The Concept of Moments of Order Invariant Quantum LÉVY Processes.

Bassi, I.G. , Markus Samaila, Okechukwu C. E.
Department of Maths/Stat, University of Maiduguri - Nigeria
E- Mail of corresponding Author bassiibrahim@yahoo.com

Abstract

Important information about stationary flows $(B_I)_{I\in I}$ is contained in moments of the corresponding Lévy process. We consider the case where we have an order invariance of the moments of the increments, in the sense that such moments do not change if we shift the increments against each other, as long as we do not change the relative position of the intervals. We then show that for large classes of quantum Lévy processes one can make precise statements about the time behaviour of their moments.

Keywords: Von Neumann algebras, Stationary flows and quantum Lévy processes, order invariant, Trace and moments.

1. Introduction

Lévy processes, i.e. processes with stationary and independent processes, or `white noises` as models for their derivatives, form one of the most important classes of classical stochastic processes, and the understanding of their structure was instrumental for many developments in classical probability theory. It is to be expected that the understanding of non- commutative versions of Lévy processes will be an important step towards a deeper theory of non – commutative stochastic processes.

Important information about stationary flows $(B_I)_{I\in I}$ is contained in moments of the corresponding Lévy process. In this we will address some of the canonical basic questions of this theory; namely how we can distinguish between different non – commutative white noises; and what can be said about the time behaviour of their moments. Even though a general answer to these problems for the class of all non – commutative Lévy process seems to be out of reach but we are able to provide answers to these questions for some quite large classes of non – commutative white noises. We want to generalize the notion of a classical process with stationary and independent increments to a non – commutative setting. In the classical setting, it is not only the process itself which is of importance, but sometimes one is more interested in the structure of the associated filtration of σ - algebras of the increments. In the same way, we find it advantageous in the non – commutative case to distinguish between the filtration generated by the process, and the process itself. In the non-commutative setting, the filtration is given by the von Neumann algebras generated by (or, in the case of unbounded operators, affiliated to) the increments of the process. We will restrict to the finite case, i.e. where the underlying state τ is a trace.

2. Moments Of Quantum Lévy Processes

2.1 Lemma

Let $B = (B_I)_{I\in I}$ be a stationary flow and $(B_t)_{t\geq 0}$ the corresponding quantum Lévy process. Then there exist constants α, β, and γ such that we have for all $t > 0$

$$\tau(B_t) = \alpha t,$$

$$\tau(B_t^2) = \alpha^2 t^2 + \beta t,$$

$$\tau(B_t^3) = \alpha^3 t^3 + 3\alpha\beta t^2 + \gamma t.$$

Proof. For all $s, t \geq 0$, we have

$$B_{s+t} = B_{[0,s)} + B_{[s,s+t),}$$

and thus

$$\tau(B_{s+t}) = \tau(B_s) + \tau(B_t),$$

which gives, by continuity, the equation for the first moment, with $\alpha = \tau(B_1)$.

For the second moment we get

$$B_{s+t}^2 = B_{[0,s)}^2 + B_{[0,s)}.B_{[s,s+t)} + B_{[s,s+t)}.B_{[0,s)} + B_{[s,s+t)}^2.$$

Pyramidal independence gives

$$\tau(B_{s+t}^2) = \tau(B_s^2) + \tau(B_t^2) + 2\tau(B_s)\tau(B_t),$$

which implies the equation for the second moment.

