# Problem of Integration with Respect to Unbounded Measures on the Set of Projections

Bassi, I.G. , Markus Samaila, Okechukwu C. E.
Department of Maths/Stat, University of Maiduguri- Nigeria
E- Mail of corresponding Author bassiibrahim@yahoo.com

**Abstract**

We note if φ is a normal weight on M, then $\varphi|_{M^n}$ is a measure on projections and if a measure on projections can be extended to a normal weight, then the problem of constructing an integral with respect to this measure reduces to the problem of constructing an integral with respect to the weight. We therefore present several methods of constructing noncommutative integration which gives a survey of the contemporary state of the theory in the von Neumann algebra (M) with respect to weightφ. For every $\alpha \in [0,1]$, the Banach space $L_{p,\alpha}(\varphi)$ is isometrically isomorphic to the space $Lp(\tau)$ and the space $L_{p,\alpha}(\varphi)$ is, by definition, the Banach space completion of $m_\alpha^{1/p}$ in the norm $\|.\|_{p,\alpha.}$ .We construct the scale of $L_p(\varphi)$ spaces $(1 \le p \le \infty)$ with respect to a faithful normal semifinite (f.n.s.) weight φ on a von Neumann algebra M. These spaces are realized by operators. This is achieved by extending the original algebra M, and the Hilbert space where M originally acted is altered, as well. In the construction of the scale, the concept of an operator-valued weight is used. We discuss the problem of integration with respect to measures on projections which remains open for unbounded measures (m(1) = +∞) and their structure has been studied only for the algebra $\mathcal{L}(\mathcal{H})$.

**Keywords**: Von Neumann algebra, Faithful normal semifinite trace (f.n.s.)τ, weight, isometrically isomorphic, projections, Banach spaces and $L_p$-spaces

## 1. Introduction

We discuss the problem of integration with respect to measures on projections. By a measure on the projections of a von Neumann algebra M we mean a mapping $m : M^n \to [0, +\infty]$ satisfying the requirements

$$p = \sum_{i \in I} p_i (p, p_i \in M^n, p_i p_j = 0\ (i \ne j)) \Rightarrow m(p) = \sum_{i \in I} m(p_i).$$

In particular, if φ is a normal weight on M, then $\varphi|_{M^n}$ is a measure on projections. Furthermore, by the spectral theorem, a normal weight is uniquely defined by its restriction $\varphi|_{M^n}$ . Conversely, if a measure on projections can be extended to a normal weight, then the problem of constructing an integral with respect to this measure reduces to the problem of constructing an integral with respect to the weight. We present several methods of constructing non-commutative integration which gives a survey of the contemporary state of the theory in the von Neumann algebra with respect to weight φ as follows:

## 2. Integration In Von Neumann Algebra With Respect To A Weight

We give several different approaches of construction to non - commutative integration with respect to a f.n.s. weight in a von Neumann algebra. They all arise as various generalizations of Segal's theory of integration with respect to a trace, which we therefore take as a natural starting point of our exposition.

Let τ be a f,n.s. weight on a von Neumann algebra M acting in a Hilbert space $\mathcal{H}$. The function $x \to \tau(|x|)$ $(x \in m_\tau)$ is a norm on the complex vector space $m_\tau$ (here $|x| = (x^*x)^{1/2}$). We denote by $L_1(\tau)$ the completion of $m_\tau$ in this norm. More precisely, $L_1(\tau)$ consists of those closed densely defined operators h affiliated with M for which $\tau(|h|) < +\infty$.

I.Segal characterized $L_1(\tau)$ as a Banach space depending only on the von Neumann algebra $M_*$. The corresponding isomorphism is defined by the formula $h \in L_1(\tau)^+ \to \tau(h\bullet) \in M_\bullet$. A Banach equivalent of $L_1(\tau)$ is the space $M_*$, which is independent of not only the trace τ, but also the Hilbert space $\mathcal{H}$ where M acts. In the following this remarkable property can be considered as the very criterion of the correctness of meaningful constructions of an $L_1$ space with respect to a weight.

We construct an $L_1$-space with respect to a weight in which the integrable elements are obtained as limits of elements of the von Neumann algebra.

