

A Semi-Analytic Method for Solving Two-Dimensional Fractional Dispersion Equation

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Abstract

In this paper, we use the analytical solution the fractional dispersion equation in two dimensions by using modified decomposition method. The fractional derivative is described in Caputo's sense. Comparing the numerical results of method with result of the exact solution we observed that the results correlate well.

Keywords: Modified decomposition method, Fractional derivative, Fractional dispersion equation.

1. Introduction

The fractional calculus is used in many fields of science and engineering [1, 2, 3]. The solution of differential equation containing fractional derivatives is much involved and its classic a analytic methods are mainly integral transforms, such as Laplace transform, Fourier transform, Mellin transform, etc.[1,2,4]

In recent years Adomian decomposition method is applied to solving fractional differential equations. This method efficiently works for initial value or boundary value problems, for linear or nonlinear, ordinary or partial differential equations, and even for stochastic systems [5] as well. By using this method Saha Ray et al [6, 7, 8, 11] solved linear differential equations containing fractional derivative of order 1/2 or 3/2, and nonlinear differential equation containing fractional derivative of order1/2.

In this paper, we consider the two-dimensional fractional dispersion equation of the form [12]:

$$D_t u(x, y, t) = a(x, y) D_x^{\beta} u(x, y, t) + b(x, y) D_y^{\gamma} u(x, y, t) + s(x, y, t) \quad (1)$$

on a finite rectangular domain $x_L < x < x_H$ and $y_L < y < y_H$, with fractional orders $1 < \beta < 2$ and $1 < \gamma < 2$, where the diffusion coefficients $a(x, y) > 0$ and $b(x, y) > 0$. The 'forcing' function $s(x, y, t)$ can be used to represent sources and sinks. We will assume that this fractional dispersion equation has a unique and sufficiently smooth solution under the following initial and boundary conditions. Assume the initial condition $u(x, y, 0) = f(x, y)$ for $x_L < x < x_H$ and $y_L < y < y_H$, and Dirichlet boundary condition $u(x, y, t) = k(x, y, t)$ on the boundary (perimeter) of the rectangular region $x_L < x < x_H$ and $y_L < y < y_H$, with the additional restriction that $K(x_L, y, t) = K(x, y_L, t) = 0$. Eq.(1) also uses Caputo fractional derivatives of order β and γ.

In the present work, we apply the modified decomposition method for solving eq.(1) and compare the results with exact solution. The paper is organized as follows. In section 2, mathematical aspects. In section 3, basic idea of modified decomposition method. In section 4 the two dimensional fractional dispersion equation and its solution by modified decomposition method. In section 5 numerical example is solved using the modified decomposition method. Finally, we present conclusion about solution the two dimensional fractional dispersion equations in section 6.

2. Mathematical Aspects

The mathematical definition of fractional calculus has been the subject of several different approaches [13, 14]. The Caputo fractional derivative operator D^{α} of order α is defined in the following form:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \qquad \alpha > 0,$$

where $m-1<\alpha<m, m\in N, x>0$.
Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation:

$$D^{\alpha}(\lambda f(x)+\mu g(x))=\lambda D^{\alpha}f(x)+\mu D^{\alpha}g(x),$$

where λ and μ are constants.
For the Caputo's derivative we have:

$$D_{L+}^{\alpha}(x-L)^{n}=\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}(x-L)^{n-\alpha}$$

and

$$D_{R-}^{\alpha}(R-x)^{n}=\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}(R-x)^{n-\alpha}$$

3. Modified Decomposition Method

Consider a general nonlinear equation, [15]

$$Lu+R(u)+F(u)=g(t) \qquad (2)$$

where L is the operator of the highest-ordered derivatives with respect to t and R is the remainder of the linear operator. The nonlinear term is represented by F(u). Thus we get

$$Lu=g(t)-R(u)-F(u) \qquad (3)$$

The inverse L^{-1} is assumed an integral operator given by

$$L^{-1}=\int_{0}^{t}(\cdot)dt\ ,$$

The operating with the operator L^{-1} on both sides of Equation (3) we have

$$u=f+L^{-1}(g(t)-R(u)-F(u)) \qquad (4)$$

where f is the solution of homogeneous equation

$$Lu=0 \qquad (5)$$

involving the constants of integration. The integration constants involved in the solution of homogeneous Equation (5) are to be determined by the initial or boundary condition according as the problem is initial-value problem or boundary-value problem.

The ADM assumes that the unknown function $u(x,t)$ can be expressed by an infinite series of the form

$$u(x,t)=\sum_{n=0}^{\infty}u_{n}(x,t)$$

and the nonlinear operator F(u) can be decomposed by an infinite series of polynomials given by

$$F(u)=\sum_{n=0}^{\infty}A_{n}$$

where $u_n(x,t)$ will be determined recurrently, and A_n are the so-called polynomials of $u_0, u_1, \ldots, u_n$ defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{n=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0,1,2,\ldots,$$

But the modified decomposition method was introduced by Wazwaz [16]. This method is based on the assumption that the function $f(x)$ can be divided into two parts, namely $f_1(x)$ and $f_2(x)$. Under this assumption we set

$$f(x) = f_1(x) + f_2(x)$$

We apply this decomposition when the function f consists of several parts and can be decomposed into two different parts. In this case, f is usually a summation of a polynomial and trigonometric or transcendental functions. A proper choice for the part f_1 is important. For the method to be more efficient, we select f_1 as one term of f or at least a number of terms if possible and f_2 consists of the remaining terms of f.