Similarly, one shows the result for the third moment as follows,

$$B_{s+t}^3 = B_{[0,s)}^3 + B_{[0,s)}^2.B_{[s,s+t)} + 2B_{[0,s)}^2 B_{[s.s+t)} + 2B_{[s,s+t)}^2.B_{[0,s)} + B_{[s,s+t)}^2 B_{[0,s)} + B_{[s,s+t)}^3,$$

$$B_{s+t}^3 = B_{[0,s)}^3 + 3B_{[0,s)}^2 B_{[s,s+t)} + 3B_{[s,s+t)}^2 B_{[0,s)} + B_{[s,s+t)}^3.$$

And thus by $\mathbb{C}$ - independence we have,

$$\tau(B_{s+t}^3) = \tau(B_s^3) + \tau(B_t^3) + 3\tau(B_s^2)\tau(B_t) + 3\tau(B_t^2)\tau(B_s). \quad \square$$

2.2 Remark: $\mathbb{C}$ - independence: for all $I, J \in I$ with $I \cap J = \varnothing$ we have that $\tau(ab) = \tau(a)\tau(b)$ for all $a \in M_I$ and $b \in M_J$, where M_I and M_J are von Neumann subalgebras of M

Notes: we can also phrase the $\mathbb{C}$ - independence in the following form: for all $s < t < u$,

$$\begin{array}{ccc} & M_{[s,t)} & \subset M_{[s,u)} \\ & \cup & \cup \\ \mathbb{C} & \subset & M_{[t,u)} \end{array}$$

is a commuting square S.Popa[1983]. If the von Neumann algebra M is commutative, $\mathbb{C}$ - independence is equivalent to the usual notion of stochastic independence in probability theory.

Note that pyramidal independence does not allow us to calculate all mixed moments of fourth and higher order: e.g., we cannot make a general statement about $\tau(B_{[0,s)}B_{[s,s+t)}B_{[0,s)}B_{[s,s+t)})$. Thus, in this generality; similar statements are not accessible for higher moments. Nevertheless, explicit polynomial bounds for the growth of higher moments are established in C Kostler[2000],[preprint],[2002].

However, if we require some more special structure, then we can say much more about the behaviour of higher moments. In this work we consider the case where we have an order invariance of the moments of the increments, in the sense that such moments do not change if we shift the increments against each other, as long as we do not change the relative position of the intervals. We first consider a discrete version of this before we treat the continuous case.

3. Limit For Order Invariant Distributions

Consider random variables $b_i^{(N)}(i, N \in \mathbb{N}, i \le N)$ living in some non – commutative probability space (M,τ).

For an n-tuple

$$\mathbf{i}:\{1,\ldots,n\} \to \{1,\ldots\ldots,N\}$$

we put

$$b_{\mathbf{i}}^{(N)} = b_{i(1)}^{(N)}, b_{i(2)}^{(N)}, \ldots\ldots, b_{i(n)}^{(N)}.$$

For an **i** as above, we denote by |**i**| the number of elements in the range of **i.**

3.1 Definitions:

(a) Let **i,j : {**1,…..,n} $\to \mathbb{N}$ be two n-tuples of indices. We say that they are ***order equivalent,*** denoted by **i ~j,** if

$$i(k) \le i(l) \Leftrightarrow j(k) \le j(l) \qquad \text{for all } k,l = 1,\ldots\ldots n.$$

We denote by O(n) the set of equivalence classes for maps $\mathbf{i}:\{1,\ldots,n\} \to N$ under this order equivalence. Note that for each n this is a finite set.

(b) We say that the distribution of the variables $b_i^{(N}$ is ***order invariant*** if we have for all $n, N \in \mathbb{N}$ and all $i, j:\{1,\ldots\ldots,n\} \to \{1,\ldots\ldots,N\}$ with $i \sim j$ that

$$\tau(b_{\mathbf{i}}^{(N)}) = \tau(b_{\mathbf{j}}^{(N)}).$$

In this case we denote, for $\sigma \in$ O(n), by $\tau(b_\sigma^{(N)})$ the common value of $\tau(b_{\mathbf{i}}^{(N)}$ for $i \in \sigma$.

3.2 Remark: Given such order invariant random variables, one can make quite precise statements about the moments of the sums $b_1^{(N)} + \ldots. + b_N^{(N)}$ in the limit $N \to \infty$.