 

As an analog of the $L_1$-norm we consider, on the real vector space $m_\varphi^+$, the convex functional

$$\|x\|_\varphi = \inf\{\varphi(x_1 \mid x_2) \mid x = x_1 - x_2 (x_1, x_2 \in m_\varphi^+)\}. \qquad (1)$$

In particular, $\|x\|_\varphi = \varphi(x)$ for $x \in m_\varphi^+$. We note that if φ is a trace, $\|x\|_\varphi = \varphi(|x|)$.

The lineal of a normal semifinite weight φ on a von Neumann algebraM acting in a Hilbert space $\mathcal{H}$ is, by definition, the lineal

$$D_\varphi = \{f \in H \mid \ni \lambda > 0 \forall x \in M^+ \quad ((xf, f) \le \lambda\varphi(x))\}$$

If now $(x_n)$ is a $\|.\|_\varphi$ – fundamental sequence in $m_\varphi^h$, for any vectors f, g∈ $D_\varphi$, the numerical sequence $(x_n f,g)$ is convergent. Therefore, on $D_\varphi$ an Hermitian bilinear form (b.f.) $a_{\{x_n\}}(f,g) \equiv \lim_n (x_n f, g) \quad (f, g \in D_\varphi)$ is defined. This motivates the following definition:

*2.1 Definition:* Let φ be a f.n.s. weight on a von Neumann algebra M acting in a Hilbert space $\mathcal{H}$. An Hermitian b.f. *a* defined on the lineal $D_\varphi$ is said to be integrable with respect to φ if a sequence $(x_n) \subset m_\varphi^+$, called a defining sequence, exists such that

*(a)* $a(f,g) = \lim_n (x_n f, g) \quad (f, g \in D_\varphi)$,

*(b)* $\|x_n - x_m\|_\varphi \to 0 \; (n, m \to \infty)$.

*2.2 Theorem:* Let φ be a f.n.s. weight on a von Neumann algebra M acting in a Hilbert space $\mathcal{H}$ and let $(x_n)$ be a defining sequence for the zero b.f. Then $\lim_n \|x_n\|_\varphi = 0$.

This theorem implies immediately that the formula $\|a\|_\varphi = \lim_n \|x_n\|_\varphi$, where a is an integrable b.f. and $x_n$ is a defining sequence of it, defines a unique extension of the norm $\|.\|_\varphi$ from $m_\varphi^h$ to the real vector space of integrable Hermitian b.f.'s and the latter is complete in this norm. Hence the space $L_1(\varphi)^h$ can be identified with the space of integrable Hermitian b.f.'s. Moreover, for an integrable Hermitian b.f. a = $a_{\{xn\}}$ the value $\varphi(a) = \lim_n \varphi(x_n)$ is unambiguously defined, which can naturally be called the integral of *a* with respect to φ.

The concept of an integrable b.f. can be extended from the Hermitian to the general case. We say that a b.f. *a* defined on $D_\varphi$ is Integrable with respect to φ if its Hermitian and skew-Hermitian parts $a_1$, $a_2$ belong to $L_1(\varphi)^h$. Then the class $L_1(\varphi)$ of integrable b.f.'s is a complex vector space and the value $\varphi(a) = \varphi(a_1) + i\varphi(a_2)$ $(a \in L_1(\varphi))$ is called the integral of *a* with respect to φ. Now we define a norm in $L_1(\varphi)$ so that it will coincide with the norm already defined on $L_1(\varphi)^h$. The following result shows that the natural requirement that $L_1(\varphi)$ be isomorphic to $M_*$ determines this norm essentially uniquely.

*2.3 Theorem:* Let φ be a f.n.s. weight on a von Neumann algebra M. There is a unique mapping $\gamma : m_\varphi \to M_*$ satisfying the conditions:

*(i)* $\gamma(x)(1) = \varphi(x) \quad x \in m_\varphi)$,

*(ii)* $\gamma(m_\varphi^+) = \{w \in M_*^+ \mid \ni \lambda > 0 \; (w \le \lambda\varphi)\}$,

*(iii)* The mapping $\{x, y\} \to \gamma(x)(y^*)$ is a nonsingular positive b.f. on $m_\varphi$.

It turns out that then $\|x\|_\varphi = \|\gamma(x)\| \quad (x \in m_\varphi^h)$ and the lineal $\gamma(m_\varphi)$ is dense in $M_*$.

