Solving Ordinary Differential Equations Using "Saxena & Gupta Transform"

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Abstract
Integral transforms play an important role in solving system of ordinary differential equations and integral equations. In the present paper we discuss some applications of new transform "Saxena & Gupta" transform is an interesting method to solve certain type of system of ODEs. This transform operates by converting a given system of ordinary differential equations into an algebraic equations. Upon solving this algebraic equations, the inverse transform is applied to yield the sought-after solution.
Keywords:Saxena & Gupta transform, inverse Saxena & Gupta transform, system of differential equation, Boundary value problems.
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I. Introduction

Fractional calculus is the branch of mathematics which deal with the investigation and applications of integrals and derivatives arbitrary order. Due to the growing range of applications, there has been significant interest in developing transforms for the solution of fractional differential equations.
Integral transforms are the most useful techniques of the mathematics which are used to find the solutions of differential equations, partial differential equations, integro-differential equations, partial integro- differential equations, delay differential equations and population growth.
In this paper we apply a new integral transform, called Saxena & Gupta transform, for solving a system of ordinary differential equations. Integral transformations essential for solving complex problems in engineering, natural sciences, computers, optical sciences, and modern mathematics to a simple system of algebraic equations that can be solved easily.

Definition :
Let $f(t)$ be a function of $t \geq 0$. The new transform of a function $f(t)$ is defined as follows, see in [7]

$$f(v) = Z[f(t)] = \frac{1}{v}\int_0^\infty f(vt)e^{-t}dt \qquad \text{……………… (1.1)}$$

The above integral convergent.

Saxena & Gupta transform of derivatives :-

i. $Z\{f'(t)\} = \frac{F(v)}{v} - \frac{f(0)}{v^2}$

ii. $Z\{f''(t)\} = \frac{F(v)}{v^2} - \frac{f(0)}{v^3} - \frac{f'(0)}{v^2}$

iii. $Z\{f^n(t)\} = \frac{F(v)}{v^n} - \frac{f(0)}{v^{n+1}} - \frac{f'(0)}{v^n} - \frac{f''(0)}{v^{n-1}}$

Table 1-- Saxena & Gupta transform of some elementary function

S.NO.	Function $f(t)$	New transform $Z[f(t)]$
1	1	$\frac{1}{v}$
2	t	1
3	t^2	$2v$
4	t^n	$u^{n-1}\Gamma(n+1)$
5	e^{at}	$\frac{1}{v\,(1-av)}$
6	$sinat$	$\frac{a}{1+a^2v^2}$
7	$cosat$	$\frac{1}{v\,(1+a^2v^2\,)}$
8	$sinhat$	$\frac{a}{1-a^2v^2}$
9	$coshat$	$\frac{1}{v\,(1-a^2v^2\,)}$

II.Application of Saxena and Gupta Transform of Certain system of Ordinary Differential Equations.

As specified in the introduction of this paper, the Saxena and Gupta transform can be used as an effective tool. For analysing the basic properties of a linear system governed by the differential equation in response to initial data. The following examples illustrate the use of the Saxena and Gupta transform in solving certain initial value problems described by system of ordinary differential equations[1],[2],[3]

Theorem (2.1)- Consider the system of differential equations

$$\frac{dx}{dt} + y = 2cost \quad \text{........................ (2.1.1)}$$

$$\frac{dy}{dt} - x = 1 \quad \text{........................ (2.1.2)}$$

with initial condition $x(0) = 1 \; and \; y(0) = 1$

Solution : To obtain the solution of system of ordinary differential equations first we Applying the Saxena & Gupta transform of both side of eq. (2.1.1) and (2.1.2)

$$Z(\frac{dx}{dt}) + Z(y) = 2Z(cost)$$

$$z\left(\frac{dy}{dt}\right) - z(x) = z(1)$$

since $z[x(t)] = G_1(u) and\; z[y(t)] = G_2(u)$

$$\frac{G_1(u)}{u} - \frac{x(0)}{u^2} + G_2(u) = \frac{2}{u(1+u^2)} \quad \text{..................(2.1.3)}$$

$$\frac{G_2(u)}{u} - \frac{y(0)}{u^2} - G_1(u) = \frac{1}{u} \quad \text{..................(2.1.4)}$$

Solving this equations for $G_1(u) and\; G_2(u)$;

$$G_1(u) = \frac{u^2}{u(1+u^2)} = \frac{1-1-u^2}{u(1+u^2)} = \frac{1}{u(1+u^2)} - \frac{1}{u} \quad \text{............... (2.1.5)}$$

$$G_2(u) = \frac{u+1}{u(1+u^2)} = \frac{1}{(1+u^2)} + \frac{1}{u(1+u^2)} \quad \text{............... (2.1.6)}$$