Consider random variables $b_i^{(N)} \in (M,\tau)$ $(i, N \in \mathbb{N}, i \le N)$, whose distribution is order invariant. Assume that for all $n \in \mathbb{N}$ and all $\sigma \in$ O(n) the following limit exists:

$$c(\sigma) = \lim_{N\to\infty} N^{|\sigma|}\tau(b_\sigma^{(N)}).$$

Define

$$S_N = b_1^{(N)} + ... + b_N^{(N)}.$$

Then we have for all $n \in \mathbb{N}$

$$\lim_{N\to\infty} \tau(S_N^n) = \sum_{\sigma \in O(n)} \alpha_\sigma c(\sigma),$$

where the α_σ are the constants,

$$\alpha_\sigma = \lim_{N\to\infty} \frac{\#\{i : \{1,....,n\} \to \{1,...,N\}}{N^{|\sigma|}} = \frac{1}{|\sigma|!}.$$

4. Moments Of Order Invariant Quantum LÉVY Processes.

In the following, we will use, for two intervals *I, J*∈ I, the notation $I < J$ to indicate that we have s < t for all s ∈ *I* and t ∈ J.

4.1 Definition

Let $(B_I)_{I\in I}$ be a flow. We say that the flow (or its corresponding quantum Lévy process) is ***order invariant*** if we have for all $I_{1,} ...., I_n \in I$ with $I_\kappa \cap I_\iota = \phi(\kappa; \iota = 1,...., n)$ that

$$\tau(B_{I_1} ....B_{I_n}) = \tau(B_{I_1+t_1} ....B_{I_n+t_n})$$

for all $t_1, ..., t_n$ with the property that, for all $\kappa; \iota = 1,...., n$, $I_\kappa < I_\iota$ implies

$$I_\kappa + t_\kappa < I_\iota + t_\iota.$$

4.2 Remark: that an order invariant flow is in particular stationary.

Consider such an order invariant flow $(B_I)_{I\in I}$. Put

$$b_i^{(N)} = B_{[\frac{i-1}{N}, \frac{i}{N})}.$$

Then we have and $S_N = b_1^{(N)} + ... + b_N^{(N)} = B_1$ for all $N \in \mathbb{N}$ and, since the distribution of the $b_i^{(N)}$ is order invariant, our limit (1), yields that

$$\tau(B_1^n) = \sum_{\sigma \in O(n)} \alpha_\sigma c(\sigma),$$

if all

$$c(\sigma) = \lim_{N\to\infty} N^{|\sigma|} \tau(b_\sigma^{(N)}) \text{ exist.}$$

4.3 Proposition

Let $(B_I)_{I\in I}$ be an order invariant flow. Then, for all $n \in \mathbb{N}$ and $\sigma \in O(n)$, the limit

$$c(\sigma) = \lim_{N\to\infty} N^{|\sigma|} \tau(b_\sigma^{(N)}) \textit{ exists.}$$

Proof. We will prove this, for fixed n, by induction over the length of σ, starting with the maximal length of σ.

Namely, fix n and consider first a σ with $|\sigma| = n$. This means that $i = (i(1),....,i(n)) \in \sigma$ is a tuple of n different numbers. By using the stochastic independence we get

$$\begin{aligned} N^{|\sigma|}\tau(b_\sigma^{(N)}) &= N^n \tau(b_{i(1)}^{(N)}....b_{i(n)}^{(N)}) \\ &= N^n \tau(b_{i(1)}^{(N)})....\tau(b_{i(n)}^{(N)}) \\ &= N^n \tau(b_1^{(N)})^n \\ &= (N\tau(B_{[0,\frac{1}{N})}))^n \\ &= \tau(B_1)^n, \end{aligned}$$

and hence the limit

$$c(\sigma) = \lim_{N\to\infty} N^{|\sigma|}\tau(b_\sigma^{(N)}) = \tau(B_1)^n \text{ exists.}$$