 

Now we define a norm in $L_1$ (φ) by the equality $\|a\|_\varphi = \lim\|\gamma(x_n + iy_n)\|$ where $(x_n)$ , $(y_n)$ are the defining sequences of the Hermitian and skew-Hermltian parts of *a*. Then the mapping $\gamma: m_\varphi \to M_\bullet$ can he extended by continuity to an isometric isomorphism (which will be denoted by the same letter in the following) of the Banach spaces $L_1(\varphi)$ and $M_*$, mapping the cone $L_l(\varphi)^+$ of integrable positive b.f.'s onto the cone $M_*^+$ and having the property $\varphi(a) = \gamma(a)(1)$ $(a \in L_1(\varphi))$.

As has already been mentioned, the space $L_1(\tau)$ associated with a f.n.s. trace τ, can be realized by operators. It is interesting to clarify when b.f.'s integrable with respect to a weight reduce to operators.
Let φ be a f.n.s, weight on a von Neumann algebra M and let h⩾0 be a self-adjoint operator. If $D_\varphi \subset D(h^{1/2})$ then the b.f. $eoh(f,g) \equiv (h^{1/2}f, h^{1/2}g)$ $(f, g \in D_\varphi)$ is unambiguously defined. We shall say that a positive b.f. *a* reduces to an operator if a self-adjoint operator h⩾0 affiliated with M exists such that *a=eoh.*

The next important step in constructing a non-commutative integration theory is the
Construction of the scale of spaces *Lp* $(1 \le p \le \infty)$.
Let M be a semifinite von Neumann algebra, τ a f.n.s. trace on M and φ=τ(h.) a f.n.s. locally measurable weight on M. Let $L_p(\tau)$ be the Banach space of measurable operators, p-th power integrable with respect to τ with the norm

$$\|s\|_p^\tau = [\tau(|s|^p)]^{1/p}, \quad s \in L_p(\tau), \quad 1 \le p < +\infty.$$

For $p = +\infty$, as usual, we set $L_\infty(\tau) = M$, $\|x\|_\infty = \|x\|$ $(x \in M)$. For every $p \ge 1$ and $\alpha \in [0,1]$. We put

$$m_\alpha^{1/p} = \{x \in M \mid h^{\alpha/p}.x.h^{(1-\alpha)/p} \in L_p(\tau)\}$$

$$\|x\|_{p,\alpha} \equiv \|h^{\alpha/p}.x.h^{(1-\alpha)/p}\|_p^\tau \quad (x \in m_\alpha^{1/p})$$

(here the product of operators is meant in the strong sense). These definitions are unambiguous (independent of the choice of τ). The space $L_{p,\alpha}(\varphi)$ is, by definition, the Banach space completion of $m_\alpha^{1/p}$ in the norm $\|.\|_{p,\alpha}$.
*2.4 Theorem:* Let p > 1 and let the f.n.s. weight φ be locally measurable. For every $\alpha \in [0,1]$, the Banach space $L_{p,\alpha}(\varphi)$ is isometrically isomorphic to the space *Lp(τ).*
Under the additional assumption of regularity of φ (i.e., local measurability of $h^{-1}$),
these spaces admit realizations by locally measurable operators. Namely, let $\mathcal{L}$ be the *-algebra of locally measurable operators with respect to M. Then the lineal $\{s \in \mathcal{L} \mid h^{\alpha/p}.s.h^{(1-\alpha)/p} \in Lp(\tau)\}$ in $\mathcal{L}$ equipped with the norm $\|s\|_{p,\alpha} \equiv [|\,\tau(h^{\alpha/p}.s.h^{(1-\alpha)/p}\,|^p)]^{1/p}$ is a Banach space isomorphic to $L_{p,\alpha}(\varphi)$.

