1.9. The spin and Hestenes-Dirac equation

The Hestenes-Dirac equation can be obtained straightaway by considering the extended particle structure and local rotations in complex vector space. Differentiating (103) with respect to time gives the Dirac equation in the rest frame of the particle.

$$\frac{\partial}{\partial t} R_0 2S = mc^2 R_0. \tag{107}$$

Using $\partial_0 = \partial / c\partial t$ and multiplying this equation from left by unit vector γ_0 gives

$$\gamma_0 \partial_0 R_0 2S - mc\gamma_0 R_0 = 0 \ . \tag{108}$$

Now, applying Lorentz boost to the above equation gives

$$L_b^{-1}\gamma_0\partial_0 L_b L_b^{-1} R_0 2S - mcL_b^{-1}\gamma_0 R_0 = 0. \tag{109}$$

The unit vector γ_0 is invariant under special rotation, $\bar{R}\gamma_0 R = \gamma_0$ and $L_b^{-1}\gamma_0\partial_0 L_b = \gamma^\mu \partial_\mu = \partial$ and using these identities in (109) we have

$$\partial\psi 2S - mc\psi\gamma_0 = 0, \tag{110}$$

The full form of Dirac equation along with the external electromagnetic potential can be obtained by adding an electromagnetic potential term $(eA/c)\psi$.

$$c\partial\psi 2S + eA\psi - mc^2\, \psi\gamma_0 = 0 \tag{111}$$

This equation is the well-known Hestenes-Dirac equation. A detailed version of the above derivation has been carried out elsewhere (Muralidhar 2014). The geometrical properties of Dirac equation were discussed elaborately by Hestenes (2003c) and Boudet (2011). In the above treatment we have considered complex rotations in Minkowski spacetime and the Dirac equation has been derived without using momentum or position operators and even the Planck constant is replaced by the bivector spin angular momentum. Thus, from a pre-quantum level classical approach, the transition to relativistic quantum physics is achieved.

2.2. Classical to quantum correspondence in the complex vector space

In this section we shall explore how the parameters $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ are connected to the operators in quantum mechanics and explore various possibilities of classical to quantum behaviour of particles. Dirac (1947) expressed a quantum Poisson bracket in the following form

$$\boldsymbol{ab}-\boldsymbol{ba}=2(\boldsymbol{a}\wedge\boldsymbol{b})=ik[\boldsymbol{a},\boldsymbol{b}]_{PB}, \tag{112}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are any two dynamical variables and k is a universal constant. If $\boldsymbol{ab}+\boldsymbol{ba}$ is real then $i(\boldsymbol{ab}-\boldsymbol{ba})$ must be real and this gives the distinction between classical and quantum approaches. Thus, the bivector product $\boldsymbol{a}\wedge\boldsymbol{b}$ plays an important role in the definition of quantum condition. Some results of this section are obtained previously (Muralidhar 2016).

In the stochastic electrodynamics approach, the position of an oscillating particle in the fluctuating random zeropoint field, can be considered as an oscillating quantity $\boldsymbol{x}(t)=\boldsymbol{x}_0\exp(\boldsymbol{i\omega}_0 t)$, where $\boldsymbol{x}_0$ is the mean value of position. The momentum of such an oscillating particle is then expressed as $\boldsymbol{p}=\boldsymbol{i\omega}_0 m\boldsymbol{x}$. Now, the left side of (112) in terms of $\boldsymbol{x}$ an $\boldsymbol{p}$ is

$$\boldsymbol{xp}-\boldsymbol{px}=2B\omega_0 mx^2, \tag{113}$$

where B is a unit bivector. Taking the stochastic average of this equation and using the stochastic average value of $\langle x^2\rangle=\hbar/2m\omega_0$ given in (A15), we have

$$\langle\boldsymbol{xp}-\boldsymbol{px}\rangle=B\hbar. \tag{114}$$

Thus, the constant k in (112) is indeed the Planck's constant. In view of the particle structure, the magnitudes of internal position and momentum are equivalent to the deviations Δx and Δp of the particle from its mean path and can be expressed as

$$\boldsymbol{\xi}=\boldsymbol{\sigma}_a(x-\bar{x});\ \boldsymbol{\pi}=\boldsymbol{\sigma}_b(p-\bar{p}), \tag{115}$$

where $\bar{x}$ and $\bar{p}$ are average values of position and momentum and $\boldsymbol{\sigma}_a$ an $\boldsymbol{\sigma}_b$ are unit vctors along $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ respectively. The bivector product representing the particle spin is then written as

$$\boldsymbol{\xi}\wedge\boldsymbol{\pi}=\boldsymbol{x}\wedge\boldsymbol{p}-\boldsymbol{\sigma}_a\wedge\boldsymbol{\sigma}_b(x\bar{p}-\bar{x}p)+\boldsymbol{\sigma}_a\wedge\boldsymbol{\sigma}_b\bar{x}\bar{p}. \tag{116}$$

Here, $\boldsymbol{\sigma}_a\wedge\boldsymbol{\sigma}_b xp$ is expressed as $\boldsymbol{x}\wedge\boldsymbol{p}$ and taking stochastic average on both sides of the above expression gives

$$2\langle\boldsymbol{xp}-\boldsymbol{px}\rangle=2\langle\boldsymbol{\xi}\wedge\boldsymbol{\pi}\rangle=2S, \tag{117}$$

where the stochastic averages $\langle x\bar{p}\rangle$, $\langle\bar{x}p\rangle$ and $\langle\bar{x}\bar{p}\rangle$ are taken to be zero. Thus, the position and momentum are non-commutative and it leads to the conclusion that the internal parameters $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ correspond to position and momentum operators respectively.

$$\boldsymbol{\xi}\rightarrow\hat{x}\ ;\quad \boldsymbol{\pi}\rightarrow\hat{p}=-i\hbar\frac{\partial}{\partial x} \tag{118}$$

This identification gives an isomorphism between operators and internal parameters and it yields the correspondence between quantum commutation relation $[\hat{x}\,,\hat{p}] = i\hbar$ and the bivector product $2\boldsymbol{\xi} \wedge \boldsymbol{\pi}$.

$$[\hat{x},\hat{p}] = i\hbar \rightarrow [\boldsymbol{\xi},\boldsymbol{\pi}\,] = 2S = \boldsymbol{i\sigma}_s\hbar \tag{119}$$

Since, the magnitudes $|\boldsymbol{\xi}|$ and $|\boldsymbol{\pi}|$ are equal to Δx and Δp and the minimum uncertainty relation between position and momentum becomes

$$\Delta x\Delta p = |\boldsymbol{\xi}||\boldsymbol{\pi}| = \hbar/2, \tag{120}$$

where for the particle with structure, $|\boldsymbol{\xi}| = \hbar/2mc$ and $|\boldsymbol{\pi}| = mc$. However, one can also find the same result by writing $\langle\xi^{\boldsymbol{2}}\rangle = \langle\Delta x^2\rangle$ and $\langle\pi^{\boldsymbol{2}}\rangle = \langle\Delta p^2\rangle$ and using the stochastic averages (A15) and (A16), $\langle\xi^{\boldsymbol{2}}\rangle = \hbar/2m\omega_0$ and $\langle\pi^{\boldsymbol{2}}\rangle = \hbar m\omega_0/2$ we get $\langle\Delta x^2\rangle\langle\Delta p^2\rangle = (\hbar/2)^2$. Thus, the extended particle approach gives directly the uncertainty relation and the quantum nature of particles in general.

2.2.1. The complex convective time derivative and the quantum potential

To further establish the position and momentum operators connection to the extended particle structure, we introduce the complex convective time derivative by considering time as a function of X and $\bar{X}$, $t = t\,(X,\bar{X})$.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{1}{2}\left(U\frac{\partial}{\partial X} + \bar{U}\frac{\partial}{\partial \bar{X}}\right) + \frac{1}{2}\left(U\frac{\partial}{\partial X} - \bar{U}\frac{\partial}{\partial \bar{X}}\right) \tag{121}$$

The last two terms in this equation represent symmetric and asymmetric parts respectively. The partial derivative $\partial/\partial X$ can be expanded in the following manner.

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial x}\left(1 + \boldsymbol{i\xi}\frac{\partial}{\partial x}\right)^{-1} = \boldsymbol{\nabla}(1 + \boldsymbol{i\xi\nabla})^{-1} \tag{122}$$

Similarly, the derivative $\partial/\partial\bar{X}$ can be obtained. Substituting these partial derivatives in (121) one can obtain

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla}\Sigma - \frac{\mathrm{S}}{\mathrm{m}}\nabla^2\Sigma + \boldsymbol{iu} \wedge \boldsymbol{\nabla}\Sigma - \boldsymbol{v} \wedge \boldsymbol{i\xi}\nabla^2\Sigma, \tag{123}$$

where we have used $S = \boldsymbol{\xi} \wedge \boldsymbol{\pi}$ and the operator $\Sigma = \sum_{\mathrm{k=0}}^{\infty}(-\xi^2\nabla^2)^k$. In the above equation the second term on right is a scalar, the last two terms are vectors and the third term is a bivector. When ξ^2 terms are suppressed in (123), we have

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla} - \frac{\mathrm{S}}{\mathrm{m}}\nabla^2 + \boldsymbol{iu} \wedge \boldsymbol{\nabla} - \boldsymbol{v} \wedge \boldsymbol{i\xi}\nabla^2. \tag{124}$$

The first two terms on right gives the normal convective time derivative as in classical mechanics

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla} \tag{125}$$

and the remaining terms are due to consideration of complex vector space. Multiplying (125) from right by probability density $\rho = \rho(\boldsymbol{x},t)$ yields the continuity equation.

$$\frac{\partial\rho}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla}\rho = 0 \tag{126}$$

When the particle is at rest, $\boldsymbol{v} = 0$ and (124) reduces to

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{iu} \wedge \boldsymbol{\nabla} - \frac{\mathrm{S}}{\mathrm{m}} \nabla^2. \tag{127}$$

In the stochastic approach to quantum mechanics (de la Pena-Auerback 1971), we find a similar expression with the identification of diffusion constant with $|\mathrm{S}|/m$. The present formulation followed here is entirely different from the method followed in the stochastic approach and in (127) the velocity $\boldsymbol{u}$ is neither diffusive nor osmotic velocity but it is the internal rotational velocity. The derivative obtained in (127) is somewhat similar to the one obtained in scale relativity theory developed by Nottale (2011). However, in the scale relativistic approach the internal velocity is identified as the velocity due to fractal fluctuations. The correspondence between probability density and the internal parameters can be obtained by multiplying (127) right by probability density ρ.

$$\frac{\partial \rho}{\partial t} + \boldsymbol{i\nabla}\left(\boldsymbol{u}\rho - \frac{|\mathrm{S}|}{\mathrm{m}} \boldsymbol{\nabla}\rho\right) = 0. \tag{128}$$

This equation is something like continuity equation and the term in brackets is like current density.

$$C = \boldsymbol{u}\rho - \frac{|\mathrm{S}|}{\mathrm{m}} \boldsymbol{\nabla}\rho \tag{129}$$

In the rest frame of the particle, the average value of this current must be zero. Squaring (129) on both sides and equating the square of current to zero gives

$$u^2 = -\left(\frac{|\mathrm{S}|}{\mathrm{m}}\right)^2 \left(\frac{\nabla\rho}{\rho}\right)^2. \tag{130}$$

Then the magnitude of internal momentum can be written as

$$\pi^2 = -|S|^2 (\nabla \ln \rho)^2. \tag{131}$$

Using the magnitude of spin $|S| = |\xi\pi|$ in this equation, the square of radius of rotation can be written as

$$\xi^2 = -(\nabla \ln \rho)^{-2}. \tag{132}$$

In the classical axiomatic approach to quantum mechanics (Olavo 1999; Schleich *et al*. 2013), the average fluctuations in momentum and position are expressed in terms of the Gaussian probability density and those are found to be similar to (131) and (132) respectively. Further, these equations are the most fundamental equations associated with uncertainty in position and momentum. A relation between internal kinetic energy of the extended particle and the probability density can be obtained by dividing (131) on both sides by $2m$.

$$\frac{\pi^2}{2m} = -\frac{|S|^2}{2m} (\nabla \ln \rho)^2 \tag{133}$$

With $|S| = \hbar/2$, this internal kinetic energy can be obtained as

$$\frac{\pi^2}{2m} = -\frac{\hbar^2}{8m}\left(\frac{\nabla\rho}{\rho}\right)^2. \tag{134}$$

The square root of (130) gives the magnitude of internal velocity and differentiating with respect to x gives

$$\nabla u = -i\nabla\left(\frac{\nabla\rho}{\rho}\right) = -i\left[\frac{\nabla^2\rho}{\rho} - \left(\frac{\nabla\rho}{\rho}\right)^2\right] = 0. \quad (135)$$

Since the velocity $\boldsymbol{u}$ is an internal parameter and independent of mean position $\boldsymbol{x}$, the product $\boldsymbol{\nabla}.\boldsymbol{u} = \nabla u = 0$ and we get

$$\frac{\nabla^2\rho}{\rho} = \left(\frac{\nabla\rho}{\rho}\right)^2. \quad (136)$$

Using this result, the quantum potential obtained in Bhom's approach to quantum mechanics can be written as

$$Q = -\frac{\hbar^2}{2m}\left[\frac{1}{2}\frac{\nabla^2\rho}{\rho} - \frac{1}{4}\left(\frac{\nabla\rho}{\rho}\right)^2\right] = -\frac{\hbar^2}{8m}\left(\frac{\nabla\rho}{\rho}\right)^2. \quad (137)$$

Comparison of (137) and (134) shows that the internal kinetic energy is equal to the quantum potential. In Bohm's theory (Bohm 1952; 1954), the particle fluctuations were attributed to a fluctuating ψ-field and the nature of such field had been considered analogous (not identical) to the electromagnetic field and interestingly in the above treatment, such fluctuating field is the zeropoint field. In a detailed study of Bohm theory, superposition of variable amplitude of plane wavefunction leads to helical motion maintained by quantum potential (Holand 1993). Thus, one can discern that complex motion of the particle in general is also connected to the quantum potential which is intricately connected to the classical approach to quantum mechanics through Hamilton-Jacobi equation.

