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A New Approach to Constants of the Motion and the Helmholtz Conditions

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Abstract

Based on the Helmholtz conditions for the existence of a Lagrangian, we discuss a new approach to constants of the motion associated with continuous symmetries of the equations of motion . It is shown that the constants so determined vanish whenever the symmetry group is related to actual invariance of the Lagrangian (whose existence is guaranteed by the validity of the Helmholtz conditions) ; in which case , however , we have available the classical Noether invariants for constants of the motion .

Keywords: : Helmholtz conditions, Conservation laws, Symmetry

1. Introduction

The study of constants of the motion for a dynamical system is a subject of utmost importance in physics. It is well known that a generalized form of Noether's Theorem relates a constant of the motion to any continuous trans-formation of the coordinates and time that leaves the action integral invariant. Any transformation which leaves the equations of motion invariant is called Noether symmetry if it also conserves the action integral. In this case it is conventional to have available the classical Noether form for constants of the motion. Recently in a new approach to constants of the motion [1,2] it has been demonstrated that they can be determined even by non -Noether symmetries , i.e. by transformations which leave the equations of motion invariant but do not preserve the action integral . The explicit form of these conserved quantities [3] will be specified below (see Eqn.(9)) from a knowledge of the symmetry group generator, without requiring any further integration of the equations of motion . Our aim in this note is to prove that the constants of the motion so determined must vanish in case the associated symmetry is Noether. We may thus state that any Noether symmetry leads only to classical Noether form for the conserved quantity.

Recall that the Helmholtz conditions for the existence of the Lagrangian $L(x,\dot{x},t)$ for an equation of motion with the form

$$\frac{d^2x^i}{dt^2}=F^i\left(\frac{dx}{dt},x,t\right)=F^i(x,x,t) \qquad (1)$$

are

$$w_{ij}=w_{ji}\,, \qquad (2)$$

$$\frac{\partial w_{ij}}{\partial x^k}=\frac{\partial w_{ik}}{\partial x^j}\,, \qquad (3)$$

$$\frac{D(w_{ij})}{dt}=-\frac{1}{2}\left[w_{ik}\frac{\partial f^k}{\partial x^j}+w_{jk}\frac{\partial f^k}{\partial x^i}\right], \qquad (4)$$

$$\frac{D(t_{ij})}{dt}\equiv\frac{1}{2}\frac{D}{dt}\left(w_{ik}\frac{\partial f^k}{\partial x^j}-w_{jk}\frac{\partial f^k}{\partial x^i}\right)=w_{ik}\frac{\partial F^k}{\partial x^j}-w_{jk}\frac{\partial F^k}{\partial x^i} \qquad (5)$$

where the elements of the nonsingular matrix w_{ij} are the integrating factors in the equation

$$w_{ij}\left[\frac{d^2x^j}{dt^2}-F^j(x,\dot{x},t)\right]=\frac{d}{dt}\left(\frac{\partial L}{\partial x^i}\right)-\frac{\partial L}{\partial x^i} \qquad (6)$$

In the following we use the overdot to denote the onshell time derivative,

$$\frac{DX}{dt}=\dot{X}=\frac{\partial X}{\partial t}+\dot{x}^l\frac{\partial X}{\partial x^l}+F^l\frac{\partial X}{\partial \dot{x}^l} \qquad (7)$$

In the case there exists a Lagrangian $L(x,\dot{x},t)$ for the dynamical system (1), it follows from (4) quite generally that

$$\frac{\partial F^l}{\partial \dot{x}^l}=-\frac{\dot{D}}{D} \qquad (8)$$

where D is the determinant of the matrix whose elements are w_{ij} . This observation leads automatically to the conserved quantity [3]

$$\varphi=\frac{E(D)}{D}+\frac{\partial \eta^l}{\partial x^l}+\frac{\partial \dot{\eta}^l}{\partial \dot{x}^l} \qquad (9)$$

for any (non - Noether) symmetry of a dynamical system with a Lagrangian , where

$$E=\eta^l(x,\dot{x},t)\frac{\partial}{\partial x^l}+\dot{\eta}^l(x,\dot{x},t)\frac{\partial}{\partial \dot{x}^l} \qquad (10)$$

generates the symmetry of the given dynamical system (1). The conservation of φ may be demonstrated by making use of a conservation law found by Hojman [1] and generalized by Gonzalez - Gascon [2] . According to Hojman's theorem ,let the function $\eta^l(x,\dot{x},t)$ determine a symmetry generator E for (1) ; if a function $\lambda(x,\dot{x},t)$ can be found such that the quantity

$$\Omega=\frac{\partial F^l}{\partial \dot{x}^l}+\frac{\dot{\lambda}}{\lambda} \qquad (11)$$

vanishes, then a constant of the motion is given by

$$\varphi=E(\ln\lambda)+\frac{\partial \eta^l}{\partial x^l}+\frac{\partial \dot{\eta}^l}{\partial \dot{x}^l}$$