Now consider an arbitrary $\sigma \in \mathcal{O}(n)$ and assumed that we have proved the existence of the limits $c(\sigma')$ for all $\sigma' \in \mathcal{O}(n)$ with $|\sigma'| > |\sigma|$. Choose an n – tuple $i = (i(1),....,i(n)) \in \sigma$ and consider

$$\tau(B_{[i(1),i(1)+1)}....B_{[i(n),i(n)+1)}).$$

We now decompose the intervals of length 1 into N subintervals of length 1/N, so that we can write this also as

$$\tau((\sum_{k(1)=1}^{N} B_{[i(1)+\frac{k(1)-1}{N},i(1)+\frac{k(1)}{N})}......(\sum_{k(n)=1}^{N} B_{[i(n)+\frac{k(n)-1}{N},i(n)+\frac{k(n)}{N})})).$$

If we multiply this out and collect terms together with the same relative position of the subintervals, then we get a sum of terms, one of which is exactly $N^{|\sigma|}\tau(b_\sigma^{(N)})$, and the other are of the form $\gamma_{\sigma'}\tau(b_{\sigma'}^{(N)})$, for σ' with $|\sigma'| > |\sigma|$. since also $\gamma_{\sigma'} \sim N^{|\sigma'|}$ for $N \to \infty$, we know by our induction hypothesis that all these other terms have a finite limit for $N \to \infty$.

Since the left hand side of our equation does not depend on N, the term $N^{|\sigma|}\tau(b_\sigma^{(N)})$ must also have a finite limit for $N \to \infty$. □

The same argument works if we replace the time 1 by an arbitrary time t. In this case, we get the existence of the limits

$$c_t(\sigma) = \lim_{N\to\infty} N^{|\sigma|}\tau(B_{[i(1),i(1)+t/N)}.......B_{[i(n),i(n)+t/N)}),$$

for $i = (i(1),....,i(n)) \in \sigma$. The remaining question is how these $c_t(\sigma)$ depend on the time t. We see this in the following:

4.4 Lemma

We have that

$$c_s(\sigma) = c(\sigma).s^{|\sigma|} \quad \text{for all } s \in \square$$

Proof. For $i = (i(1),....,i(n)) \in \sigma$ and $t \in \square$, we have

$$c_{2t}(\sigma) = \lim_{N\to\infty} N^{|\sigma|}\tau(B_{[i(1),i(1)+2t/N)}.....B_{[i(n),i(n)+2t/N)})$$

$$= \lim_{N\to\infty} N^{|\sigma|}\tau((B_{[i(1),i(1)+t/N)} + B_{[i(1),i(1)+2t/N)})$$

$$......(B_{[i(n),i(n)+t/N)} + (B_{[i(n),i(n)+t/N)} + B_{[i(n),i(n)+2t/N)})$$

$$= 2^{|\sigma|}c_t(\sigma).$$

Note that for each block of σ we can choose either the increments from i to i + t/N or the increments from i + t/n to i + 2t/N to match up, i.e. each block of σ contributes a factor 2. On the other hand, terms which match for some block an increment from i to i + t/N with an increment from i + t/N to i + t/N to i + 2t/N vanish in the limit, because they correspond to a σ' with $|\sigma'| > |\sigma|$, and so they have to be multiplied with a higher power of N to give a non – trivial limit.

In the same way one can see that for any $k \in \square$ and any $t \in \square$ we have

$$c_{kt}(\sigma) = k^{|\sigma|}c_t(\sigma).$$

This finally yields the assertion. □

5. Conclusion

By invoking t for each block of σ one could also derive functional equations for quantities which, together with the fact that they are measurable, would extend the statement of lemma above to all $t \in \square$. However, we do not need this because the continuity of the moments $\tau(B_t^n)$ allows us to extend the statement from rational to all real times t as can be summarize in the following;

5.1 Theorem

Let $(B_t)_{t\geq 0}$ be an order invariant quantum Lévy process. Then there exist constants $c(\sigma)$ for all $\sigma \in O$ such that for all $n \in \square$ and all $t \geq 0$

$$\tau(B_t^n) = \sum_{\sigma\in O(n)} \frac{1}{|\sigma|!} c(\sigma)t^{|\sigma|}.$$

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