2.2.2. Weyl form of commutation relations

In (36), the vector terms have certain significance. These terms give correspondence between classical to quantum formulation through the identification of the internal parameters $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ with the position and momentum operators. The vector terms in (36) can be expressed as

$$\frac{1}{2}[(X \wedge \bar{P}) - (\bar{X} \wedge P)] = \boldsymbol{i}(\boldsymbol{p} \wedge \boldsymbol{\xi} - \boldsymbol{x} \wedge \boldsymbol{\pi}). \quad (138)$$

To reveal the quantum nature of a quantum system under consideration, one can define a unitary function in the following form.

$$\mathcal{U}(\boldsymbol{x}, \boldsymbol{p}) = \exp[-i(\boldsymbol{p} \wedge \boldsymbol{\xi} - \boldsymbol{x} \wedge \boldsymbol{\pi})/\hbar], \quad (139)$$

where the pseudoscalar is replaced by the unit imaginary. The unitary function satisfies the relation $\mathcal{U}\bar{\mathcal{U}} = 1$. Now, it is trivial to find out the following differential equations.

$$-i\hbar\frac{\partial\mathcal{U}}{\partial\boldsymbol{x}} = \boldsymbol{\pi}\mathcal{U}; \qquad i\hbar\frac{\partial\mathcal{U}}{\partial\boldsymbol{p}} = \boldsymbol{\xi}\mathcal{U} \quad (140)$$

These equations suggestively represent the correlation between operators and internal parameters. Here, the momentum operator is defined in position space and position operator in momentum space. Using the correspondence between internal parameters and position and momentum operators given in (118), without any hesitation, one can replace internal parameters by position and momentum operators in (139) to obtain Weyl form of commutation relations.

$$\mathcal{U}(\boldsymbol{x}, \boldsymbol{p}) = \exp[-i(\boldsymbol{x}\hat{p} - \boldsymbol{p}\hat{x})/\hbar] \quad (141)$$

This is an equivalent form of abstract function that was developed by Klauder (2000) to pursue the classical approach to quantum mechanics. The function in (139) is solely a classical function and it does not have any operators. Further, the function reveals the required quantum results. Introducing complex vectors in the form $Z = (2\hbar)^{-1/2}(\boldsymbol{\xi} + \boldsymbol{i\pi})$ and $Y = (2\hbar)^{-1/2}(\boldsymbol{x} + \boldsymbol{ip})$ the unitary function can also be expressed as

$$\mathcal{U}(\boldsymbol{x}, \boldsymbol{p}) = \exp(Y\bar{Z} - \bar{Y}Z). \quad (142)$$

It is trivial to find the subsequent commutation relations from the above equation.

$$\left[\frac{\partial \mathcal{U}}{\partial \bar{Y}}, \frac{\partial \mathcal{U}}{\partial Y}\right] = \frac{2\boldsymbol{i}S}{\hbar}\ ; \qquad \left[\frac{\partial \mathcal{U}}{\partial \bar{Z}}, \frac{\partial \mathcal{U}}{\partial Z}\right] = \frac{2\boldsymbol{i}L}{\hbar} \quad (143)$$

Here, S and L are spin and orbital angular momentum bivectors respectively. The relations developed above elucidates how position and momentum operators are connected with the internal parameters $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$. The necessity of introducing the operators in quantum mechanics stems from the internal structure of the particle.

3. Lagrangian and Hamiltonian approaches in the complex vector space: Applications to micro and macro systems

The particle trajectory in Newtonian dynamics is a local path approach. However, the Lagrangian approach is global in the sense it gives the entire trajectory of the particle in terms of canonical coordinates. To find the particle path in classical mechanics we normally introduce a Lagrangian as a function of canonical coordinates $\boldsymbol{q}$ and velocities $\dot{\boldsymbol{q}}$, where over dot denotes differentiation with respect to time. In the presence of zeropoint field, the charged particle fluctuations lead to the consideration of average deviation in the position of center of mass of the particle. We have seen in the section 2 that such deviations may be considered for both micro and macro-particles such that the particle paths do not trace any definite trajectory. To account for the particle internal fluctuations in general we have introduced the complex position vector which takes care of both definite path of center of mass and mean deviations that arise due to the fluctuations. Then the dynamics of the particle under consideration can be achieved by introducing functions which depend on complex position and velocity variables.

3.1. Lagrangian formalism in complex vector space

A short version of Lagrangian method to the extended particle has been discussed recently (Muralidhar 2021) and a complete version is presented in this section. The position and velocity of extended particle are given by (16) and (20). Because of two-fold nature of complex vector space, the path of the particle is described by both X and and its conjugate $\bar{X}$ and therefore, the Lagrangian of a single extended particle system must be a function of X, $\bar{X}$, U and $\bar{U}$.

$$\mathcal{L} = \mathcal{L}(X, \bar{X}\, U, \bar{U}, t) \quad (144)$$

Now, the action can be defined as

$$\mathcal{A}(X, \bar{X}, U, \bar{U}) = \int_{t_i}^{t_f} \mathcal{L}(X, \bar{X}\, U, \bar{U}, t)dt. \quad (145)$$

Where t_i and t_f are the times at initial and final states. The condition for minimum path of the particle in complex vector space is obtained by imposing the extremum condition of action. Dividing the particle path into small elements represented by variations δX and $\delta \bar{X}$, one can find the variation in action in the form

$$\delta\mathcal{A} = \int_{t_i}^{t_f} \left[\frac{\partial \mathcal{L}}{\partial X}\delta X + \frac{\partial \mathcal{L}}{\partial \bar{X}}\delta \bar{X} + \frac{\partial \mathcal{L}}{\partial U}\delta U + \frac{\partial \mathcal{L}}{\partial \bar{U}}\delta \bar{U}\right] dt. \quad (146)$$

After performing partial integration of last two terms in the integrad and considering $\delta X(t_i) = \delta X(t_f)$ and $\delta \bar{X}(t_i) = \delta \bar{X}(t_f)$ and setting $\delta\mathcal{A} = 0$ gives

$$0 = \int_{t_i}^{t_f} \left\{\left[\frac{\partial \mathcal{L}}{\partial X} - \frac{D}{Dt}\left(\frac{\partial \mathcal{L}}{\partial U}\right)\right]\delta X + \left[\frac{\partial \mathcal{L}}{\partial \bar{X}} - \frac{D}{Dt}\left(\frac{\partial \mathcal{L}}{\partial \bar{U}}\right)\right]\delta \bar{X}\right\} dt. \quad (147)$$

Since time is also a function of both X and $\bar{X}$, the total time derivative is replaced by the complex time derivative in the above equation. At the initial and final points, it is expected that $\delta\boldsymbol{\xi} \to 0$ and therefore at the end points we have $\delta X \to \delta\boldsymbol{x}$ and $\delta\bar{X} \to \delta\boldsymbol{x}$. Since the variation in the center of mass position $\delta\boldsymbol{x}$ is not zero, finally we get the complex Euler-Lagrange equation.

$$\frac{1}{2}\left(\frac{\partial \mathcal{L}}{\partial X} + \frac{\partial \mathcal{L}}{\partial \bar{X}}\right) - \frac{1}{2}\frac{D}{Dt}\left(\frac{\partial \mathcal{L}}{\partial U} + \frac{\partial \mathcal{L}}{\partial \bar{U}}\right) = 0 \quad (148)$$

To account for the two-fold nature of complex vector space a factor $1/2$ is introduced in (148). In general, the Lagrangian is defined as $T - V$, where T and V are kinetic and potential energies. Normally, the potential energy is a function of position and in the complex vector space we consider it as a function of X and $\bar{X}$ and similarly the kinetic energy may be considered as a function of U and $\bar{U}$ and therefore we define the Lagrangian in the following form

$$\mathcal{L}(X, \bar{X}\ U, \bar{U}, t) = \gamma m U\bar{U} - V(X, \bar{X}). \quad (149)$$

Now, the second term in (148) becomes

$$\frac{1}{2}\frac{D}{Dt}\left(\frac{\partial \mathcal{L}}{\partial U} + \frac{\partial \mathcal{L}}{\partial \bar{U}}\right) = \frac{D\gamma m\boldsymbol{v}}{Dt} = \frac{D\boldsymbol{p}}{Dt}. \quad (150)$$

Using the expansion of complex vector time derivative given in (124), the right side of the above equation is expressed as

$$\frac{D\boldsymbol{p}}{Dt} = \frac{\partial \boldsymbol{p}}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla}\Sigma\boldsymbol{p} - \frac{\mathrm{S}}{\mathrm{m}}\nabla^2\Sigma\boldsymbol{p} + \boldsymbol{iu}\wedge\boldsymbol{\nabla}\Sigma\boldsymbol{p} + \boldsymbol{v}\wedge\boldsymbol{i\xi}\nabla^2\Sigma\boldsymbol{p}. \quad (151)$$

The last two terms in the above equation goes to zero (see Appendix). Expanding the partial derivatives $\partial/\partial X$ and $\partial/\partial\bar{X}$ as in the previous section, the first term in (148) can be written as

$$\frac{1}{2}\left(\frac{\partial \mathcal{L}}{\partial X} + \frac{\partial \mathcal{L}}{\partial \bar{X}}\right) = \Sigma\boldsymbol{\nabla}\mathcal{L} = -\Sigma\boldsymbol{\nabla}V(X, \bar{X}). \quad (152)$$

Substituting (150) and (152) in (148) gives a generalized Newton's equation

$$\frac{D\boldsymbol{p}}{Dt} = -\nabla\Sigma V(X, \bar{X}). \quad (153)$$

The expansion of $V(X, \bar{X})$ depends on the type of potential under consideration and when the ξ^2 terms are neglected $V(X, \bar{X}) \to V(x)$. In the case of harmonic oscillator, the potential is symmetric and it is a linear function of x^2. In the complex vector approach, we replace x^2 by $X\bar{X} = x^2 + \xi^2$ and the potential is given by (74). However, for other types of potential functions the expression for complex potential would be different.

Combining (151) and (153) gives

$$\frac{\partial \boldsymbol{p}}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla}\Sigma\boldsymbol{p} - \frac{\mathrm{S}}{m}\nabla^2\Sigma\boldsymbol{p} = -\boldsymbol{\nabla}\Sigma V(X,\bar{X}). \tag{154}$$

Since the magnitude of ξ is very small and it is of the order of Compton wavelength of the particle. Neglecting ξ^2 and higher order terms in the above equation gives

$$\frac{d\boldsymbol{p}}{dt} = -\boldsymbol{\nabla}V(x) + \frac{\mathrm{S}}{m}\nabla^2\boldsymbol{p}, \tag{155}$$

where $\frac{d\boldsymbol{p}}{dt} = \frac{\partial \boldsymbol{p}}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla}\boldsymbol{p}$.

However, in the point particle limit the spin term, the last term in the above equation, can be neglected and (155) reduces to Newtonian equation.

$$\frac{d\boldsymbol{p}}{dt} = -\boldsymbol{\nabla}V(x) \tag{156}$$

Using the generalised theory developed above, some of the applications of micro-systems and macro-systems will be elaborated in the following sub sections.

3.1.1. Generalized Newton's equation for micro-particles

In the micro limit, for particles like electrons, the wavefunction representing the particle state is given by (93) and the momentum $\boldsymbol{p}$ is given by (94) and substituting it in (151) yields a generalized Schrödinger equation (see Appendix for detailed calculation) and from (A.22) we have

$$2S\frac{\partial\psi}{\partial t} = \frac{(2\mathrm{S})^2}{2\mathrm{m}}\nabla^2\Sigma\psi + \Sigma V(X,\bar{X})\psi + \Sigma Q\psi. \tag{157}$$

Since the complex vector approach stems from the random fluctuations of the zeropoint field, the above equation may be called stochastic Schrödinger equation. The generalized Schrödinger equation obtained in (157) has been derived by the author (Muralidhar 2017) using a different approach without considering Euler-Lagrange equations and the method followed here is more close to the classical approach. When higher order terms in (157) are neglected, we get the time-dependent Schrödinger equation. Note that the usual $i\hbar$ is being replaced by $2S$ which shows that the unit imaginaly in quantum mechanics indicates the particle spin and (157) contains internal structure in disguise. The correspondence between spin angular momentum and unit imaginary in quantum mechanics was first pointed out by Hestenes (1979). The operator Σ in (157) acting on ψ gives the higher order relativistic correction terms. Let us consider the expansion of the term $\bar{\psi}\Sigma\psi$.

$$\bar{\psi}\sum_{\mathrm{k}=0}^{\infty}(-\xi^2\nabla^2)^k\,\psi = \rho\left[1-\left(\frac{\beta}{2}\right)^2\right]^{-1} \tag{158}$$

Using this result in the first term on right of (157) one can obtain

$$\bar{\psi}[(2S)^2/2m\rho]\nabla^2\psi(\bar{\psi}\Sigma\psi) = \frac{p^2}{2m}\left[1-\left(\frac{\beta}{2}\right)^2\right]^{-1}. \tag{159}$$

In the above equation, after expanding the square brackets, the second term onwards gives higher order corrections to non-relativistic kinetic energy $p^2/2m$ and the second term $p^4/8m^3c^2$ turns out to be of the same relativistic energy term obtained after taking Foldy-Wouthuysen transformation to Dirac equation.

One can also find higher order correction terms in quantum potential term. In the present formulation there appear higher order correction terms in the potential and it leads to a correction in the energy of the particle. The complex potential in (157) can be expanded and to the order ξ^2, it can be obtained as

$$\Sigma V(X,\bar{X}) = \left(1-\frac{3}{2}\xi^2\nabla^2\right)\mathrm{V}(\boldsymbol{x}). \tag{160}$$

Since the orbital velocity of electron is very small, while considering the internal high frequency oscillations, the electron may be treated almost stationary. Using $|\xi|\sim\hbar/2mc$, the relativistic correction to potential energy term in (160) can be written as

$$\Delta V = -\frac{1}{8}\left(\frac{\hbar}{\mathrm{mc}}\right)^2\nabla^2\mathrm{V}(\boldsymbol{r}), \tag{161}$$

where, a factor $1/3$ is introduced in the above expression due to the fact that the coulomb potential $\mathrm{V}(\boldsymbol{r}) = -\mathrm{Ze}^2/\boldsymbol{r}$ is symmetric. The potential energy in (161) is simply the Darwin term. In usual relativistic quantum theory, the Darwin term is attributed to the *zitterbewegung* oscillations of the electron (Sakurai 2007). Thus, the correction terms in (157) are relativistic in nature.