$$\dot{\varphi}=0 \qquad (12)$$

In view of (8) , we may choose $\lambda=D$, and therefore the condition $\Omega=0$ can always be satisfied for a dynamical system with Lagrangian leading to (9). We will now prove that φ vanishes if the associated symmetry is a symmetry of the Lagrangian system. Our proof will make heavy use of the Helmholtz conditions (2) - (5) which are necessary and sufficient for the existence of Lagrangian for a dynamical system given by (1).
Given a constant of the motion $C(x,\dot{x},t)$, the necessary and sufficient conditions for its being related to symmetry of the Lagrangian are [5]

$$\eta^l=g^{lm}\frac{\partial C}{\partial \dot{x}^m}$$

$$\dot{C}=0 \qquad (13)$$

Here we are considering invariance of (1) under the infinitesimal coordinate transformation $x^l\rightarrow x^{l'}=x^l+\varepsilon\eta^l(x,\dot{x},t)$
(ε is a small parameter) and g denotes the inverse matrix to w , i.e. the matrix of integrating factors for the Helmholtz conditions . Without restricting ourselves to any specific choice of the w_{ij} in the system of equation (2)-(5) , it can be proved that $(F_l=w_{lm}F^m)$

$$\dot{\eta}^l=-g^{lm}\frac{\partial C}{\partial x^m}+\frac{1}{2}g^{lm}g^{np}\left(\frac{\partial F_m}{\partial \dot{x}^p}-\frac{\partial F_p}{\partial \dot{x}^m}\right)\frac{\partial C}{\partial \dot{x}^n} \qquad (14)$$

$$\ddot{\eta}^l=E(F^l) \qquad (15)$$

The result in (15) is the general condition for the invariance [4] of (1) under the infinitesimal transformation considered with the symmetry generator E given by (10),(13) and (14). From (13) and (14) it is straightforward to verify that

$$\frac{\partial\eta^l}{\partial x^l}+\frac{\partial\eta^l}{\partial x^l}+\eta^l g^{mn}\frac{\partial w_{nm}}{\partial x^l}=-(\frac{\partial g^{lm}}{\partial x^l})\frac{\partial C}{\partial x^m}+\frac{1}{2}\frac{\partial(g^{lm}g^{np})}{\partial x^l}(\frac{\partial F_m}{\partial x^n}-\frac{\partial F_n}{\partial x^m})\frac{\partial C}{\partial x^p} \qquad (16)$$

$$\dot{\eta}^l g^{mn}\frac{\partial w_{nm}}{\partial x^l}=(\frac{\partial g^{lm}}{\partial x^l})\frac{\partial C}{\partial x^m}+\frac{1}{2}(\frac{\partial g^{lm}}{\partial x^n})g^{pn}(\frac{\partial F_p}{\partial x^l}-\frac{\partial F_l}{\partial x^p})\frac{\partial C}{\partial x^m}$$

$$-\frac{1}{2}\frac{\partial(g^{lm}g^{np})}{\partial x^l}(\frac{\partial F_m}{\partial x^n}-\frac{\partial F_n}{\partial x^m})\frac{\partial C}{\partial x^p} \qquad (17)$$

Adding (16) and (17) we are led to

$$\varphi=\frac{\partial\eta^l}{\partial x^l}+\frac{\partial\eta^l}{\partial x^l}+E(\ln D)$$

$$=\frac{\partial\eta^l}{\partial x^l}+\frac{\partial\eta^l}{\partial x^l}+\eta^l g^{mn}\frac{\partial w_{nm}}{\partial x^l}+\dot{\eta}^l g^{mn}\frac{\partial w_{nm}}{\partial x^l}=\frac{1}{2}[(\frac{\partial g^{lm}}{\partial x^n})g^{pn}](\frac{\partial F_p}{\partial x^l}-\frac{\partial F_l}{\partial x^p})\frac{\partial C}{\partial x^m}$$

This demonstrates that φ vanishes because it turns out that given the equations (2) and (3) the tensor in the square brackets is symmetric under the exchange of p and l (for each m), while the tensor in the parenthesis is antisymmetric under that exchange .

In this note we have considered the application of Helmholtz conditions to the analysis of conserved quantities which are associated with the continuous symmetries of dynamical systems with Lagrangian. We have shown that the conserved quantity vanishes whenever the symmetry group for a dynamical system is related to actual invariance of the action integral. In that case we utilize the classical Noether invariants.

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Mukesh Kumar, born in New Delhi (India) on 12 October 1976, an honours graduate and postgraduate from University of Delhi. He was awarded Ph.D. by University of Delhi in 2005. He was awarded JRF ans SRF fellowship by CSIR. In his more than 8 years of teaching career at Swami Shraddhanad College, University of Delhi, he has taught various courses to undergraduate students. He is actively involved in research in the fields of classical mechanics. He has published many papers and organized conferences.