Applying inverse Saxena & Gupta transforms

$$z^{-1}(G_1(u)) = z^{-1}\left(\frac{1}{u(1+u^2)}\right) - z^{-1}\left(\frac{1}{u}\right)$$

$z^{-1}(G_2(u)) = z^{-1}\left(\frac{1}{(1+u^2)}\right)\frac{1}{(1+u^2)} + z^{-1}\left(\frac{1}{u(1+u^2)}\right)$

since $z^{-1}(G_1(u)) = x(t) and\ z^{-1}(G_2(u)) = y(t)$

Thus required solution of given differential equations are

$x(t) = tcost - 1$ ………….. (2.1.7)

$y(t) = tsint + cost$ …………… (2.1.8)

Theorem -(2.2)- Find the solution of the system of the differential equations

$\frac{dx}{dt} + \alpha y = 0$ …………… (2.2.1)

$\frac{dy}{dt} - \alpha x = 0$ ……....... (2.2.2)

with initial conditions$x(0) = c_1 and y(0) = c_2$, where $c_1 and c_2$ are arbitrary constants

Solution : To obtain the solution of system of ordinary differential equations first we applying the Saxena & Gupta transform of both sides of eq. (2.2.1) and (2.2.2), we get

$$Z(\frac{dx}{dt}) + \alpha Z(y) = 0$$

$$z\left(\frac{dy}{dt}\right) - \alpha z(x) = 0$$

since $z[x(t)] = G_1(u) and\ z[y(t)] = G_2(u)$

$\frac{G_1(u)}{u} - \frac{x(0)}{u^2} + \alpha G_2(u) = 0$ ………..(2.2.3)

$\frac{G_2(u)}{u} - \frac{y(0)}{u^2} - \alpha G_1(u) = 0$ … … … … … … . . (2.2.4)

Solving this equations for $G_1(u) and\ G_2(u)$;

$G_1(u) = \frac{c_1}{u(1+\alpha^2u^2)} - \frac{\alpha c_2}{(1+\alpha^2u^2)}$ ………………. (2.2.5)

$G_2(u) = \frac{\alpha c_1}{(1+\alpha^2u^2)} + \frac{c_2}{u(1+\alpha^2u^2)}$ ………………… (2.2.6)

Applying the inverse Saxena & Gupta transform both sides of the equation (2.2.5) and (2.2.6)

$$z^{-1}(G_1(u)) = z^{-1}\left(\frac{c_1}{u(1+\alpha^2u^2)}\right) - z^{-1}\left(\frac{\alpha c_2}{(1+\alpha^2u^2)}\right)$$

$$z^{-1}(G_2(u)) = z^{-1}\left(\frac{\alpha c_1}{(1+\alpha^2u^2)}\right) + z^{-1}\left(\frac{c_2}{u(1+\alpha^2u^2)}\right)$$

since $z^{-1}(G_1(u)) = x(t) and\ z^{-1}(G_2(u)) = y(t)$

thus required solution of given differential equations are

$x(t) = c_1 cos\alpha t - c_2 sin\alpha t$ ……………. (2.2.7)

$y(t) = c_1 sin\alpha t + c_2 cos\alpha t$ …………….. (2.2.8)

Theorem -(2.3)- Find the solution of the system of ordinary differential equations

$\frac{dx}{dt} - 2y = cos2t$ …………… (2.3.1)

$\frac{dy}{dt} + 2x = sin2t$ ……..........(2.3.2)

with initial conditions $x(0) = 1\ and\ y(0) = 0$

Solution : To obtain the solution of system of ordinary differential equations first we applying the Saxena & Gupta transform of both sides of eq. (2.3.1) and (2.3.2), we get

$$Z\left(\frac{dx}{dt}\right) - Z(2y) = Z(cos2t)$$

$$z\left(\frac{dy}{dt}\right) + z(2x) = z(sin2t)$$

since $z[x(t)] = G_1(u) and\ z[y(t)] = G_2(u)$

$\frac{G_1(u)}{u} - \frac{x(0)}{u^2} - 2G_2(u) = \frac{1}{u(1+4u^2)}$ …………….. (2.3.3)

$\frac{G_2(u)}{u} - \frac{y(0)}{u^2} + 2G_1(u) = \frac{2}{(1+4u^2)}$ ………………(2.3.4)

Solving this equations for $G_1(u) and\ G_2(u)$;

$G_1(u) = \frac{2}{2(1+4u^2)} + \frac{1}{u(1+4u^2)}$ ………………. (2.3.5)