3.1.2. Effect of higher order corrections on operators and uncertainty relation

One can expect the presence of correction terms in (157) leads to changes even in the operators in quantum mechanics. From the normal Schrödinger equation, it is easy to show that the probability density $\rho = \psi\bar{\psi}$ and the current density $j = (|S|/m)(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi})$. Consideration of higher order terms in stochastic Schrödinger equation leads to additional terms in the probability current density and to the first order, it is

$$J = j + \xi^2\nabla^2 j. \tag{162}$$

However, the form of continuity equation is unchanged with the replacement of j with J. Using (159), the momentum operator can be written as

$$\hat{p} = -2S\nabla\left[1-\left(\frac{\beta}{2}\right)^2\right]^{-1/2}. \tag{163}$$

Using this relation, the commutation relation can be expressed as

$$[\hat{x},\hat{p}] = 2S\left[1-\left(\frac{\beta}{2}\right)^2\right]^{-1/2}. \tag{164}$$

Retaining only up to β^2 terms, this equation can be written in the form

$$[\hat{x},\hat{p}] = 2|S|(1+\kappa^2 p^2), \tag{165}$$

where $\kappa^2 = 1/8m^2c^2$. This type of commutation relation was long back derived by Snyder (1947) and showed that the spacetime is not continuous but discrete. Further one can expect the minimal length uncertainty relation in the form

$$\Delta x \Delta p \sim 2|S|(1 + \kappa^2 p^2). \quad (166)$$

However, similar expressions have been derived considering the gravitational fluctuations in spacetime and in this case the constant like κ will be proportional to the Planck length and such relations are called generalized uncertainty relations which are considered in the development of quantum gravity, black hole dynamics and string theory (Kempf *et al*. 1995; Maggiore 1993; Chang *et al*. 2011).

3.1.3. Klein-Gordon equation

Since the total energy must contain the rest mass energy, we add $mc^2\, \psi(\boldsymbol{x}, t)$ to the right of (157) and neglect the external potential and quantum potential terms.

$$2S \frac{\partial}{\partial t} \psi = \frac{(2S)^2}{2m} \nabla^2 \Sigma \psi(\boldsymbol{x}, t) + mc^2\, \psi \quad (167)$$

Multiplying this equation from left by the operator

$$2S \frac{\partial}{\partial t} = \frac{(2S)^2}{2m} \nabla^2 \Sigma + mc^2 \quad (168)$$

and 2ψ from right, we get

$$(2S)^2 \partial_t^2 \varphi \approx [(2S)^2 c^2 \nabla^2 \Sigma + m^2 c^4\,] \varphi, \quad (169)$$

where $\varphi = 2\psi^2$. Using plane wave form of ψ gives

$$\varphi = \rho' \exp[(Et - \boldsymbol{p}.x)/S], \quad (170)$$

where $\rho' = 2\rho$ and $\bar{\varphi}\varphi = \rho'^2$. Now the energy and momentum of free particle becomes $2E$, and $2\boldsymbol{p}$ respectively. However, the transformation $\psi \to \varphi$ gives the wavefunction representing spin one particles. The Σ term in (169) represents a series of higher order relativistic terms which converge to the square of Lorentz factor.

$$(1/\rho'^2) \bar{\varphi} \Sigma \varphi = (1 - \beta^2)^{-1} = \gamma^2 \quad . \quad (171)$$

Substituting this result in (169) gives

$$\partial_0^2 \varphi = [\gamma^2 \nabla^2 + \mu^2], \quad (172)$$

where $\mu^2 = (mc/2S)^2$, $x_0 = ct$. After rearranging the terms,

$$(\partial^2 - \mu^2)\varphi = 0 \quad (173)$$

and the wavefunction is expressed as

$$\varphi = \varphi_0 \exp(p.x/S)\,, \quad (174)$$

where $p = (\gamma \boldsymbol{p}, E/c)$ and $S = \boldsymbol{i\sigma}_s \hbar$. Since, the adjoint $\bar{\varphi}$ is also a solution of (173) and gives the same result as that of φ the wavefunction of (174) may be written as a sum of φ and $\bar{\varphi}$.

$$\phi = \varphi + \bar{\varphi} \quad (175)$$

Above procedure elucidates the fact that squaring the generalized Schrödinger equation yields the Klein-Gordon equation.

3.2. Hamiltonian formalism in complex vector space

The Hamiltonian of a system is a function of momentum and velocity in classical mechanics. In complex vector space, the Hamiltonian can be considered as a function of complex momentum P, complex velocity U and their conjugates, $\mathcal{H} = \mathcal{H}(P, \bar{P}, U, \bar{U})$ and it can be defined in the form

$$\mathcal{H} = \frac{1}{2}(U\bar{P} + \bar{U}P) + \frac{1}{2}(U\bar{P} - \bar{U}P) - \mathcal{L}. \tag{176}$$

The first and second terms on left are symmetric and asymmetric products respectively. The above form of Hamiltonian gives the following set of equations of motion.

$$\frac{1}{2}\left(\frac{\partial\mathcal{H}}{\partial\bar{P}} + \frac{\partial\mathcal{H}}{\partial P}\right) = \boldsymbol{v} \tag{177}$$

$$\frac{1}{2}\left(\frac{\partial\mathcal{H}}{\partial\bar{P}} - \frac{\partial\mathcal{H}}{\partial P}\right) = \boldsymbol{iu} \tag{178}$$

$$\frac{1}{2}\left(\frac{\partial\mathcal{H}}{\partial X} + \frac{\partial\mathcal{H}}{\partial\bar{X}}\right) = -\Sigma\nabla\mathcal{L} \tag{179}$$

These partial derivatives can be simplified by introducing the following notation of complex derivatives.

$$\frac{D\mathcal{H}}{D\boldsymbol{p}} = \frac{1}{2}\left(\frac{\partial\mathcal{H}}{\partial\bar{P}} + \frac{\partial\mathcal{H}}{\partial P}\right) + \frac{1}{2}\left(\frac{\partial\mathcal{H}}{\partial\bar{P}} - \frac{\partial\mathcal{H}}{\partial P}\right) = U \tag{180}$$

$$\frac{D\mathcal{H}}{D\boldsymbol{x}} = \frac{1}{2}\left(\frac{\partial\mathcal{H}}{\partial X} + \frac{\partial\mathcal{H}}{\partial\bar{X}}\right) = \Sigma\nabla\mathcal{L} = -\nabla\Sigma V(X, \bar{X}) \tag{181}$$

With these at hand, a modified Poisson bracket can be defined in form

$$\{f, g\} = \left(\frac{Df}{D\boldsymbol{x}}\frac{Dg}{D\boldsymbol{p}} - \frac{Dg}{D\boldsymbol{x}}\frac{Df}{D\boldsymbol{p}}\right). \tag{182}$$

Using (180), (181) and (153), the Hamiltonian equations of motion in terms of complex derivatives can be written in the form

$$\frac{D\boldsymbol{p}}{Dt} = -\{p, \mathcal{H}\} \tag{183}$$

$$U = \frac{D\mathcal{H}}{D\boldsymbol{p}} = \{x, \mathcal{H}\} \tag{184}$$

Note that equation (183) is simply the generalized Newton's equation derived in the Lagrangian approach. Expanding partial derivatives in (180) and (181) gives

$$\frac{1}{2}\left(\frac{\partial\mathcal{H}}{\partial\bar{P}} + \frac{\partial\mathcal{H}}{\partial P}\right) \cong \frac{\partial\mathcal{H}}{\partial\boldsymbol{p}} - O(\pi^2) \tag{185}$$

$$\frac{1}{2}\left(\frac{\partial \mathcal{H}}{\partial \bar{P}} - \frac{\partial \mathcal{H}}{\partial P}\right) \cong \boldsymbol{i\pi}\frac{\partial^2 \mathcal{H}}{\partial p^2} - O(\pi^2) \tag{186}$$

$$\frac{1}{2}\left(\frac{\partial \mathcal{H}}{\partial X} + \frac{\partial \mathcal{H}}{\partial \bar{X}}\right) \cong \frac{\partial \mathcal{H}}{\partial x} - O(\xi^2) \tag{187}$$

It is interesting to find that by suppressing π^2 and ξ^2 in the above equations gives the classical Hamiltonian equations.

$$\frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} = \boldsymbol{v} \ ; \qquad \frac{\partial \mathcal{H}}{\partial x} = \frac{d\boldsymbol{p}}{dt} \ ; \qquad \frac{\partial^2 \mathcal{H}}{\partial p^2} = \frac{1}{m} \tag{188}$$

In this case the Hamiltonian reduces to $\mathcal{H} = (p^2/2m) + V(x)$. Thus, in the point particle limit we retrieve the classical Hamiltonian equations of motion.

3.2.1. Complex Hamilton-Jacobi equation

Let us consider the momentum as a function of position and energy, $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{x}, E)$. This is one of the important transformations that have been considered in the development of quantum mechanics from classical theories most probably from the beginning of wave mechanics by Schrödinger and subsequent development by Dirac, Jordon and Feynman and Bohmian mechanics of classical approach to quantum mechanics. Now, the partial time derivative of momentum may be expressed in the form $\partial \boldsymbol{p}/\partial t = -\boldsymbol{\nabla} E$ and with this result, (154) can be expressed as

$$E = \Sigma V(X, \bar{X}) + \Sigma \frac{p^2}{2m} - \frac{2S}{2m}\Sigma \boldsymbol{\nabla p}. \tag{189}$$

This equation is the quantum Hamilton-Jacobi equation in complex vector space. Substituting the wavefunction $\psi = \rho \exp \mathcal{S}/2S$ in the generalized Schrödinger equation (157) also gives (189). In the non-relativistic limit both γ and Σ reduce to unity and (189) can be expressed in the form

$$\frac{p^2}{2m} - \frac{2S\boldsymbol{\nabla p}}{2m} = E - V(X, \bar{X}). \tag{190}$$

Using (115) one can approximate $\boldsymbol{\nabla p} \sim \Delta \boldsymbol{p}/\Delta \boldsymbol{x} \sim \boldsymbol{\pi}/\boldsymbol{\xi}$ and therefore the second term on left of (190) can be approximated as $-2S\boldsymbol{\nabla p}/2m \sim \pi^2/2m$ which is the internal kinetic energy. Using the relation $P.\bar{P} = p^2 + \pi^2$, the terms on left of (190) can be expressed in a compact form.

$$\frac{P.\bar{P}}{2m} = \frac{p^2}{2m} - \frac{2S\boldsymbol{\nabla p}}{2m} \tag{191}$$

Substituting this result in (190), we get

$$P.\bar{P} = 2m[E - V(X, \bar{X})]. \tag{192}$$

In the case of harmonic oscillator, the above equation transforms into (75). Since the ground state energy of internal oscillator system $E_0 = \hbar\omega_0/2$, (192) can be written as

$$\frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 = E - E_0 \ . \tag{193}$$

The classical Hamilton-Jacobi approach gives the energy $E - E_0 = J\omega_0$, where J is the action. The momentum in terms of particle wavefunction is given by (94) and in the case of constant probability density, the quantum action function can be defined as

$$J = \frac{1}{2\pi}\oint p(x,E)\,dx = \frac{-2S}{2\pi}\oint \frac{d\psi}{\psi}. \qquad (194)$$

For a periodic motion of the particle, the wavefunction ψ takes the same value at the initial and final points. Replacing 2S by $i\hbar$ and solving the integral in (194), we obtain the quantum action $J = n\hbar$. This method of obtaining Bohr-Sommerfeld condition was due to Tarnovskii (1990). Since the total energy $E_n = J\omega_0 + E_0$, the harmonic oscillator energy becomes $E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0$.

The classical momentum of the particle $p_c = \sqrt{2m[E - V(x)]}$. Now, the replacement of $2S$ by $i\hbar$ in (190) gives the Riccati equation.

$$p^2 - i\hbar\nabla\boldsymbol{p} = p_c^2 \qquad (195)$$

This equation allows determining the exact bound state energy levels and related eigen functions without solving Schrodinger equation (Leacock & Padgett 1983). Considering the boundary conditions, the classical action can be defined as

$$J = \frac{1}{2\pi}\oint_C p(x,E)\,dx. \qquad (196)$$

Here C is a counter-clockwise contour in the complex x-plane enclosing the real line between classical turning points $x = \pm\sqrt{2mE}/m\omega_0$, where $p_c = 0$. In such a case the wavefunction is known to have nodes between these classical tuning points. When left hand side of (195) is zero, the quantum momentum function can be obtained in the form $p = i\hbar/(x - x_0)$ and the classical momentum goes to zero at finite number of singular points on the real x-axis for $n = 1,2,3 \ldots$ and these singular points are known as moving singularities and these correspond to poles of quantum momentum function. The quantum number corresponds to number of poles of $p(x,E)$ inside the contour C and performing the contour integration in the counter-clockwise direction gives $J = n\hbar$. A convenient choice of quantum momentum function and proper choice of values of constants leads to correct eigen values for the periodic potentials under consideration. Several authors studied such quantum Hamilton-Jacobi equation for solving the problems like potential well, harmonic oscillator, hydrogen atom etc., (Leacock & Padgett1983; Bhalla *et al*. 1997).