In this section we construct the scale of $L_p(\varphi)$ spaces $(1 \le p \le \infty)$ with respect to a f.n.s. weight φ on a von Neumann algebra M. These spaces are realized by operators. This is achieved by extending the original algebra M, and the Hilbert space where M originally acted is altered, as well. In the construction of the scale, the concept of an operator-valued weight is used.
The extended positive part of the von Neumann algebra M (denoted by $\hat{M}^+$) is, by definition, the set of lower semi continuous additive and positively homogeneous mappings $m: M_\bullet^+ \to [0.+\infty]$. The cone $M^+$ can be considered as part of $\hat{M}^+$ if we identify $x \in M^+$ with $m_x$ , where $m_x(\eta) = \eta(x)$ $(\eta \in M_\bullet^+)$. For $a \in M$, $m \in \hat{M}^+$ we define
$a^*ma \in \hat{M}^+$ by putting $a^*ma(\eta) \equiv m(a\eta a^*)$, $\eta \in M_\bullet^+$, where $a\eta a^*(.) \equiv \eta(a^*(.)a)$.
Every normal weight ψ on M possesses a unique extension (also denoted
by ψ) to $\hat{M}^+$, such that

$$\psi(\lambda m) = \lambda\psi(m), \quad \psi(m+n) = \psi(m) + \psi(n) \quad (\lambda \ge 0; m, n \in \hat{M}^+),$$

 

$$m_i \uparrow m \Rightarrow \psi(m_i) \uparrow \psi(m),$$

Where $m_i \uparrow m$ means that $w(m_i) \uparrow w(m)$ for any $w \in M_{\bullet}^{+}$.

Now let N be a von Neumann subalgebra of M. An operator-valued weight from M into N, by definition, is an additive positively homogeneous mapping $T: M^{+} \to N^{+}$, such that

$T(a^{*}xa) = a^{*}T(x)a \;\; (x \in M, a \in N)$. The operator-valued weight T is said to be

-normal if $x_i \uparrow x \;\; (x_i, x \in M^{+}) \Rightarrow T(x_i) \uparrow T(x)$,

- f a i t h f u l if $T(x^{*}x)=0 \Rightarrow x=0$,

- semifinite if $n_T = \{x \in M \mid \|T(x^{*}x)\| < \infty\}$ is ultraweakly dense in M.

Let $\varphi$ be a f.n.s. weight on a von Neumann algebra M and let $\Sigma$ be the group of modular automorphisms of M. With this group is associated a one-parameter group $(U(t))_{t\in R}$ of unitary operators, such that $\sigma_i^{*}(x) = U(t)xU(t)^{*} \;\; (x \in M)$. Let $R(M,\Sigma)$ be the crossed product of M and $\Sigma$. $R(M,\Sigma)$ is automatically a semifinite von Neumann algebra and M can be considered as a subalgebra of it [more precisely, M is isomorphic to a subalgebra of $R(M,\Sigma)$ ]. With these agreements, a one-parameter group $(\theta_s)_{s\in R}$ is defined in $R(M,\Sigma)$ (dual action), which is characterized by the equalities (for all $s\in R$):

$$\theta_s(x) = x \;\; (x \in M), \;\; \theta_s(U(t)) = e^{-ist}U(t) \;\; t \in R).$$

In particular, M is the largest subalgebra of $R(M,\Sigma)$ invariant under $\theta$. The equality

$$Tx \equiv \int_{-\infty}^{+\infty} \theta_s(x)ds, \;\; x \in R(M,\Sigma)^{+},$$

defines a f.n.s. operator-valued weight from $R(M,\Sigma)$ into M. Moreover, $R(M,\Sigma)$ possesses a unique f.n.s. weight $\tau$ such that $\varphi \circ T = \tau(h.)$ where $h \geq 0$ is a self-adjoint operator affiliated with $R(M,\Sigma)$, such that $h^{it}$ = $U(t)$. The weight $\tau$ satisfies the equality

$\tau \circ \theta_s = e^{-s}\tau \;\; (s \in R)$. The group $(\theta_s)_{s\in R}$ can be naturally extended to an automorphism group of $\widehat{R(M,\Sigma)}^{+}$.

For every normal semifinite weight $\psi$ we put $\tilde{\psi} = \psi \circ T$. Let $h_\varphi$ be the Radon--Nikodym derivative of $\tilde{\psi}$ with respect to $\tau$, i.e., $\tilde{\psi} = \tau(h_\varphi .)$, where $h_\varphi \geq 0$ is a self-adjoint operator affiliated with $R(M,\Sigma)$ and $\theta_s(h) = e^{-s}h \;\; (s \in R)$. In particular, $h_\varphi$ is $\tau$-measurable if and only if $\psi$ is bounded.