$G_2(u) = \frac{-2}{(1+4u^2)}$ ……………….. (2.3.6) Applying the inverse Saxena & Gupta transform both sides of the equation (2.3.5) and (2.3.6)

$$z^{-1}(G_1(u)) = z^{-1}\left(\frac{2}{2(1+4u^2)}\right) + z^{-1}\left(\frac{1}{u(1+4u^2)}\right)$$

$$z^{-1}(G_2(u)) = z^{-1}\left(\frac{-2}{(1+4u^2)}\right)$$

since $z^{-1}(G_1(u)) = x(t) and\ z^{-1}(G_2(u)) = y(t)$

thus required solution of given differential equations are

$x(t) = sin2t + cos2t$ …… (2.3.7)

$y(t) = -sin2t$ …….. (2.3.8)

Theorem -(2.4)- Find the solution of the system of ordinary differential equations

$\frac{dx}{dt} + y = sint$ …………… (2.4.1)

$\frac{dy}{dt} + x = cost$ ……..........(2.4.2)

with initial conditions $x(0) = 0$ *and* $y(0) = 2$

Solution : To obtain the solution of system of ordinary differential equations first we applying the Saxena & Gupta transform of both sides of eq. (2.4.1) and (2.4.2)

$$Z\left(\frac{dx}{dt}\right) + Z(y) = Z(sint)$$

$$z\left(\frac{dy}{dt}\right) + z(x) = z(cost)$$

since $z[x(t)] = G_1(u)$*and* $z[y(t)] = G_2(u)$

$\frac{G_1(u)}{u} - \frac{x(0)}{u^2} + G_2(u) = \frac{1}{(1+u^2)}$ …….. (2.4.3)

$\frac{G_2(u)}{u} - \frac{y(0)}{u^2} + G_1(u) = \frac{1}{u(1+u^2)}$ ………. (2.4.4)

Solving this equations for $G_1(u)$*and* $G_2(u)$;

$G_1(u) = \frac{-2}{(1-u^2)}$ ………. (2.4.5)

$G_2(u) = \frac{1}{(1+u^2)} + \frac{2}{u(1-u^2)}$ …….. (2.4.6)

Applying the inverse Saxena & Gupta transform both side of the equation (2.4.5) and (2.4.6)

$$z^{-1}(G_1(u)) = z^{-1}\left(\frac{-2}{(1-u^2)}\right)$$

$$z^{-1}(G_2(u)) = z^{-1}\left(\frac{1}{(1+u^2)}\right) + z^{-1}\left(\frac{2}{u(1-u^2)}\right)$$

since $z^{-1}(G_1(u)) = x(t)$*and* $z^{-1}(G_2(u)) = y(t)$

thus required solution of given differential equations are

$x(t) = -2sinht$ …………… (2.4.7)

$y(t) = sint + 2cosht$ ……………. (2.4.8)

Theorem -(2.5)- Find the solution of the system of ordinal differential equations

$\frac{dx}{dt} = x + y$ …………… (2.5.1)

$\frac{dy}{dt} = 2x + 4y$ ..……...........(2.5.2)

with initial conditions $x(0) = 1$*and* $y(0) = 2$

Solution:- : To obtain the solution of system of ordinary differential equations first we applying the Saxena & Gupta transform of both side of eq. (2.5.1) and (2.5.2)

$$Z\left(\frac{dx}{dt}\right) = Z(x) + Z(y)$$

$$z\left(\frac{dy}{dt}\right) = 2z(x) + 4z(y)$$

since $z[x(t)] = G_1(u)$*and* $z[y(t)] = G_2(u)$

$\frac{G_1(u)}{u} - \frac{x(0)}{u^2} = G_1(u) + G_2(u)$ …….. (2.5.3)

$\frac{G_2(u)}{u} - \frac{y(0)}{u^2} = 2G_1(u) + 4G_2(u)$ …. (2.5.4)

Solving this equations for $G_1(u)$*and* $G_2(u)$ then applying inverse transforms we get the solution of given differential equations are

$x(t) = e^{2t} - 2e^{-t} - 2t + 1$ ……………… (2.5.5)

$x(t) = e^{2t} + 4e^{-t} + 2t - 3$ ………………. (2.5.6)

Theorem -(2.6)- Find the solution of the system of the equations

$\frac{dx}{dt} + y = e^t$ …………… (2.6.1)

$\frac{dy}{dt} - x = -t$ ……..........(2.6.2)