3.3. Generalized Newton's equation for macro-particles

A macro-particle may be visualised to contain large number of micro-particles. As discussed in section 2, the effective deviation of the point at center of mass and the position of that point can be represented by a complex vector. The influence of zeropoint fluctuations may then be considered to produce modification in the potential term in (153) and it is considered as a function of complex position vector of the macro-particle. Since the orientation of spin angular momentum of each micro-particle is at random inside a macro-particle, the effective or total internal spin vanishes and the spin terms in (153) can be neglected. Then the generalized Newton's equation becomes

$$\frac{d\boldsymbol{p}}{dt} = -\nabla\Sigma V(X,\bar{X}). \qquad (197)$$

Expanding the complex potential term for central potential $V(\boldsymbol{x}) = -k/\boldsymbol{x}$ and keeping terms up to the order ξ^2 we find

$$\Sigma V(X, \bar{X}) = \mathrm{V}(\boldsymbol{x}) \left(1 - \frac{3\xi^2}{x^2}\right). \tag{198}$$

Now, extending the approximation in (32) to the macro-particles, the modified central potential can be expressed in the form

$$V'(\boldsymbol{x}) = \mathrm{V}(\boldsymbol{x})(1 - 3\beta^2). \tag{199}$$

This potential is similar to the one obtained in Weber's electrodynamics studied by Assis (1994; 1989) with the difference that the integer 3 in (199) is replaced by $n/2$ and it was assumed $n = 6$ for gravitational potential and $n = 1$ for Coulomb potential. The additional term in (198) is a consequence of the presence of zeropoint field and the modified potential is obtained naturally. Now, the central force acting on a test particle can be obtained by substituting (199) in (153).

$$\frac{d\boldsymbol{p}}{dt} = -\widehat{\boldsymbol{x}} \frac{k}{x^2}\left[1 - 3\beta^2 + \frac{6\,(\boldsymbol{x}.\boldsymbol{a})}{c^2}\right] \tag{200}$$

For a large spherically symmetric gravitating mass M with its center of mass at $x = 0$, the constant $k = GMm$, where G is the gravitational constant and m is the mass of a particle. Rearranging the terms in (200), the acceleration acting on the particle can be written in the form

$$a = -g(1 - 3\beta^2) + 3g\left(\frac{r_0}{x}\right)(1 - 3\beta^2), \tag{201}$$

where $g = GM/x^2$ and $r_0 = 2GM/c^2$. At velocities much smaller than the velocity of light, $\beta^2 \ll 1$, this equation reduces to Newtonian acceleration due to gravity. However, the first term on right side of (201) represents both attractive and velocity dependent repulsive gravitation. The acceleration is predominantly repulsive when the particle velocity is comparable with the velocity of light. When $\beta^2 > 1/3$, the gravitation is purely repulsive. In the extreme case $\beta^2 = 1$ for particles moving with velocity of light, the acceleration becomes $2g$. The repulsive gravitation is a century old concept and interestingly it not only depends on radial distance but also on the particle velocity. In 1917 D. Hilbert discussed about the repulsive gravitational force in the Schwarzschild field and showed the gravitational repulsion inequality $\beta^2 > 1/3$ (Gorkavyi & Vasilkov 2016). In due course several researchers advocated the gravitational repulsive force. A historical review of gravitational repulsion and the pros and cons were nicely discussed by McGruder (1982). Recently, Gorkavyi and Vasilkov (2016) concluded that cosmic repulsive gravitation can be obtained from the classical Einstein theory of general relativity.

The validity of (201) can be confirmed when it is compared with that of general relativistic form. Here, we shall follow the method given by McVittie (1964) to derive a three-dimensional acceleration from Einstein general theory of relativity. The fundamental paths in Riemann space are called geodesics and they have properties similar to those of straight lines in Euclidean space. The Riemann space is in general n-dimensional and its reduction to four dimensions is employed in relativistic theories. The differential equations of such geodesic in four dimensional spacetime is given by

$$\frac{d}{ds}\left(g_{\mu\mu}\frac{dx^\mu}{ds}\right) - \frac{1}{2}\frac{\partial g_{\lambda\lambda}}{\partial x^\mu}\left(\frac{dx^\lambda}{ds}\right)^2 = 0, \tag{202}$$

where $ds^2 = c^2 d\tau^2$, $g_{\mu\nu}$ is the metric tensor and the indices $\mu = 0,1,2,3$.

Let us consider a massive object of mass M with its center of mass at $r = 0$ and the space around it is empty. The gravitational field around such an object is spherically symmetric and the spacetime metric in such spherically symmetric gravitational field deviates from the Minkowski metric. Such metric was derived by Schwarzschild and the components of metric tensor are given by

$$g_{00} = -\left(1 - \frac{r_0}{r}\right) \; ; \quad g_{11} = \frac{1}{c^2}\left(1 - \frac{r_0}{r}\right)^{-1} ; \; g_{22} = \frac{r^2}{c^2} \; ; \; g_{33} = \frac{r^2}{c^2} sin^2 \theta \qquad (203)$$

Substituting these metric components in (202), the radial acceleration on a test particle of mass m at a distance r is obtained in the appendix and it is given by (A27).

$$a = 2g\left(1 - \frac{r_0}{r}\right) - 3g(1 - \beta^2)\left(1 - \frac{r_0}{r}\right)^2 \qquad (204)$$

Rearranging the terms in this equation gives

$$a = -g(1 - 3\beta^2) + 4g\left(\frac{r_0}{r}\right)\left(1 - \frac{3\beta^2}{2}\right), \qquad (205)$$

where the higher order terms are neglected. Comparison between (201) and (205) shows a striking similarity between these equations: the first term on the right is the same and except for a constant numerical value the second term is also similar. This comparison further establishes the validity of the approximation (32) for macro-particles. In general theory of relativity, gravitation is spacetime curvature and the similarity between (201) and (205) univocally suggests that the curvature of spacetime may be due to the presence of fluctuating random zeropoint fields.

3.3.1. Advance of perihelion of mercury

In the case of planetary motion, the central potential $V(\boldsymbol{x}) = -GMm/\boldsymbol{x}$, where M and m are masses of Sun and planet respectively. Choosing the polar axis normal to the plane of mechanical motion and using polar coordinates (r, θ) as it is usually done in classical mechanics (Goldstein *et al*. 2000), one can find the modified orbital equation by eliminating dt.

Using $\frac{d}{dt} = \frac{h}{r^2}\frac{d}{d\theta}$ and denoting the constant $h = \frac{l}{m}$, (200) can be expressed in the form

$$\frac{d^2u}{d\theta^2} + u = B - \frac{3r_0}{2}\left(\frac{du}{d\theta}\right)^2 + 3r_0 u \frac{d^2u}{d\theta^2}, \qquad (206)$$

where and $u = 1/r$ and $B = GM/h^2$. This equation contains additional terms to the Newtonian expression and in general relativity these are replaced by the term $3GMu^2/c^2$. Considering $u = u_0 + u_1$ and $u_0 \gg u_1$, (206) can be solved iteratively. Using the condition $du/d\theta = 0$ at turning points, the perihelion advance $\Delta\theta$ has been calculated in the appendix and from (A28)

$$\Delta\theta = 3\pi B r_0 = \frac{6\pi G^2 M^2}{c^2 h^2}. \qquad (207)$$

Using the relation $h^2/GM = a_m(1-\epsilon^2)$, this equation can be expressed in the familiar form

$$\Delta\theta = \frac{6\pi GM}{c^2 a_m(1-\epsilon^2)}, \tag{208}$$

where a_m is the semi-major axis and ϵ the eccentricity. The data on the observations of mercury was available from 1765. The analysis of the data by Simon Newcomb in 1882 gave the centennial precession of $43''$. Reanalysis of the observations by Clemence in 1943 revealed a more precise value $43.11 \pm 045''$ (Weinberg 1972). Mercury makes 415 revolutions per century and using the values of semi-major axis $a = 57.91 \times 10^9\ m$ and eccentricity $\epsilon = 0.2056$ we have $(a(1-\epsilon^2) = 55.46 \times 10^9\ m$ (Balogh & Giampieri 2002). With standard values of $= 6.67428 \times 10^{-11} m^3 kg^{-1} s^{-2}$, $c = 299792458\ m\, s^{-1}$ and mass of the Sun $M = 1.9891 \times 10^{30}\ kg$, the calculation of $\Delta\theta$ per century gives the result $42.98''$.

3.2.2. Gravitational deflection of light

To find gravitational deflection of light near a massive object, consider a particle moving parallel to y-axis and passing through mass M at a distance $x = R$. From the modified force equation (200), the acceleration in x-direction can be expressed in the following form

$$\frac{d^2x}{dt^2} \sim -\frac{GMx}{r^3}(1-3\beta^2)\left(1-\frac{3r_0 x}{r^2}\right). \tag{209}$$

For a light particle moving almost parallel to the y-axis, we have $dy/dt = c$ and $d^2y/dt^2 = 0$.

Using the identity $\frac{d^2x}{dt^2} = \frac{d^2x}{dy^2}c^2$, replacing $v = c$ and $x = R$ in (209) gives

$$\frac{d^2x}{dy^2} = \frac{r_0 R}{(R^2+y^2)^{3/2}} - \frac{3r_0^2 R^2}{(R^2+y^2)^{5/2}}. \tag{210}$$

Integrating this expression on both sides we get the slope dx/dy. From which we find the total angle of deflection 2α and it is given by (A37).

$$2\alpha = 2\frac{r_0}{R} - 4\left(\frac{r_0}{R}\right)^2 \tag{211}$$

This coincides with the general theory of relativity result when the second term on right is neglected. Using the value of radius of Sun $R = 6.9551 \times 10^8\ m$ and with the remaining values as given above, we find $2r_0/R = 4GM/c^2R = 8.4949 \times 10^{-6}\ rad = 1.7522''$. Further, the value of the term $4(r_0/R)^2$gives a small correction in the fifth decimal pace of the result. There appears to be several methods of finding deflection of light for example null geodesic and material medium methods (Roy and Sen 2019). One of the famous predictions of general theory of relativity is the gravitational deflection of light and it led to the idea of gravitational lensing which is the growing subject in astronomy (Wambsganss 1998)

4. Stochastic electrodynamics in complex vector space

In the introduction, we have mentioned that the stochastic electrodynamics approach is one of the best classical approaches to the development of quantum mechanics and recently Boyer (2019) stressed this view. However, complex vector approach to the motion of extended particles in zeropoint field yields the results those obtained in quantum electrodynamics. In this section we analyse how the extended particle motion in the zeropoint field leads to the important results like mass correction and charge correction such that one can estimate the anomalous magnetic moment and Lamb shift. In the classical stochastic electrodynamics, the center of charge and center of mass are not considered as separate but the charged particle is treated as a point particle. However, the influence of zeropoint field on extended particle gives different results.

In the point particle limit, normally the particle motion in the zeropoint field is described by Abraham-Lorentz equation (de La Pena & Cetto 1996).

$$m\ddot{\boldsymbol{x}} = -m\omega_0^2\boldsymbol{x} + \Gamma_{\mathrm{a}} m\dddot{\boldsymbol{x}} + e\boldsymbol{E}_{zp}(\boldsymbol{x},t) \tag{212}$$

Where, $\Gamma_{\mathrm{a}} = \mathrm{e}^2/6\pi\epsilon_0 \mathrm{mc}^3$ is the radiation damping constant. The force term contains three parts: the binding force $m\omega_0^2\boldsymbol{x}$, radiation damping force $\Gamma_{\mathrm{a}} m\dddot{\boldsymbol{x}}$ and external force $e\boldsymbol{E}_{zp}$. The binding and damping force terms are found to be much smaller than the external force term. The strength of these terms can be expressed in the following order.

$$m\omega_0^2\boldsymbol{x} < \Gamma_{\mathrm{a}} m\dddot{\boldsymbol{x}} < e\boldsymbol{E}_{zp} \tag{213}$$

Using certain approximations, the energy of a particle oscillator in the zeropoint field was derived by several authors. One of the important derivations of energy of harmonic oscillator was due to Boyer (1975a). Since the radiation damping term is very small compared to the external force, $\Gamma_{\mathrm{a}}\omega_0 \ll 1$ and in the limit $\omega_0\tau \ll 1$, the mean square displacements of position and momentum were found to be very small for a particle oscillator with mass m, frequency of oscillations ω_0 and the characteristic time τ. The zeropoint energy of the particle oscillator was found to be equal to the ground state energy of the harmonic oscillator in quantum mechanics. For reference, a complete derivation of Boyer's approach to harmonic oscillator is given in the appendix.

In the later studies Rueda (1981) considered the charged particle as a homogeneously charged rigid sphere. The energy absorbed by the particle has been estimated using Abraham-Lorentz equation by neglecting both binding and radiation damping terms. The cut-off wavelength is assumed to be much smaller than the particle size. To account for finite size of the particle, a convergence form factor was introduced. The average zeropoint energy of the oscillator has been calculated to be

$$\langle \Delta E \rangle = \frac{\Gamma_a \hbar \omega_c^2}{\pi}\eta(\omega), \tag{214}$$

where the convergence form factor $\eta(\omega)$ takes the values between 0 and 1. Haitch *et al.,* (1994) obtained the acceleration of a charged particle in the presence of zeropoint field and showed that the inertial mass of the particle is proportional to the particle acceleration. Further, they suggested that the radiation damping constant in the zeropoint fields as Γ_z (*zitterbewegung* damping constant) which is not necessarily equal to the damping constant Γ_a of Larmor formula of power radiated by an accelerated charged particle (Haisch *et al*. 2001) and by choosing $\eta(\omega_c)\Gamma_z\omega_c \sim 1$, the ground state energy of the oscillating particle in zeropoint field in (214) becomes $\hbar\omega_c/\pi$ which is similar to the *zitterbewegung* energy of the charged particle provided the frequency equals the frequency of *zitterbewegung* oscillations.

4.1. Equations of motion of center of mass and center of charge

In the complex vector space, the extended particle motion is a combined motion of both center of mass and center of charge. The zeropoint electric field vector may be considered as a function of complex position vector, $\boldsymbol{E}_{zp} = \boldsymbol{E}_{zp}(X,t)$ and its conjugate $\bar{\boldsymbol{E}}_{zp} = \boldsymbol{E}_{zp}(\bar{X},t)$. Expanding $\boldsymbol{E}_{zp}(X,t)$ using Taylor series we find

$$\boldsymbol{E}_{zp}(X,t) = \boldsymbol{E}_{zp}(\boldsymbol{x},t) + \boldsymbol{i\xi}\left|\partial \boldsymbol{E}_{zp}/\partial \boldsymbol{x}\right|_{x\to 0} + \cdots \tag{215}$$

The second term on left is a function of $\boldsymbol{\xi}$ only and neglecting higher order terms the complex electric vector can be expressed in the form

$$\boldsymbol{E}_{zp}(X,t) = \boldsymbol{E}_{zp}(\boldsymbol{x},t) + \boldsymbol{i}\boldsymbol{E}_{zp}(\boldsymbol{\xi},t) \tag{216}$$

and performing the reversion operation on $\boldsymbol{E}_{zp}(X,t)$ gives its complex conjugate

$$\boldsymbol{E}_{ZP}(\bar{X},t) = \boldsymbol{E}_{ZP}(\boldsymbol{x},t) - \boldsymbol{i}\boldsymbol{E}_{ZP}(\boldsymbol{\xi},t). \tag{217}$$

Now, the equation of motion of center of mass can be expressed as

$$m\frac{d^2(X+\bar{X})}{dt^2} = e[\boldsymbol{E}_{ZP}(X,t) + \boldsymbol{E}_{ZP}(\bar{X},t)], \tag{218}$$

where we have neglected the binding ad radiation damping terms. For the zeropoint field acting on center of mass, the particle mass and charge appear at the same point and the equation of center mass becomes

$$m\ddot{\boldsymbol{x}} = e\boldsymbol{E}_{ZP}(\boldsymbol{x},t). \tag{219}$$

In the case of center of charge, effective mass m_z may be perceived as the electromagnetic mass which is proportional to the potential due to charge e at the center of mass. The equation of motion of center of charge is then expressed in the form

$$m_z\frac{d^2(X-\bar{X})}{dt^2} = e[\boldsymbol{E}_{ZP}(X,t) - \boldsymbol{E}_{ZP}(\bar{X},t)], \tag{220}$$

which yields the equation of motion of center of charge in the particle rest frame.

$$m_z\ddot{\boldsymbol{\xi}} = e\boldsymbol{E}_{ZP}(\boldsymbol{\xi},t) \tag{221}$$

Thus, we have separated the complex motion of a charged particle on the zeropoint field into center of mass motion and center of charge motion and these are basically two different equations. In the following sub-sections, using stochastic electrodynamics, the energy of the particle will be estimated using these equations of motion.