Spaces $L_p(\varphi)$ are now defined in the following manner. The space $L_p(\varphi) \;\; (1 \leq p < +\infty)$ consists of all $\tau$-measurable operators affiliated with $R(M,\Sigma)$ such that $\theta_s(h) = e^{-s/p}(s \in R)$. The space $L_\infty(\varphi)$ consists of all $\tau$-measurable operators affiliated with $R(M,\Sigma)$ such that $\theta_s(h) = h \;\; (s \in R)$ (By what was said above, $L_\infty(\varphi)$ = M.). It follows from the above construction that

(a) $L_p(\varphi)L_q(\varphi) = \{0\}$ if $p \neq q$,

(b) for $p<\infty$, all nonzero operators from $L_p(\varphi)$ are unbounded.

In particular $L_1(\varphi)$ consists of all $\tau$-measurable operators affiliated with $R(M,\Sigma)$ such that $\theta_s(h) = e^{-s}h$ and $L_1(\varphi)^{+} = \{h_\psi \mid \psi \in M_{\bullet}^{+}\}$. On the space $L_1(\varphi)$ the linear functional Tr is unambiguously defined by the equality $Tr(h_\psi) = \psi(1)$. This implies the compatibility of Haagerup`s construction.

*2.5 Theorem:* The space $L_1(\varphi)$ equipped with the norm $\|h\|_1 = Tr(|h|)$ is a Banach space isometrically isomorphic to M.

The norms $\|h\|_p \equiv [Tr(|h|^p)]^{1/p} \; (1 \leq p < \infty)$, $\|h\|_\infty \equiv \|h\|$ enable us to speak of the scale of the

Banach spaces $\bar{L}_p(\varphi)$. The Holder inequality

 

$$\|hk\|_1 \leq \|h\|_p \|k\|_q \quad (\frac{1}{p}+\frac{1}{q}=1, h\in L_p(\varphi), k\in L_q(\varphi), p,q\in[1,+\infty))$$

is satisfied and the following analog of classical duality holds: if $p\in[1,+\infty)$ and $q=\frac{p}{p-1}$ then every operator $h\in L_q(\varphi)$ defines a functional

$$\langle ., h\rangle \in L_p(\varphi)^* : \langle x, h\rangle \equiv Tr(hx) \quad (x\in L_p(\varphi)).$$

Where the mapping $h\rightarrow\langle ., h\rangle$ is an isometric isomorphism of $L_q(\varphi)$ onto $L_p(\varphi)^*$.

Another construction of the spaces $L_p(\varphi)$ are realized by closed operators in the Hilbert space where M acts. However, these operators are not affiliated with M in general.
Let $\varphi$ be a f.n.s. weight on a von Neumann algebra M acting in a Hiibert space $\mathcal{H}$ and let $\Gamma$ be the completion of $n_\varphi$, where $n_\varphi \equiv \{x\in M \mid \varphi(xx^*)<+\infty\}$ is a left ideal in M. To every vector f from the lineal $D_\varphi$ we assign an operator R(f): $\Gamma\rightarrow\mathcal{H}$ defined by the formula $R(f)\equiv xf \ (x\in n_\varphi)$. This operator is bounded and, by the same token, unambigously defined on Γ. For every normal semifinite weight ψ on M', we can define, by the same token, a positive b.f. $a_{\psi}$,:

$$D(a_{\psi},) = \{f\in D_\psi \mid \psi(R(f)R(f)^*)<+\infty\},$$
$$a_\psi(f,g) = (R(f)R(g)^*) \quad (f, g\in D(a_\psi)).$$

This b.f. turns out to be closable, and consequently, there is a self-adjoint operator $\frac{d\psi}{d\varphi}\geq 0$ such that $a_\psi(f, f)$ $=\left\|\left(\frac{d\psi}{d\varphi}\right)^{1/2} f\right\|^2$ $(f\in D(a_\psi))$.

The operator $\frac{d\psi}{d\varphi}$ is called the "spatial" Radon - Nikodym derivative of ψ with respect to φ. The space $L_p(\varphi)$ is now defined as the set of closed operators h acting in $\mathcal{H}$, such that $|h|^p=\frac{d\psi}{d\varphi}$ for some $\psi\in M_*^+$.