With initial conditions $x(0) = 0$*and* $y(0) = 0$

Solution:- : To obtain the solution of system of ordinary differential equations first we applying the Saxena & Gupta transform of both sides of eq. (2.6.1) and (2.6.2)

$$Z\left(\frac{dx}{dt}\right) + Z(y) = z(e^t)$$

$$z\left(\frac{dy}{dt}\right) - z(x) = z(-t)$$

since $z[x(t)] = G_1(u) and\ z[y(t)] = G_2(u)$

$$\frac{G_1(u)}{u} - \frac{x(0)}{u^2} + G_2(u) = \frac{1}{u(1-u)} \quad \text{….. (2.6.3)}$$

$$\frac{G_2(u)}{u} - \frac{y(0)}{u^2} - G_1(u) = -1 \quad \text{… (2.6.4)}$$

Solving this equations for $G_1(u) and\ G_2(u)$;

$$G_1(u) = \frac{1+u^2-u^3}{(1+u^2)} = 1 - u + \frac{u}{1+u^2} = 1 - \frac{2u}{2} + \frac{2u}{2(1+u^2)} \quad \text{………. (2.6.5)}$$

$$G_2(u) = \frac{-u^4}{1+u^2} = -u^2 + 1 + \frac{-1}{1+u^2} \quad \text{……….. (2.6.6)}$$

Applying the inverse Saxena & Gupta transform both side of the equation (2.6.5) and (2.6.6)

$$z^{-1}(G_1(u)) = z^{-1}\left(1 - \frac{2u}{2} + \frac{2u}{2(1+u^2)}\right)$$

$$= z^{-1}(1) - z^{-1}\left(\frac{2u}{2}\right) + z^{-1}\left(\frac{2u}{2(1+u^2)}\right)$$

$$z^{-1}(G_2(u)) = z^{-1}\left(-u^2 + 1 + \frac{-1}{1+u^2}\right)$$

$$= z^{-1}(-u^2) - z^{-1}(1) - z^{-1}\left(\frac{-1}{1+u^2}\right)$$

since $z^{-1}(G_1(u)) = x(t) and\ z^{-1}(G_2(u)) = y(t)$

thus required solution of given differential equations are

$$x(t) = t - \frac{t^2}{2} + \frac{tsint}{2} \quad \text{……………. (2.6.7)}$$

$$y(t) = \frac{-1}{4}t^4 + t - sint \quad \text{………… (2.6.8)}$$

Conclusion

This innovative technique demonstrates greater effectiveness and ease of use in handling ordinary differential equations compared to conventional methods. Also this method is very efficient, simple and engineering applications , with the potential to extend its utility to a wide array of problems across various domains. The main goal of this research is to solve certain system of ordinary differential equations.

Refrences

1. A. M. Takate and D.P.Pati, S. R. Kushare Vidyabharati ,Comparison Between Laplace, Elzaki And Mahgoub Transforms For Solving System Of First Order First Degree Differential Equations Vidyabharati International Interdisciplinary Research Journal (Special Issue) ISSN 2319-4979
2. D.P Patil, Aboodh and Mahgoub transform in boundary value problems of system of ordinary differential equation, International Journal of advanced Research in Science, Communication and technology, Vol.6, Issue 1(2021) pp.67-75
3. Dr. D. P. Patil, Aboodh and Mahgoub Transform in Boundary Value Problems of System of Ordinary Differential Equations, International Journal of Advanced Research in Science, Communication and Technology (IJARSCT) Volume 6, Issue 1, June 2021
4. G. K. Watugala, “Sumudu transform: a new integral transform to solve differential equations and control engineering problems,” *International Journal of Mathematical Education in Science and Technology*, vol. 24, no. 1, pp. 35–43, 1993.
5. Hassan Eltayeb and AdemKilicman, (2010), A Note on the Sumudu Transforms and differential Equations, Applied Mathematical Sciences, VOL, 4, no. 22, 1089-1098.
6. Hemlata saxena, Sakshi gupta, A new integral transform called “Saxena & Gupta Transform” and relation new transform and other integral transforms. GJSFR volume 23 Issue 4 version 1.0 year 2023
7. Ordinary Differential Equations Laplace Transforms And Numerical Methods For Engineers By Steven J. Desjardins And R´Emi Vaillancourt Notes for the course MAT 2384 3X Spring 2011 D´epartement de math´ematiques et de statistique Department of Mathematics and Statistics Universit´e d’Ottawa / University of Ottawa Ottawa, ON, Canada K1N 6N5
8. P. V. Pawani, U. L. Priya and B. A. Reddy: Solving Differential Equations by using Laplace Transform, International Journal of Research and Analytical Reviews, Vol. 5, Issue 3 , pp 1796-1799.
9. Sudhanshu Aggarwal, A Comparative Study of Mohand and Mahgoub Transforms, Journal of Advanced Research in Applied Mathematics and Statistics, Volume 4,Issue 1-2019,pg.no.1-7.Peer Reviewed Journal.