4.2. Energy absorbed by the extended particle in its rest frame: zitterbewegung

The energy absorbed by the particle in its rest frame can be obtained by finding the stochastic average of square of internal velocity. The plane wave form of electric field vector $\boldsymbol{E}_{zp}(\boldsymbol{\xi},t)$ can be defined in following form.

$$\boldsymbol{E}_{zp}(\boldsymbol{\xi},t) = Re\sum_{\lambda=1}^{2}\int d^3\,k\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)H(\omega)\left[ae^{i\phi} + a^*e^{-i\phi}\right]. \tag{222}$$

Where $\phi = \boldsymbol{k}.\boldsymbol{\xi} - \omega t$ is the phase angle,$a = \exp[-i\theta(\boldsymbol{k},\lambda)]$, $a^* = \exp[i\theta(\boldsymbol{k},\lambda)]$, $\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)$ is the polarisation vector. The parameter $\lambda = 1, 2$ represents the zeropoint field modes. The angle $\theta(\boldsymbol{k},\lambda)$ reresents a set of random variables uniformly distributed between 0 and 2π and further, these random variables are mutually independent for each choice of $\boldsymbol{k}$ and λ. The stochastic nature of the field lies in these random variables and an average of these random angles gives the stochastic average of the field. The characteristic function $H(\omega)$ in stochastic electrodynamics is obtained as $H^2(\omega) = \hbar\omega\eta(\omega)/8\pi^3\epsilon_0$, where a convergence form factor $\eta(\omega)$ is introduced to account for extended nature of the particle and it takes the value between 0 and 1. Considering extended particle structure, one can impose a cut off frequency equivalent to the internal frequency of oscillations ω_0. Integrating the expression (222) with respect to time gives the internal velocity.

$$\boldsymbol{u} = \dot{\boldsymbol{\xi}} = \frac{e}{m_Z}\int_0^{\tau} \boldsymbol{E}_{ZP}(\boldsymbol{\xi}, t)dt = \frac{e}{m_Z} I \qquad (223)$$

Where, the characteristic time is defined as $\tau = 2\pi/\omega_0$ and it is the characteristic time required for the electromagnetic wave to traverse the distance equal to the size of the particle and

$$I = \sum_{\lambda=1}^{2} \int d^3 k\, \boldsymbol{\varepsilon}(\boldsymbol{k}, \lambda) H(\omega) \times \frac{1}{2}\left[a e^{i\boldsymbol{k}.\boldsymbol{\xi}}\left(\frac{e^{i\omega\tau}-1}{i\omega}\right) + a^* e^{-i\boldsymbol{k}.\boldsymbol{\xi}}\left(\frac{e^{-i\omega\tau}-1}{-i\omega}\right)\right]. \qquad (224)$$

A detailed calculation is carried out in the appendix and the stochastic average of square of velocity can be obtained in the form

$$\langle \boldsymbol{u}^2 \rangle = (e^2\hbar/3\pi^2\epsilon_0 m_Z^2 c^3)\eta(\omega_0)\int_0^{\omega_0}(1-\cos\omega\tau)\ \omega d\omega, \qquad (225)$$

where the upper cut-off frequency ω_0 is chosen to be the frequency of oscillations of the particle in the zeropoint field. Carrying out the integration in (225) and using the approximation $\omega_0\tau \sim 2\pi$ gives

$$\langle \Delta E_0 \rangle = \eta(\omega_0)\frac{\Gamma_Z \hbar \omega_0^2}{\pi}, \qquad (226)$$

where $\Gamma_z = e^2/6\pi\epsilon_0 m_z c^3$. Approximating $\eta(\omega_0)\Gamma_z\omega_0 = 1$ in the above equation gives the average zeropoint energy acquired by the particle in its rest frame.

$$\langle \Delta E_0 \rangle = \frac{\hbar\omega_0}{\pi}. \qquad (227)$$

This energy of the particle coincides with the quantum result the *zitterbewegung* energy. According to Sakurai's interpretation, *zitterbewegung* arises from the influence of virtual electron-positron pairs or vacuum fluctuations on the electron (Sakurai 2007). However, the above derivation shows that the *zitterbewegung* oscillations of electron are mainly due to the fluctuations of the particle immersed in the zeropoint field. In the Boyer's calculation what we obtain is the energy of the particle oscillator per mode and it is equal to the ground state energy of the oscillator. The energy obtained here is the average energy and it differs by a factor $\pi/2$.

4.3. Energy absorbed by the particle due to its center of mass motion: mass and charge corrections

The zeropoint electric field vector $\boldsymbol{E}_{ZP}(\boldsymbol{x}, t)$ can be expressed in the plane wave form.

$$\boldsymbol{E}_{zp}(\boldsymbol{x}, t) = Re\sum_{\lambda=1}^{2}\int d^3 k \boldsymbol{\varepsilon}(\boldsymbol{k}, \lambda) H(\omega)\left[a e^{i\phi} + a^* e^{-i\phi}\right] \qquad (228)$$

Where $\phi = \boldsymbol{k}.\boldsymbol{x} - \omega t$ is the phase angle and the other parameters are defined as in the previous subsection. As the center of mass moves in the zeropoint field, the zeropoint field surrounding the particle will be modified and the cut-off frequency may be considered as a modulated frequency due to the internal frequency of oscillations and the de Broglie frequency corresponding to the particle center of mass motion and one can impose cut off frequency equivalent to the frequency ω_c given by (22). Integrating (221) with respect to time gives the velocity of center of mass motion. The stochastic average of square of velocity can be obtained in the same method given in the previous subsection and the detailed calculation is given in the appendix. The final result is given by (A42).

$$\Delta E_c = \frac{2\alpha}{3\pi}\frac{(\hbar\omega_c)^2}{mc^2}\eta(\omega_c) \tag{229}$$

Now, substituting $\omega_c^2 = \omega_0^2 + \omega_B^2 = \omega_0^2(1+\beta^2)$ and $\hbar\omega_0 = mc^2$, we get

$$\Delta E_c = \frac{2\alpha}{3\pi}\eta(\omega_c)(1+\beta^2)mc^2. \tag{230}$$

Comparison of this expression with (34) gives $\eta(\omega_c) = 3/4$ and the energy represents the mass correction obtained via stochastic electrodynamics approach.

$$\Delta E_c = \frac{\alpha}{2\pi}(1+\beta^2)mc^2 \tag{231}$$

When the particle moves with velocity, the particle itself induces certain modifications in the zeropoint field which gives correction terms to mass. It is quite interesting that the mass correction depends on the particle velocity and such dependence was not found previously in quantum electrodynamics.

In the purview of quantum electrodynamics, one can perceive that both mass and charge corrections are obtained from the common origin (Milonni 1994) and in the point particle limit, Dirac theory suggests that the positive energy electron should electrostatically repel the negative energy electrons in the Dirac Sea which leads to vacuum polarization. In other words, due to vacuum polarization the effective electron charge is smaller in the magnitude than the bare charge of electron. From the energy time uncertainty principle, the electron-positron virtual pairs may exist for times of the order of $\hbar/mc^2$ or the virtual particles are separated over a distance $\hbar/mc$. The interaction of electromagnetic field with a charged particle leads to the vacuum polarisation which corresponds to second order photon self-energy and the point particle limit naturally leads to spectral divergence. However, considering the oscillations spread over a region, one can impose cut off frequency equivalent to the frequency of oscillation ω_0. Then the point particle limit is obtained by choosing $\eta(\omega_0) = 1$ and $\omega_c = \omega_0$ in (230) and the energy absorbed by the point particle in the zeropoint field is equal to

$$\Delta E_0 = \frac{2\alpha}{3\pi}\frac{(\hbar\omega_0)^2}{mc^2}. \tag{232}$$

This energy corresponds to the charge spread of the oscillating point particle. In the Casimir semi-classical shell model, the electron is considered as a charge shell and the charge density is taken to be sufficiently dense. In that case the vacuum fluctuation field completely vanishes inside the shell and there could be an inward radiation pressure counteracting the outwardly acting Coulomb force such that the electron has stable configuration. The deficit zeropoint energy of the shell was calculated by Puthoff (2007). Under the pressure balance conditions and as the shell radius tends to zero, it was found that the zeropoint energy of the shell is equal to the Coulomb energy. Therefore, it may be expected that the zeropoint energy associated with an electron in the point particle limit may be attributed to the charge correction rather than mass correction. Hence it may be appropriate to consider the ratio $\Delta E_0/\hbar\omega_0$ as a ratio of change in the fine structure constant $\Delta\alpha$ and α.

$$\frac{\Delta\alpha}{\alpha} = \frac{\Delta E_0}{\hbar\omega_0} = \frac{2\alpha}{3\pi} \tag{233}$$

Now, it can be proposed that in the point particle limit and in the absence of both radiation damping and binding terms, the energy of the particle oscillator gives charge correction. The reduced fine structure constant is then expressed as

$$\alpha \to \alpha\left(1 - \frac{2\alpha}{3\pi}\right). \qquad (234)$$

These mass and charge corrections can be used while calculating anomalous magnetic moment and Lamb shift of electron. These calculations were performed previously (Muralidhar & Rajendra 2018) and a short version of which is given below.

4.3.1. Anomalous magnetic moment of electron

A small deviation of electron magnetic moment $a_e = (g-2)/2$ is known as anomalous magnetic moment and this is absent in the relativistic theory of Dirac electron. The first experimental observation of anomalous magnetic moment was due to Kusch and Foley (1948). Theoretically, Schwinger (1948) estimated $a_e = \alpha/2\pi$ in the purview of quantum electrodynamics. Later an accurate estimate considering several Feynman diagrams led to an accurate value and an excellent review of the subject was given by Kinoshita (1990). A recent measurement of anomalous magnetic moment was given by Henneke *et al.*, (2008) $a_e(exp) = 1.15965218073(28) \times 10^{-3}$.

In section 2 the anomalous magnetic moment of electron a_e has been shown to be equal to the ratio $\delta m/m$. Approximating $\beta^2 = 2\alpha^2/3$ in (32) or in (237) and using the charge correction for fine structure constant given by (240), the anomalous magnetic moment can be expressed as a series expansion in α/π.

$$a_e = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 + \frac{\pi^2}{3}\left(\frac{\alpha}{\pi}\right)^3 - \frac{2\pi^2}{3}\left(\frac{\alpha}{\pi}\right)^4 + \cdots \qquad (235)$$

Using the recommended fine structure constant value $\alpha = 7.2973525376(50) \times 10^{-3}$ (Mohr *et al.* 2008), the anomalous magnetic moment of electron is estimated to be $a_e(th) = 1.15965227095 \times 10^{-3}$. The difference between theoretical and recent experimental value is found to be $a_e(th) - a_e(exp) = 90.22 \times 10^{-12}$. It shows that the present stochastic electrodynamics approach with particle extended structure gives the result agreeing with experimental value up to tenth decimal place. However, the present calculation depends on the approximation of particle velocity in terms of fine structure constant.

4.3.2. Lamb shift

In the relativistic the theory of Dirac electron, the energy levels $2S_{1/2}$ and $2P_{1/2}$ are found to be degenerate. Lamb and Rutherford (1947), using a microwave method, found the energy level shift of $2S_{1/2}$-$2P_{1/2}$ to be $1058.27 + 1.0\ MHz$. In the case of hydrogen like atoms, the orbital electron spreads out around the coulomb source and the average deviation is considered to be of the order of 2ξ. Note that the deviation from the path of electron brings a change in the orbital radius. Now the potential given in (153), can be expanded with ξ replaced by 2ξ.

$$\Sigma V(X, \bar{X}) = \mathrm{V}(\boldsymbol{r}) - 2\xi^2\nabla^2\mathrm{V}(\boldsymbol{r}) \qquad (236)$$

Where, the potential is considered to be symmetric and an additional multiplying factor 1/3 has been introduced in the second term on right of (236). Since the vector $\boldsymbol{\xi}$ is visualized to rotate in the spin plane, it can be expressed as an oscillating quantity $\boldsymbol{\xi} = |\boldsymbol{\xi}|exp(-\boldsymbol{i\sigma}_s\omega_0 t)$. The stochastic average $\langle\xi\rangle = 0$ and $\langle\xi^2\rangle = \langle|\boldsymbol{\xi}|^2\rangle/2$. Now, taking the stochastic average on both sides of (236) and substituting

$$\langle\xi^2\rangle = -\langle|\dot{\boldsymbol{\xi}}|^2\rangle/2\omega_0^2,$$

the change in the potential energy can be expressed as

$$\Delta E_L = \langle \Sigma V(X, \bar{X}) - \mathrm{V}(\boldsymbol{r}) \rangle = \left(\langle |\dot{\boldsymbol{\xi}}|^2 \rangle / \omega_0^2 \right) |\nabla^2 \mathrm{V}(\boldsymbol{r})|. \tag{237}$$

The energy obtained is similar to the one obtained in the Welton's method (Welton 1948) with a difference that ξ is replaced by the deviation of electron path δx and a multiplying factor $1/6$. The energy in (237) corresponds to the Lamb shift in the energy levels due to the interaction of electron with the zeropoint field. In the case of an atomic electron, the modifications in the zeropoint field are caused by the motion of orbital electron and therefore the cut-off frequency in the calculation of energy may be chosen to be the de Broglie frequency ω_B. Such low frequency cut-off was not considered previously either in stochastic electrodynamics or in quantum electrodynamics calculations. The stochastic average $\langle \dot{\boldsymbol{\xi}}^2 \rangle$ can be calculated in the appendix and from (A.46) we have

$$\langle |\dot{\boldsymbol{\xi}}|^2 \rangle = \frac{2\alpha}{3\pi} (\hbar^2 \omega_B^2 / m_r^2 c^2). \tag{238}$$

For an orbital electron in a circular orbit, the magnitude of Coulomb potential is equal to twice the kinetic energy of the electron and therefore, the potential energy is expressed as $V(r) = Ze^2/r = m_r v^2 = m_r \omega_B^2 r^2$. Now, we have

$$\nabla^2 \mathrm{V}(\boldsymbol{r}) = 2 m_r \omega_B^2. \tag{239}$$

Considering $\hbar\omega_0 = m_r c^2$ and substituting (240) and (239) in (238) gives

$$\Delta E_L = \frac{4\alpha}{3\pi} (\omega_B/\omega_0)^4 m_r c^2. \tag{240}$$

The orbital electron velocity $v \sim Z\alpha c$ and using the approximation $(\omega_B/\omega_0)^4 = \beta^4 \sim (Z\alpha)^4$, the Lamb shift energy can be expressed in the form

$$\Delta E_L = \frac{4Z^4\alpha^5}{3\pi} m_r c^2. \tag{241}$$

This gives a complete classical calculation of Lamb shift and it differs from that of Welton (1948) by a term $(1/n^3) ln(2/16.55\alpha^2) \sim 1$ for $n = 2$. In the case of atomic electron, the charge correction may be expected three times that of the free electron given in (234). Now, the fine structure constant in (242) will be replaced by $\alpha \to \alpha[1 - (2\alpha/\pi)]$ and using the mass correction for reduced mass the Lamb shift can be calculated. The calculated Lamb shift in hydrogen spectra is found to be $1058.3696\ MHz$ which is in agreement with the Lamb and Rutherford value. The present calculation is considerably in agreement with the standard value of Lamb shift $1057.8439\ MHz$ (Mohr *et al.* 2008) and the difference $0.5257\ MHz$ may be attributed to the finite size of the proton.