## 3. Problem of Integration

In the general case it is natural to first answer the question of whether a measure on the projections of von Neumann algebra can be extended to a weight. This is the well-known problem of linearity. A positive solution for unitarily invariant measures on projections had already been obtained by Murray and von Neumann. (long before the general problem of linearity was posed). A program of constructing an integral with respect to such measures was realized by Segal (the result mentioned above was in fact obtained by Segal using the method of extending a measure on projections to an integral). The next major step is Gleason's famous theorem describing all finite measures on the projections of a factor of type In ($3\leq n\leq\infty$) in a separable Hilbert space $\mathcal{H}$. Namely, for every measure m defined on $\mathcal{L}(\mathcal{H})^n$($m(1)<+\infty$) a unique kernel operator $k\geqslant 0$ acting in $\mathcal{H}$ exists such that $m(p)=Tr(kp)$ $(p\in\mathcal{L}^n)$, then the equality $\varphi(x)=Tr(kx)$ $(x\in\mathcal{L})$ defines an extension of m to a normal functional φ on $\mathcal{L}$.
After the efforts of several mathematicians, the problem of linearity has, at the present
time, an exhaustive solution. Namely, if the von Neumann algebra M does not contain direct summands of type $I_2$ then every finite measure on $M^n$ can be extended to a normal functional.

Therefore, the problem of integration with respect to measures on projections remains open for unbounded measures ($m(1)=+\infty$) . These measures are much more complicated than normal weights and their structure has been studied only for the algebra $\mathcal{L}(\mathcal{H})$.

Here we give a structure theorem for unbounded measures in a somewhat more general form than that in Segal. A measure m on the projections of a von Neumann algebra M is said to be semifinite if a family $(p_i)_{i\in I}$ of mutually orthogonal projections in M exists such that $1=\sum_{i\in I} p_i$ and $m(p_i)<+\infty\ (i\in I)$. In particular, if I = N, then the measure is said to be σ-finite.

Theorem: Let $\mathcal{H}$ be a Hilbert space (dim$\mathcal{H}\geqslant 3$) and m a semifinite measure on $\mathcal{L}(\mathcal{H})^n$. Then a unique densely defined positive b.f. t exists such that

$$m(p)=\begin{cases}\mathrm{Tr}(t\circ p), \text{if } t\circ p\in \mathfrak{S}_I,\\ +\infty \text{ otherwise.}\end{cases} \qquad (*)$$

[Here $t\circ p\in \mathfrak{S}_I$ means that (1) $p\mathcal{H}$ is contained in the lineal D(t) – the domain of the b.f. t, (2) the b.f. $t\circ p(f,f)\equiv t(pf,pf)\ (f\in\mathcal{H})$ is defined by some kernel operator which is identified with this form.]

In particular, if the b.f. t is closed, then the measure can be extended to a normal weight on $\mathcal{L}(\mathcal{H})$. However, the class of semifinite measures is significantly larger than that of normal semifinite weights on $\mathcal{L}(\mathcal{H})$. We would like to note that it is not at all true that all positive b.f.'s t define semifinite measures by means of equality (*). A condition characterizing the b.f.'s defining σ-finite measures has been obtained by G.D.Lugovaya. The same condition characterizes the b.f.'s defining semifinite measures. We note the remarkable peculiarity of the measures being studied. If m is a semifinite measure on $\mathcal{L}^n$, then among the normal weights whose restrictions to $\mathcal{L}^n$ are majorized by m, there is a largest normal (necessarily semifinite) weight φ, called the regular component of m. If $\varphi|\mathcal{L}^n \neq m$, then $\varphi(1)=+\infty$ and φ automatically cannot be the additive summand of m.

**4. Conclusion**

On the basis of the above results on the structure of unbounded measures it is possible to introduce the space $L_1$ for a large class of unbounded measures on the projections of $\mathcal{L}^n$ and realize its predual by an appropriate von Neumann algebra.We shall say that a semifinite measure m: $\mathcal{L}^n\to[0,+\infty]$ is absolutely faithful if the regular component weight φ of m is faithful. For the positive b.f. t associated with m, we denote by $K_t^+$ the cone of operators x of finite rank in the Hilbert space $\mathcal{H}$, such that the range $\mathfrak{R}(x)\subset D(t)$. Let $K_t$ ($K_t^r$) be the linear (real linear) hull of $K_t^+$. The results imply that m can be extended to a linear functional (also denoted by m) on $K_t$. On $K_t^r$ we define the convex functional [compare with (1)].

$$x\to\|x\|_m=\inf\{m(x_1)+m(x_2)\mid x=x_1-x_2, x_i\in K_t^+\}.$$

For an absolutely f a i t h f u l measure, $\|.\|_m$ is a norm. We denote by $L_1(m)^r$ the Banach space completion of $K_t^r$ in the norm $\|.\|_m$. For the formulation of the result on the realization of the space $L_1(m)^r$, we need the following analog of the GNS construction.