4. Solution Two-Dimensional Fractional Dispersion Equation by Modified Adomian's Decomposition Method

We adopt modified decomposition method for solving Equation (1). In the light of this method we assume that

$$u = \sum_{n=0}^{\infty} u_n$$

Now, Equation (1) can be rewritten as

$$Lu(x, y, t) = a(x, y)D_x^{\beta} u(x, y, t) + b(x, y)D_y^{\gamma} u(x, y, t) + s(x, y, t)$$

where $L_t = D_t$ which is an easily invertible linear operator, D_x^{β} and D_y^{γ} are the caputo derivative of order β, γ.

Therefore, we can write,

$$u(x, y, t) = u(x, y, 0) + L_t^{-1}\left(a(x, y)D_x^{\beta}\left(\sum_{n=0}^{\infty} u_n\right)\right) + L_t^{-1}\left(b(x, y)D_y^{\gamma}\left(\sum_{n=0}^{\infty} u_n\right)\right) + L^{-1}(s(x, y, t)) \quad (6)$$

In [9], he assumed that if the zeroth component $u_0 = f$ and the function f is possible to divide into two parts such as f_1 and f_2, then one can formulate the recursive algorithm for u_0 and general term u_{n+1} in a form of the modified decomposition method recursive scheme as follows:

$$u_0 = f_1$$

$$u_1 = f_2 + L_t^{-1}\left(a(x, y)D_x^{\beta} u_0\right) + L_t^{-1}\left(b(x, y)D_y^{\gamma} u_0\right)$$

$$u_{n+1} = L_t^{-1}\left(a(x, y)D_x^{\beta}\left(\sum_{n=0}^{\infty} u_n\right)\right) + L_t^{-1}\left(b(x, y)D_y^{\gamma}\left(\sum_{n=0}^{\infty} u_n\right)\right), n \geq 0$$

5. Application:

Consider the problem (1) with the following boundary and initial conditions:

$$D_t u(x,y,t) = a(x,y) D_x^{1.8} u(x,y,t) + b(x,y) D_y^{1.6} u(x,y,t) + s(x,y,t)$$

with the coefficient function: $a(x,y) = \Gamma(2.2) x^{2.8} y/6$, and $b(x,y) = 2x^{2.6}\, y/\Gamma(4.6)$, and the source function: $s(x,y,t) = -(1+2xy)e^{-t}x^3y^{3.6}$, subject to the initial condition $u(x,y,0) = x^3y^{3.6}$, $0 < x < 1$, and Dirichlet boundary conditions $u(x,0,t) = u(0,y,t) = 0$, $u(x,1,t) = e^{-t}x^3$, and $u(1,y,t) = e^{-t}y^{3.6}$, $t \geq 0$. Note that the exact solution to this problem is: $u(x,y,t) = e^{-t}\, x^3\, y^{3.6}$.

Table 1 shows the analytical solutions for fractional dispersion equation obtained for different values and comparison between exact solution and analytical solution.

Table 1. Comparison between exact solution and analytical solution when $\beta = 1.8, \gamma = 1.6$ for fractional dispersion equation

x = y	t	Exact Solution	Modified Adomian Decomposition Method	$\lvert u_{ex} - u_{MADM} \rvert$
0	1	0.00000000000	0.00000000000	0.00000000
0.1	1	0.00000009241	0.00000009241	0.00000000
0.2	1	0.00000089640	0.00000089640	0.00000000
0.3	1	0.00013020000	0.00013020000	0.00000000
0.4	1	0.00086960000	0.00086960000	0.00000000
0.5	1	0.00379200000	0.00379200000	0.00000000
0.6	1	0.01300000000	0.01300000000	0.00000000
0.7	1	0.03500000000	0.03500000000	0.00000000
0.8	1	0.08400000000	0.08400000000	0.00000000
0.9	1	0.18400000000	0.18400000000	0.00000000
1	1	0.36800000000	0.36800000000	0.00000000
0	2	0.00000000000	0.00000000000	0.00000000
0.1	2	0.00000003399	0.00000003399	0.00000000
0.2	2	0.00000032980	0.00000032980	0.00000000
0.3	2	0.00004791000	0.00004791000	0.00000000
0.4	2	0.00031990000	0.00031990000	0.00000000
0.5	2	0.00139500000	0.00139500000	0.00000000
0.6	2	0.00464700000	0.00464700000	0.00000000
0.7	2	0.01300000000	0.01300000000	0.00000000
0.8	2	0.03100000000	0.03100000000	0.00000000
0.9	2	0.06800000000	0.06800000000	0.00000000
1	2	0.13500000000	0.13500000000	0.00000000
0	3	0.00000000000	0.00000000000	0.00000000
0.1	3	0.00000001251	0.00000001251	0.00000000
0.2	3	0.00000121300	0.00000121300	0.00000000
0.3	3	0.00000176200	0.00000176200	0.00000000
0.4	3	0.00011770000	0.00011770000	0.00000000
0.5	3	0.00051320000	0.00051320000	0.00000000
0.6	3	0.00171000000	0.00171000000	0.00000000
0.7	3	0.00472900000	0.00472900000	0.00000000
0.8	3	0.01100000000	0.01100000000	0.00000000
0.9	3	0.02500000000	0.02500000000	0.00000000
1	3	0.05000000000	0.05000000000	0.00000000

6. Conclusion

in this paper modified decomposition method is proposed for solving fractional dispersion equation with boundary conditions and initial condition. The results obtained show that the modified decomposition methods provide us an exact solution to the above problems. It is observed that the computation of the solution and its components are easy and take less time as compare to the traditional techniques.

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