5. Summary and conclusions

The presence of zeropoint field means the space is not continuous but discrete and the particle may be visualised to have internal structure in complex vector space. Charged particles like electrons are not point like but having substructure, where the center of mass and center of charge are separate and the center of charge oscillations are considered as complex rotations in complex vector space. Such internal structure cannot be detected even by modern scattering experiments due to the fact that the center of charge rotation is at the speed of light. The particles are thus visualised as extended and its position is represented by a complex vector which is a combination of position vector of center of mass and a bivector representing the plane of charge rotation. The vector part of complex vector is symmetric and since the bivector changes its sign on reversion, it is asymmetric and because of this reason the complex vector space is twofold in nature. The angular momentum of complex internal rotation is the bivector spin angular momentum of the particle and it has been shown that it plays an important role in the development of micro mechanics. The wavefunction representing the state of the particle is intricately connected to the internal complex rotation. The mass of the particle is attributed to the local internal rotational energy and a relation between spin and mass is derived. When the center of mass moves with certain velocity, the charge rotation may be visualised as helical and gives the relativistic effects. It has been shown that the interaction of extended particle with zeropoint fields leads to mass correction. The Hestenes-Dirac equation can be obtained straightaway by considering the extended particle structure and the local rotations in complex vector space. The correspondence between internal parameters and operators in quantum mechanics is established and the internal kinetic energy is shown to be equal to the quantum potential. Considering time as a function of complex position vector, a complex vector time derivative is obtained and it replaces the total time derivative in complex vector approach.

Implementing classical Lagrangian approach in the complex vector space, complex Euler-Lagrange equations are derived. With a proper Lagrangian function, a generalized Newton's equation has been derived. The presence of zeropoint field allows presuming that the space itself is fine grained at micro level and such small discrete structure of spacetime corresponds to additional correction terms of Newtonian equation of motion. Expressing momentum in terms of wavefunction of the particle, a generalized Schrödinger equation is derived and it contains additional relativistic terms both in kinetic and potential energies and in the limiting case it reduces to usual Schrödinger equation in quantum mechanics. Squaring the generalized Schrödinger equation gives the Klein-Gordon equation. The Hamiltonian approach in the complex vector space yields complex Hamiltonian equations of motion. A semi-classical approach to quantum mechanics can be obtained from the complex Hamilton-Jacobi equation.

A macro object contains a conglomeration of micro-particles and in the presence of zeropoint field each micro-particle has a deviation from its center of mass. The effective deviation of all such micro-particles may be considered at the point of center of mass of the macro object. It has been shown that such effective deviation is of the order of average individual deviation of a micro-particle. Therefore, the position of point at center of mass of an object can be expressed as a complex vector. The influence of zeropoint field on a macro-particle is to produce modifications in the potential term. Since the individual particle spin orientations are at random, the effective spin of a macro-particle turns out to be zero. Then the generalized Newton's equation contains a modified potential term without any spin term and it represents the equation of motion of macro-particles. It is found that the modified potential depends on the particle velocity and as a consequence the particle acceleration in the gravitational field contains both attractive and velocity dependent repulsive gravitational terms. The radial acceleration obtained in the present complex vector approach is found to be equivalent to that obtained in the general theory of relativity. Applying generalized Newton's equation to the planetary system, the precession of perihelion of planets is derived and shown to be equivalent to the one obtained in general theory of relativity. Similarly, the gravitational deflection of light has been calculated and found to contain higher order correction terms. The estimated total angle of deflection is found to be in agreement with the observed value.

The stochastic electrodynamics approach is one of the best classical approaches to the development of quantum mechanics and the extended particle motion in the presence of zeropoint fields gives the mass correction and charge correction. These corrections terms are in agreement with those found in quantum electrodynamics. Using these corrections, anomalous magnetic moment and lamb shift have been estimated. The theoretical value of anomalous magnetic moment is found to be correct up to the tenth decimal place of the observed value. The value of Lamb shift is found to be in agreement with the standard value.

A sub-quantum level theory developed here is based on the concept that the entire universe is filled with zeropoint field and the particles are treated as extended objects and each particle position has been represented by a complex vector. The complex vector algebra is found to be an interesting mathematical tool which paves the way for better understanding of quantum foundations. Further, it gives an easy transition from micro mechanics to macro mechanics and allows us to predict new aspects of nature such as particle mass is fundamentally a field quantity and purely electromagnetic in nature. At macro level the additional terms those appear in the modified acceleration equation are found to be the manifestation of zeropoint field and such additional terms are actually responsible for the effects like advance of perihelion and gravitational deflection of light. Therefore, it may be speculated that presence of zeropoint field may be the cause of spacetime curvature. However, it requires further investigation to assert such conclusion and the present theory may hopefully leads to certain advancement in this aspect. Thus the fundamental theory developed here may open up further research towards micro and macro physics.

6. Appendix: Derivations

For the sake of conceptual continuity, most of the equations in the main text are directly given without explicit derivation. However, some are quite simple but many are not. The lengthy derivations in the text are addressed in this appendix for lucid understanding.

6.1. Derivation of equation (27)

The complex vector potential $\boldsymbol{A}_{zp}(X,t)$ can be expanded using Taylor series,

$$\boldsymbol{A}_{zp}(\boldsymbol{x}+\boldsymbol{i\xi},t)=\boldsymbol{A}_{zp}(\boldsymbol{x},t)+\boldsymbol{i\xi\nabla A}_{zp}(x,t)+\cdots \tag{A1}$$

and its conjugate $\overline{\boldsymbol{A}}_{zp}(X,t)=\boldsymbol{A}_{zp}(\bar{X},t)$ is

$$\boldsymbol{A}_{zp}(\boldsymbol{x}-\boldsymbol{i\xi},t)=\boldsymbol{A}_{zp}(\boldsymbol{x},t)-\boldsymbol{i\xi\nabla A}_{zp}(x,t)+\cdots \tag{A2}$$

Using complex momentum $P=(\boldsymbol{p}+\boldsymbol{i\pi})-(e\boldsymbol{A}_{zp}/c)$ and its conjugate, the total energy is $E^2=P\bar{P}c^2$.

$$E^2=\left[(\boldsymbol{p}+\boldsymbol{i\pi})-\tfrac{e}{c}\boldsymbol{A}_{zp}(X,t)\right]\times\left[(\boldsymbol{p}-\boldsymbol{i\pi})-\tfrac{e}{c}\boldsymbol{A}_{zp}(\bar{X},t)\right]c^2. \tag{A3}$$

Multiplying term by term gives

$$E^2=p^2c^2+m^2c^4-ec(\boldsymbol{p}+\boldsymbol{i\pi})\boldsymbol{A}_{zp}(\bar{X},t)-ec\boldsymbol{A}_{zp}(X,t)(\boldsymbol{p}-\boldsymbol{i\pi})+e^2\boldsymbol{A}_{zp}(X,t)\boldsymbol{A}_{zp}(\bar{X},t) \tag{A4}$$

Substituting (A1) and (A2) in the third and fourth terms on right of above equation, we get

$$2ec[\boldsymbol{A}_{zp}(x,t).\boldsymbol{p}+(\boldsymbol{\xi}\wedge\boldsymbol{\pi})\boldsymbol{\nabla A}_{zp}(\boldsymbol{x},t)-\boldsymbol{i\xi p}.\boldsymbol{\nabla A}_{zp}(\boldsymbol{x},t)],$$

where the terms of order ξ^2 are neglected. Noting the identities

$$\mathbf{2}\boldsymbol{A}_{zp}=-\boldsymbol{x}\wedge \boldsymbol{iB}_{zp};\ \boldsymbol{\nabla}.\boldsymbol{A}_{zp}=0\ ;\ \boldsymbol{\nabla}\wedge\boldsymbol{A}_{zp}=-\boldsymbol{iB}_{zp}\ ;\ \ \boldsymbol{p}.\boldsymbol{\nabla}\wedge\boldsymbol{A}_{zp}=\boldsymbol{p}\wedge\ \boldsymbol{\nabla}.\boldsymbol{A}_{zp}=0$$

and using the relations L= $\boldsymbol{x}\wedge\boldsymbol{p}$ and $S=\boldsymbol{\xi}\wedge\boldsymbol{\pi}$, the third and fourth terms on right of (A4) reduce to

$$-ec(L+2S).\boldsymbol{iB}_{zp}(\boldsymbol{x},t).$$

The last term in (A4) becomes

$$e^2A_{zp}^2(\boldsymbol{x},t)+e^2\xi^2\left|\boldsymbol{\nabla A}_{zp}(\boldsymbol{x},t)\right|^2.$$

Combining all the terms we get

$$E^2=p^2c^2+E_0^2, \tag{A5}$$

where $E_0^2=m^2c^4+e^2A_{zp}^2+e^2\xi^2\left|\nabla\boldsymbol{A}_{zp}\right|^2-ec(L+2S).\boldsymbol{iB}_{zp}$.

6.2. Calculation of stochastic average $\langle A_{zp}^2\rangle$

The stochastic average $\langle A_{zp}^2\rangle$ has been calculated by Haisch *et al.*, (1994) and we follow their method here. The plane wave form expansion of electromagnetic vector potential can be expressed as

$$\boldsymbol{A}_{ZP}(\boldsymbol{x},t)=Re\ \sum_{\lambda=1}^{2}\int d^3\,k\ \boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)\frac{ic}{\omega}H(\omega)\left[ae^{i\phi}-a^*e^{-i\phi}\right].$$

where $\phi=\boldsymbol{k}.\boldsymbol{x}-\omega t$, $\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)$ is the polarization vector, $\lambda=1,2$ represent orthogonal polarisations, Re represents the real part and the normalization constant is set equal to unity.

The phase factors $a=e^{-i\theta(\boldsymbol{k},\lambda)}$ and $a^*=e^{i\theta(\boldsymbol{k},\lambda)}$ and the phase angle $\theta(\boldsymbol{k},\lambda)$ represents a set of random variables uniformly distributed between 0 and 2π and are mutually independent for each choice of wave vector $\boldsymbol{k}$ and λ.

The characteristic function $H^2(\omega)=\hbar\omega/8\pi^3\epsilon_0$ and ϵ_0 is the permittivity of free space. Multiplying $\boldsymbol{A}_{zp}(\boldsymbol{x},t)$ with its complex conjugate and taking stochastic average on both sides yields

$$\langle\boldsymbol{A}_{zp}(\boldsymbol{x},t)\boldsymbol{A}_{zp}^*(\boldsymbol{x}',t')\rangle=\sum_{\lambda,\lambda'=1}^{2}\int d^3k\,d^3k'\langle\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda).\boldsymbol{\varepsilon}^*(\boldsymbol{k}',\lambda')\rangle\frac{c^2}{\omega^2}H^2(\omega)\langle aa^*\rangle.$$

The averages $\langle aa\rangle=\langle a^*a^*\rangle=0$ and $\langle aa^*\rangle=\langle a^*a\rangle=\delta(\lambda-\lambda')\delta^3(k-k')$. The stochastic average of the scalar product of polarisation vectors $\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)$ and $\boldsymbol{\varepsilon}^*(\boldsymbol{k},\lambda)$ can be obtained as $\sum_{\lambda=1}^{2}\langle\boldsymbol{\varepsilon}_i(\boldsymbol{k},\lambda).\boldsymbol{\varepsilon}_j^*(\boldsymbol{k},\lambda)\rangle=\delta_{ij}$. Using $\int d\Omega=4\pi$, $kc=\omega$, and the integral

$$\int d^3k=\int d\Omega\,k^2dk=\frac{4\pi}{c^3}\int\omega^2\,d\omega$$

in the above equation and using the upper limit of integration equal to the cut-off frequency ω_0 gives

$$\langle\left|\boldsymbol{A}_{zp}(\boldsymbol{x},t)\right|^2\rangle=\frac{1}{2\pi^2\varepsilon_0}\frac{\hbar}{c}\int_0^{\omega_0}\omega\,d\omega=\frac{1}{4\pi^2\varepsilon_0}\frac{\hbar\omega_0^2}{c}.$$

Now, the second term on right of (29) can be written as

$$\frac{e^2}{2mc^2}\langle A_{zp}^2\rangle = \frac{\alpha}{2\pi}\frac{(\hbar\omega_0)^2}{mc^2}, \tag{A6}$$

where $\alpha = e^2/4\pi\varepsilon_0\hbar c$ is the fine structure constant.