Let $n_m$ be a left ideal [in the algebra $\mathcal{L}(\mathcal{H})$] of operators x of finite rank, such that $x^*x\in K_t^+$. Then the equality

$$(x\mid y)\equiv m(y^*x)\ \ (x,y\in n_m)$$

defines a scalar product on $n_m$. Let $\mathfrak{N}$ be a Hilbert space completion of $n_m$ with respect to this scalar product. The mapping $\pi_m:\mathcal{L}(\mathcal{H})\to\mathcal{L}(\mathfrak{N})$ defined by the equality

$$\pi_m(a)x\equiv ax\ \ (a\in\mathcal{L}(\mathcal{H}),\ x\in n_m)$$

is a morphism of von Neumann algebras. Moreover $\pi_m$ is isometry.The equality

$$\beta(y^*x)(a')\equiv(a'x\mid y)\ \ (a'\in\pi_m(\mathcal{L}(\mathcal{H})'_*;\ x,y\in n_m)$$

 

uniquely defines a linear mapping $\beta : K_t \to \pi_m$ ($\mathcal{L}(\mathcal{H})$' such that $\|\beta(a)\| = \|a\|_m$ $(a \in K_t^r)$.

Theorem: Let m: $\mathcal{L}(\mathcal{H})^n \to [0, +\infty]$ be an absolutely faithful measure. Then the Banach space $L_1(m)r$ is isometric ally isomorphic to the space of $\pi_m(\mathcal{L}(\mathcal{H})_*^r$.

Finally, we define the Banach space $L_1(m)$ as the completion of $K_t$, with respect to norm $a \to \|\beta(a)\|$ $(a \in K_t)$. Then $L_1(m)$ is the complexification of the real Banach space $L_1(m)^r$ Coinciding norms and it is isometrically isomorphic to the space $\pi_m(\mathcal{L}(\mathcal{H})_*^{'}$.

## References

A. M. Gleason, (1957): "Measures on the closed subspaces of a Hilbert space," *J. Math. Mech.,* **6,** (6), 885-893

Bassi I.G, Y.A. Moh`d and Mshelia I.B. (2009): **"**On the theory of Non-commutative Integration in Von Neumann JBW algebras with respect to a weight" *International Journal of Mathematical science* **1** (1) 23 – 30.

Einstein A. (1905): **"**Functional integration and quantum physics" *Journal of Mathematical Physics* **5** 332 – 353

G.D.Lugovaya, (1983):"Unbounded Measures on the projections on a von Neumann algebra", *Thesis, kazan*.

G.D. Lugovaya and A.N. Sherstnev, (1980): "On Gleason's theorem for unbounded measures," *Izv. Vyssh. Uchebn. Zaved.,* Mat., **12,** 30-32 .

G.D. Lugovaya (1983): "Unbounded measures on the projections of von Neumann algebra" *Kazan*,.

G.D. Lugovaya and A.N. Sherstnev, (1981): "On the integration of bounded operators with respect to measures on ideals of projections," *Konstruktivn. Teoriya Funktsii I Funkts. Analiz.* **3,** 44-50.

Haagerup,U. (1975): "Normal weights on $w^*$- algebras", *Journal of Functional Analysis,* **19,** (3) 302 – 317.

HALMOS, P. R. (1950): Measure Theory**,** Van Nostrand, New York,

Lugovaya (1980): "Projection on "Unbounded Measure space",*Journal of Functional Analysis*,**1** 45- 53

Nelson E. (1974): "Notes on Non-commutative integration" *Journal of Functional Analysis* .**12** (14) 267 – 479.

Segal, I. E. (1953): "A non-commutative extension of abstract integration", *Annals of Mathematics.* **57** 401 - 457

Segal, I. E. (1974): **"**Notes on Non- commutative integration", *Journal of Functional Analysis* **15**, 103 – 116.