6.3. Energy of a point charged particle in zeropointfield: Boyer's approach

Here, we follow the method of calculation by Boyer (1975a). The equation of motion of electron is given by

$$m\ddot{\boldsymbol{x}} = -m\omega_0^2\boldsymbol{x} + \Gamma_a m\dddot{\boldsymbol{x}} + e\boldsymbol{E}_{ZP}(\boldsymbol{x},t), \tag{A7}$$

where $\Gamma_a = e^2/6\pi\epsilon_0 mc^3$. The plane wave form of zeropoint electric vector $\boldsymbol{E}_{ZP}(\boldsymbol{x},t)$ can be expressed as

$$\boldsymbol{E}_{zp}(\boldsymbol{x},t) = Re\ \sum_{\lambda=1}^{2}\int d^3\,k\ \boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)H(\omega)\left[ae^{i\phi} + a^*e^{-i\phi}\right]. \tag{A8}$$

The Fourier transforms for $\boldsymbol{x}(t)$ and $\boldsymbol{E}_{zp}(x,t)$ are taken in the form

$$\boldsymbol{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\boldsymbol{x}(\omega)e^{-i\omega t}\,dt\ \ ;\ \boldsymbol{E}_{zp}(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\boldsymbol{E}_{zp}(\boldsymbol{x},\omega)e^{-i\omega t}\,dt.$$

Substituting these transforms in (A7), we have

$$x(\omega) = \frac{e}{m}\frac{\hat{\boldsymbol{x}}.\boldsymbol{E}(\boldsymbol{x},\omega)}{C},$$

where $C = \omega_0^2 - \omega^2 + i\Gamma_a\omega^3$ and $\hat{\boldsymbol{x}}$ is the unit vector along the direction of $\boldsymbol{x}$. In the treatment of stochastic electrodynamics, the ensemble averages are assumed to be equal to the time averages and such process is called ergodic. Using the identity $\langle\boldsymbol{E}(\boldsymbol{x},\omega)\boldsymbol{E}^*(\boldsymbol{x},\omega)\rangle = \langle\boldsymbol{E}(\boldsymbol{x},t)\boldsymbol{E}^*(\boldsymbol{x},t)\rangle$. Now, we write the product

$$\left[\frac{\hat{\boldsymbol{x}}.\boldsymbol{E}(\boldsymbol{x},\omega)}{C}\right]\left[\frac{\hat{\boldsymbol{x}}.\boldsymbol{E}(\boldsymbol{x},\omega)}{C}\right]^* = \sum_{\lambda,\lambda'=1}^{2}\int d^3\,k\ d^3k'[\hat{\boldsymbol{x}}.\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)][\hat{\boldsymbol{x}}.\boldsymbol{\varepsilon}^*(\boldsymbol{k}',\lambda')]\frac{H^2(\omega)}{CC^*}\frac{1}{2}[aa^* + a^*a + aa + a^*a^*].$$

Now, the stochastic average of square of the position coordinate can be written in the form

$$\langle x^2\rangle = \frac{e^2}{m^2}\sum_{\lambda,\lambda'=1}^{2}\int d^3\,k\ d^3k'\langle[\hat{\boldsymbol{x}}.\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)][\hat{\boldsymbol{x}}.\boldsymbol{\varepsilon}^*(\boldsymbol{k}',\lambda')]\rangle\frac{H^2(\omega)}{CC^*}\times\frac{1}{2}[2\langle aa^*\rangle + \langle aa\rangle + \langle a^*a^*\rangle].$$

Using the following the stochastic averages and integrals

$$\langle aa\rangle = \langle a^*a^*\rangle = 0 \tag{A9}$$

$$\langle aa^*\rangle = \langle a^*a\rangle = \delta(\lambda-\lambda')\delta^3(k-k') \tag{A10}$$

$$\int f(k')\delta^3(k-k')d^3k' = f(k) \tag{A11}$$

$$\int d^3\,k = \int d\Omega\,k^2dk = \frac{4\pi}{c^3}\int\omega^2\,d\omega \tag{A12}$$

$$\sum_{\lambda=1}^{2}\langle[\hat{\boldsymbol{x}}_i.\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)]\left[\hat{\boldsymbol{x}}_j.\boldsymbol{\varepsilon}^*(\boldsymbol{k},\lambda)\right]\rangle = \delta_{ij} - \frac{k_ik_j}{|k|^2} = \frac{2}{3}\qquad (for\ i=j) \tag{A13}$$

and substituting the above identities in the equation of $\langle x^2\rangle$, we get

$$\langle x^2 \rangle = \frac{1}{4\pi\epsilon_0} \frac{2e^2}{3mc^3} \frac{2\hbar}{\pi m} \int_0^{\infty} (\omega^3 d\omega) / \left[\left(\omega_0^2 - \omega^2 \right)^2 + \Gamma_a^2 \omega^6 \right] .$$

In the limit $\Gamma_a \omega_0 \ll 1$ and at the resonance condition of the oscillator we approximate $\omega = \omega_0$ in the integrand except in the term $\omega - \omega_0$.

$$\langle x^2 \rangle = \Gamma_a \frac{\hbar\omega_0}{4\pi m} \int_{-\infty}^{\infty} d\omega \,/ \left[(\omega - \omega_0)^2 + \left(\frac{\Gamma_a \omega_0^2}{2} \right)^2 \right] \tag{A14}$$

Using the value of definite integral $\int_{-\infty}^{\infty} dy/(y^2 + b^2) = \pi/b$, we arrive at the desired result

$$\langle x^2 \rangle = \hbar/2m\omega_0 \ . \tag{A15}$$

Since the time derivatives of coordinate x gives multiplication by $i\omega$, the spectrum of v^2 is ω^2 times x^2. Thus, one can obtain the stochastic average of momentum $\langle p^2 \rangle$ in the same procedure as above.

$$\langle p^2 \rangle = \hbar m \omega_0 / 2. \tag{A16}$$

The energy of the charged point particle oscillator in the presence of zeropoint field is the sum of kinetic and potential energies.

$$E_0 = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2} m\omega_0 \langle x^2 \rangle = \frac{\hbar\omega_0}{2}. \tag{A17}$$

It can be seen that the energy derived here is the energy per mode of the internal oscillator system and the stochastic averages $\langle x^2 \rangle = \langle \delta x^2 \rangle = \langle \xi^2 \rangle$ and $\langle p^2 \rangle = \langle \delta p^2 \rangle = \langle \pi^2 \rangle$.

6.4. Derivation of equation (157)

The momentum $\boldsymbol{p}$ is given by (94) and squaring it on both sides gives

$$p^{\mathbf{2}} = (2S)^2 (\nabla \,ln\, \psi)^2 + (S)^2 (\nabla \ln \rho)^2 - 4(S\boldsymbol{\nabla}\, ln\, \psi).(S\boldsymbol{\nabla} \ln \rho).$$

The last term is a scalar product and it can be written as

$$(4S\boldsymbol{\nabla}\, ln\, \psi).(S\boldsymbol{\nabla} \ln \rho) = \frac{(2S)^2}{2} (\boldsymbol{\nabla} \ln \rho)^{\mathbf{2}},$$

where we have used the relation $\boldsymbol{p}.S = 0$. Then the kinetic energy can be expressed in the form

$$\frac{p^{\mathbf{2}}}{2m} = \frac{(2S)^2}{2m} (\nabla \,ln\, \psi)^2 - \frac{(2S)^2}{8m} (\nabla \ln \rho)^2. \tag{A18}$$

The operators $\boldsymbol{\nabla}$ and Σ commute and $\boldsymbol{\nabla}\Sigma = \Sigma\boldsymbol{\nabla}$is used in the following expansions. Using (94) and (A18), the terms in (151) can be expressed in the following form.

$$\frac{\partial \boldsymbol{p}}{\partial t} = -2 \frac{\partial S}{\partial t} \nabla \ln \psi - 2S\nabla \left(\frac{1}{\psi} \frac{\partial \psi}{\partial t} \right) \tag{A19}$$

$$\boldsymbol{v}\boldsymbol{\nabla}\Sigma\boldsymbol{p} = \boldsymbol{\nabla}\Sigma \left[\frac{(2S)^2}{2m} (\nabla \ln \psi)^2 + \frac{(2S)^2}{8m} (\nabla \ln \rho)^2 \right]$$

$$-\frac{2S}{2m}\Sigma\nabla^2\boldsymbol{p} = \boldsymbol{\nabla}\Sigma\left[\frac{(2S)^2}{2m}\nabla^2\ln\psi + \frac{(2S)^2}{4m}\nabla^2\ln\rho\right]$$

Using the identity $(\nabla\ln f)^2 + \nabla^2\ln f = \nabla^2 f/f$ the last two equations combined to give

$$\boldsymbol{v}\boldsymbol{\nabla}\Sigma\boldsymbol{p} - \frac{2S}{2m}\Sigma\nabla^2\boldsymbol{p} = \boldsymbol{\nabla}\Sigma\left\{\frac{(2S)^2}{2m}\frac{\nabla^2\psi(x,t)}{\psi} + Q\right\}, \tag{A20}$$

where, the quantum potential

$$Q = \frac{(2S)^2}{4m}\left[\frac{\nabla^2\rho}{\rho} - \frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2\right].$$

The last two terms in (151) vanish.

$$\boldsymbol{iu}\wedge\boldsymbol{\nabla p}\Sigma = \boldsymbol{iu}\wedge S[-2\boldsymbol{\nabla}\,ln\,\psi(\boldsymbol{x},t) + \boldsymbol{\nabla}\ln\rho] = 0$$

$$\boldsymbol{v}\wedge\boldsymbol{i\xi}\nabla^2\boldsymbol{p}\Sigma = \boldsymbol{v}\wedge\boldsymbol{i\xi}\wedge S\nabla^2[-2\boldsymbol{\nabla}\,ln\,\psi(\boldsymbol{x},t) + \boldsymbol{\nabla}\ln\rho] = 0$$

where we have used the identity $\boldsymbol{ia}\wedge\boldsymbol{b} = \boldsymbol{i}(\boldsymbol{a}.\boldsymbol{b})$ such that products $\boldsymbol{iu}\wedge S = \boldsymbol{i}(\boldsymbol{u}.S) = 0$ and $\boldsymbol{i\xi}\wedge S = \boldsymbol{i}(\boldsymbol{\xi}.S) = 0$. Now, substituting (A19) and (A20) in (151) gives

$$2\frac{\partial S}{\partial t}(\ln\psi)\psi + 2S\boldsymbol{\nabla}\left(\frac{1}{\psi}\frac{\partial\psi}{\partial t}\right) = \boldsymbol{\nabla}\Sigma V(X,\bar{X}) + \boldsymbol{\nabla}\Sigma\left[\frac{(2S)^2}{2m}\frac{\nabla^2\psi(x,t)}{\psi} + \mathrm{Q}\right]. \tag{A21}$$

Using a simple approximation $\ln\psi \sim \theta/2S$, one can see the first term on left of (A21) vanishes due to the fact that S^2 is a constant of motion.

$$2\frac{\partial S}{\partial t}(\ln\psi)\psi = \frac{\partial S^2}{\partial t}\frac{2\theta}{(2S)^2}\psi = 0$$

Now, (A21) gives a generalized Schrödinger equation in complex vector space.

$$2S\frac{\partial}{\partial t}\psi(\boldsymbol{x},t) = \frac{(2S)^2}{2m}\nabla^2\Sigma\psi(\boldsymbol{x},t) + \Sigma V(X,\bar{X})\psi(\boldsymbol{x},t) + \Sigma Q\psi \tag{A22}$$

6.5. Modified central potential

The complex potential $V(X,\bar{X}) = V(\boldsymbol{x} + \boldsymbol{i\xi},\ \boldsymbol{x} - \boldsymbol{i\xi})$ can be expanded in Taylor series in the following form.

$$V(X,\bar{X}) = V(\boldsymbol{x}) - \frac{\xi^2}{2}\nabla^2 V(\boldsymbol{x}) + O(\xi^4).$$

Using the expansion for $\Sigma = \sum_{k=0}^{\infty}(-\xi^2\nabla^2)^k$ and retaining terms up to the order ξ^2,

$$\Sigma V(X,\bar{X}) = (1-\xi^2\nabla^2)\left(1 - \frac{\xi^2}{2}\nabla^2\right)\mathrm{V}(\boldsymbol{x}) = \left(1 - \frac{3\xi^2}{2}\nabla^2\right)\mathrm{V}(\boldsymbol{x}).$$

For central potential $\mathrm{V}(\boldsymbol{x}) = -k/\boldsymbol{x}$, we have

$$\Sigma V(X,\bar{X}) = \left(1 - \frac{3\xi^2}{x^2}\right)\mathrm{V}(\boldsymbol{x}).$$

Now, using (32) the modified central potential can be expressed in the form

$$V'(\boldsymbol{x}) = \mathrm{V}(\boldsymbol{x})\left(1-\frac{3v^2}{c^2}\right) = \mathrm{V}(\boldsymbol{x})(1-3\beta^2). \qquad \text{(A.23)}$$

6.6. Radial acceleration in general relativity

Suppose that center of spherical mass M is located at $r = 0$ and let the space around it is empty. The Schwarzschild metric in spherical polar coordinates is given by

$$ds^2 = e^{\nu}dt^2 - \frac{1}{c^2}[dr^2 e^{-\nu} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2], \qquad \text{(A24)}$$

where $e^{\nu} = \left(1-\frac{r_0}{r}\right)$, $r_0 = 2GM/c^2$. The values of $g_{\mu\mu}$ can be read off from (A24).

$$g_{00} = e^{\nu} \;\; ; \; g_{11} = -\frac{1}{c^2}e^{-\nu} \;\; ; \; g_{22} = -\frac{1}{c^2}r^2 \;\; ; \; g_{33} = -\frac{1}{c^2}r^2\sin^2\theta$$

The fundamental paths in Riemannian spacetime are called geodesics. The geodesic of a test particle in the curved spacetime is given by McVittie (1964).

$$\frac{d}{ds}\left(g_{\mu\mu}\frac{dx^{\mu}}{ds}\right) - \frac{1}{2}\frac{\partial g_{\lambda\lambda}}{\partial x^{\mu}}\left(\frac{dx^{\lambda}}{ds}\right)^2 = 0 \quad (\mu = 0,1,2,3).$$

The geodesic equation for $\mu = 0$ is

$$\frac{d}{ds}\left[\left(1-\frac{r_0}{r}\right)\frac{dt}{ds}\right] = 0$$

Integrating this expression, we get

$$\frac{dt}{ds} = \gamma\left(1-\frac{r_0}{r}\right)^{-1}, \qquad \text{(A25)}$$

where γ is a constant. If the motion is purely radial, we neglect θ and ϕ terms and we write the following integral equation from (A24) in the form

$$1 = \left(1-\frac{r_0}{r}\right)\left(\frac{dt}{ds}\right)^2 - \frac{1}{c^2}\left[\left(\frac{dr}{ds}\right)^2\left(1-\frac{r_0}{r}\right)^{-1}\right].$$

Using the identity $\frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds}$, and substituting (A25) in the above equation gives

$$\left(\frac{dr}{dt}\right)^2 = c^2\left(1-\frac{r_0}{r}\right)^2 - \frac{c^2}{\gamma^2}\left(1-\frac{r_0}{r}\right)^3. \qquad \text{(A26)}$$

When $r \to \infty$ we have $\frac{r_0}{r} = 0$ and the constant $\gamma = \left(1-\frac{v^2}{c^2}\right)^{-1/2} = (1-\beta^2)^{-1/2}$.

Differentiating the expression (A26) with respect to time gives the radial acceleration

$$\frac{d^2r}{dt^2} = 2g\left(1 - \frac{r_0}{r}\right) - 3g(1-\beta^2)\left(1 - \frac{r_0}{r}\right)^2. \quad \text{(A27)}$$

6.7. Advance of perihelion

Substituting (A23) in (197) the central force can be expressed in the form

$$\frac{d\boldsymbol{p}}{dt} = -\boldsymbol{\nabla}V'(\boldsymbol{x}) = -\hat{\boldsymbol{x}}\frac{k}{x^2}\left[1 - \frac{3v^2}{c^2} + \frac{6\,(\boldsymbol{x}.\boldsymbol{a})}{c^2}\right], \quad \text{(A28)}$$

where we have used $\frac{\partial v}{\partial \mathrm{x}} = \frac{\partial v}{\partial t}\frac{\partial t}{\partial x} = \frac{a}{v}$.

For planetary motion $k = GMm$, where M and m are masses of Sun and planet respectively.

The angular momentum bivector $L = \boldsymbol{r} \wedge \boldsymbol{p}$ and in polar coordinates, we have $mr^2\dot{\theta} = l$ is a constant of motion, where over dot represents differentiation with respect to time.

Now, (A28) can be expressed as

$$\frac{dp}{dt} = m\ddot{\boldsymbol{r}} - \frac{l^2}{mr^3} = -\frac{k}{r^2}\left[1 - \frac{3v^2}{c^2} + \frac{6\,(\boldsymbol{r}.\boldsymbol{a})}{c^2}\right]. \quad \text{(A29)}$$

To find the equation of the planetary motion, we eliminate dt using $\frac{d}{dt} = \frac{h}{r^2}\frac{d}{d\theta}$

and changing the variable $u = 1/r$ we have

$$v = \dot{r} = -\frac{h}{r^2}\frac{dr}{d\theta} = h\frac{du}{d\theta}\ ;\ a = \ddot{r} = -u^2h^2\frac{d^2u}{d\theta^2}\ ;\ \boldsymbol{r}.\boldsymbol{a} = -uh^2\frac{d^2u}{d\theta^2},$$

where $h = l/m$. Substituting all these terms in (A29) gives

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2} - \frac{3GM}{c^2}\left(\frac{du}{d\theta}\right)^2 + \frac{6GM}{c^2}u\frac{d^2u}{d\theta^2}. \quad \text{(A30)}$$

The solution of this equation can be solved iteratively. Let $u = u_0 + u_1$ and $u_0 \gg u_1$. We get the following two equations.

$$\frac{d^2u_0}{d\theta^2} + u_0 = \frac{GM}{h^2} = B \quad \text{(A31)}$$

$$\frac{d^2u_1}{d\theta^2} + u_1 = -D\left(\frac{du_0}{d\theta}\right)^2 + 2Du_0\frac{d^2u_0}{d\theta^2} \quad \text{(A32)}$$

Where $= \frac{3GM}{c^2}$. Now, (A31) has the solution of the form

$$u_0 = B + A\cos(\theta - \theta_0).$$

Substituting above solution in (A32) and using a particular solution of the form

$$u_1 = A_1 + B_1(\theta - \theta_0)\, sin(\theta - \theta_0) + C_1 cos^2(\theta - \theta_0),$$

we get the constants $B_1 = -DAB$; $C_1 = DA^2$and $A_1 = -DA^2$.

The advance of perihelion can be calculated using the condition $du/d\theta = 0$ at turning points and evaluating $du_0/d\theta$ at $\theta = \theta_0 + \Delta\theta$ and $du_1/d\theta$ at $\theta = \theta_0 + 2\pi$.

$$\frac{du}{d\theta} = (du_0/d\theta)_{\theta=\theta_0+\Delta\theta} - (du_1/d\theta)_{\theta=\theta_0+2\pi} = -A\Delta\theta - 2\pi B_1 = 0$$

This gives

$$\Delta\theta = -\frac{2\pi B_1}{A} = 2\pi DB = \frac{6\pi G^2 M^2}{c^2 h^2}. \quad \text{(A33)}$$

In terms of semi-major axis a_m and eccentricity ϵ, $(h^2/GM) = a_m(1-\epsilon^2)$, the above equation can be expressed in the familiar form

$$\Delta\theta = \frac{6\pi GM}{c^2 a_m(1-\epsilon^2)} \ . \quad \text{(A34)}$$

6.8. Gravitational deflection of light

From the modified force equation given by (200), the acceleration in the x-direction can be expressed in the following form

$$\frac{d^2x}{dt^2} = -GM\frac{x}{r^3}\left[1 - \frac{3v^2}{c^2} + \frac{6r}{c^2}\frac{d^2x}{dt^2}\right]. \quad \text{(A35)}$$

This can be approximated as

$$\frac{d^2x}{dt^2} \sim -GM\frac{x}{r^3}\left(1 - \frac{3v^2}{c^2}\right)\left(1 - \frac{6GM}{c^2}\frac{x}{r^2}\right), \quad \text{(A36)}$$

where $r = (x^2 + y^2)^{1/2}$.

For a light particle moving almost parallel to the y-axis we have $(dy/dt) = c$ and $(d^2y/dt^2) = 0$. Using the identity

$$\frac{dx}{dt} = \frac{dx}{dy}\frac{dy}{dt} = \frac{dx}{dy}c$$

and differentiating this expression with respect to time gives

$$\frac{d^2x}{dt^2} = \frac{d^2x}{dy^2}\left(\frac{dy}{dt}\right)^2 = \frac{d^2x}{dy^2}c^2.$$

Using this relation and replacing $v = c$ in (A36) and for $x = R$ gives

$$\frac{d^2x}{dy^2} = \frac{2GM}{c^2}\left(R/(R^2+y^2)^{3/2}\right) - 3\left(\frac{2GM}{c^2}\right)^2\left(R^2/(R^2+y^2)^{5/2}\right)$$

Integrating with respect to y gives

$$\frac{dx}{dy} = \left[\frac{2GM}{c^2R} - 3\left(\frac{2GM}{c^2R}\right)^2\right]\left[y/(R^2+y^2)^{1/2}\right] + \left(\frac{2GM}{c^2R}\right)^2\left[y^3/(R^2+y^2)^{3/2}\right] + C, \qquad \text{(A37)}$$

where C is a constant of integration. For $y = 0$, $x = R$ and $(dx/dy) = 0$ and we get $C = 0$. Since $(dx/dy) = \tan\alpha$ and for large value of y and for small value of α we get

$$[dx/dy]_{y\to-\infty}^{y\to\infty} = \tan\alpha - \tan(-\alpha) \sim 2\alpha,$$

and applying the same limits on the right of (A37) we get

$$2\alpha = \frac{2GM}{c^2R}(\pm 2) - 2\left(\frac{2GM}{c^2R}\right)^2(\pm 2).$$

This gives the angle of deflection of light.

$$2\alpha = 2\frac{r_0}{R} - 4\left(\frac{r_0}{R}\right)^2. \qquad \text{(A38)}$$

6.9. Zitterbewegung energy

In the following derivation we follow the method used by Rueda (1981) and further to account for extended structure of the particle a convergence form factor has been introduced. The equation of motion of the particle in its rest frame is given by (222)

$$m_z\ddot{\boldsymbol{\xi}} = e\boldsymbol{E}_{ZP}(\boldsymbol{\xi}, t). \qquad \text{(A38)}$$

The electromagnetic field is now considered as a function of $\boldsymbol{\xi}$ and t and expressed in the plane wave form given by (A8) replacing $\boldsymbol{x}$ by $\boldsymbol{\xi}$. Integrating the expression (A38) we find

$$\boldsymbol{u} = \dot{\boldsymbol{\xi}} = (e/m_z)\int_0^\tau \boldsymbol{E}_{ZP}(\boldsymbol{\xi}, t)dt = (e/m_z)I,$$

where τ is the characteristic time required for an electromagnetic wave to traverse a distance equal to the size of the particle and

$$I = \sum_{\lambda=1}^{2}\int d^3\,k\,\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)H(\omega)\frac{1}{2}\left[ae^{ik.x}\left(\frac{e^{i\omega\tau}-1}{i\omega}\right) + a^*e^{-k.x}\left(\frac{e^{-i\omega\tau}-1}{-i\omega}\right)\right].$$

A factor ½ is introduced in the above integral to account for the exponential form of derivation. The stochastic average of square of velocity is then obtained as

$$\langle\boldsymbol{u}^2\rangle = (e^2/m_z^2)\langle II^*\rangle.$$

While finding the product $\langle II^*\rangle$we come across with the terms like

$$aa^*\left(\frac{e^{i\omega\tau}-1}{i\omega}\right)\left(\frac{e^{-i\omega\tau}-1}{-i\omega}\right) = aa^*\frac{2(1-\cos\omega\tau)}{\omega^2}$$

and similarly, the terms containing a^*a , aa and a^*a^*. Using the stochastic averages, given by (A9) and

(A10) and the identities (A11) to (A13), we find

$$\langle \boldsymbol{u}^2 \rangle = (e^2/m_z^2)(\hbar/8\pi^3\epsilon_0)\frac{8\pi}{3c^3}\eta(\omega_0)\int_0^{\omega_0}(1-\cos\omega\tau)\ \omega d\omega, \tag{A39}$$

where the cut-off frequency ω_0 is chosen to be the upper limit of the integral. Carrying out the integration yields

$$\langle \boldsymbol{u}^2 \rangle = (e^2\hbar\omega_0^2\,/6\pi^2\epsilon_0 m_z^2 c^3)\eta(\omega_0)[1+(1/\omega_0^2\tau^2)\{1-\cos\omega_0\tau-\omega_0\tau\sin\omega_0\tau\}].$$

Using the approximation $\omega_0\tau \sim 2\pi$, the energy of the oscillator in its rest frame can be obtained as

$$\mathrm{m_z}\langle \boldsymbol{u}^2 \rangle = \eta(\omega_0)\frac{\Gamma_z\hbar\omega_0^2}{\pi},$$

where $\Gamma_z = e^2/6\pi\epsilon_0 m_z c^3$. Using the approximation $\Gamma_z\omega_0\eta(\omega_0) = 1$ we get

$$\Delta E = \frac{\hbar\omega_0}{\pi}\,. \tag{A40}$$

6.10. Energy of the extended particle due to center of mass motion in the zeropoint field

The equation of center of mass motion is given by (220)

$$m\ddot{\boldsymbol{x}} = e\boldsymbol{E}_{ZP}(\boldsymbol{x},t) \tag{A41}$$

The plane wave form of zeropoint electric vector $\boldsymbol{E}_{ZP}(\boldsymbol{x},t)$ is given by (A8) and integrating the expression (A41) on both sides gives the particle velocity.

$$\dot{\boldsymbol{x}} = \frac{e}{m}\int_0^{\tau}\boldsymbol{E}_{ZP}(\boldsymbol{x},t)dt = \frac{e}{m}I$$

The stochastic average $\langle \dot{\boldsymbol{x}}^2 \rangle$ of the oscillating particle can be obtained by integrating the electric field vector from 0 to τ and using the stochastic averages and identities given in (A9) to (A13).

We choose the upper limit of integration to be ω_c and a similar calculation as above gives

$$\langle \dot{\boldsymbol{x}}^2 \rangle = \frac{2\alpha}{3\pi}\frac{(\hbar\omega_c)^2}{m^2c^2}\eta(\omega_c)[1+(1/\omega_0^2\tau^2)\{1-\cos\omega_c\tau-\omega_c\tau\sin\omega_c\tau\}].$$

Using the condition $\omega_c\tau \sim 2\pi$, the energy of the oscillator in the zeropoint field is obtained as

$$\Delta E_0 = m\langle \dot{\boldsymbol{x}}^2 \rangle = \frac{2\alpha}{3\pi}\frac{(\hbar\omega_c)^2}{mc^2}\eta(\omega_c) \tag{A42}$$

Now, substituting $\omega_c^2 = \omega_0^2 + \omega_B^2 = \omega_0^2(1+\beta^2)$ and $\hbar\omega_0 = mc^2$, we get

$$\Delta E_0 = \frac{2\alpha}{3\pi}\eta(\omega_c)(1+\beta^2)mc^2.$$

Comparison of this expression with (32) gives $\eta(\omega_c) = 3/4$ and we get

$$\Delta E_0 = \frac{\alpha}{2\pi}(1+\beta^2)mc^2. \tag{A43}$$

In the point particle limit, we set $\omega_c = \omega_0$ and $\eta(\omega_c) = 1$ in (A42) and the energy absorbed by the point particle in the zeropoint field is then equal to

$$\Delta E_0 = \frac{2\alpha}{3\pi}\frac{(\hbar\omega_0)^2}{mc^2}. \tag{A44}$$

The energy obtained in this approach is entirely different from the one obtained by Boyer and the difference is due to the imposition of upper cut-off frequency.

6.11. Derivation of (238)

In the case of atomic electron the cut-off frequency in (A.39) is considered to be equal to the de Broglie frequency.

$$\langle|\dot{\xi}|^2\rangle = (1/4\pi\epsilon_0)\frac{2e^2}{3mc^3}\frac{4\hbar}{\pi}\int_0^{\omega_B}\omega\cos\omega\tau\ d\omega, \tag{A45}$$

Integrating this expression gives

$$m\langle|\dot{\xi}|^2\rangle = \frac{2\alpha}{3\pi}\frac{(\hbar\omega_B)^2}{mc^2}\eta(\omega_B)[1 + (2/\omega_B^2\tau^2)(1 - \cos\omega_B\tau - \omega_B\tau\sin\omega_B\tau)]$$

Since $\omega_B\tau \ll 1$, we neglect the terms in square brackets and for low values of ω_B, the converging form factor $\eta(\omega_B)$~1. This gives the required result

$$\langle|\dot{\xi}|^2\rangle = \frac{2\alpha}{3\pi}\frac{(\hbar\omega_B)^2}{m^2c^2}\ . \tag{A46}$$

Declarations

Availability of data: Standard data is used in the article in performing the calculations.

Author's contribution: The theory presented here is conceived and developed by the author.

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