SOME COMMON FIXED POINT THEOREMS IN BANACH SPACES

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Abstract

Our aim of this paper is to obtain some fixed point and common fixed point theorems in Banach space satisfying different rational contractive conditions.

Key Words: Banach space, fixed point, common fixed point

1 INTRODUCTION & Preliminaries

Fixed point has become attractive to the authors working in non linear analysis when the study of non-expansive mappings concerning the existence of the fixed point. Since the non expansive mappings include contraction as well as contractive mappings. Browder [1] was the first mathematician who studies non –expansive mapping where he applied the results for proving the existence of solution of certain integral equations. Recently study of fixed point theorems in Banach spaces is very interesting. In this paper we prove some fixed point theorems and common fixed point theorems in Banach spaces. Our results are generalization and extension of various known results.

First we recall some known definitions and results which are helpful for proving our results.

Definition 1.1 Let S and T are self maps of a Banach space X. If $w = Sx = Tx$ for some $x \in X$, then x is called a coincidence point of S and T, and w is called a point of coincidence of S and T.

Definition 1.2 Let S and T are self maps of a Banach space X, then S and T are said to be weakly compatible if

$$\lim_{n\to\infty}||STx_n - TSx_n|| = 0$$

whenever $\{x_n\}$ is sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$$

for some $x \in X$.

Definition 1.3 Let S and T are self maps of a Banach space X, then S and T are said to be weakly compatible if they commute at their coincidence points; i.e. if $Tx = Sx$ for some $x \in X$ then $TSx = STx$.

Definition 1.4 Let Φ be the set of real functions

$$\phi(t_1, t_2, t_3, t_4, t_5): [0,\infty)^5 \to [0,\infty)$$

satisfying the following conditions:

i. ϕ is non increasing in variables t_4 and t_5.

ii. There is an $h_1 > 0$ and $h_2 > 0$ such that $h = h_1h_2 < 1$ and if $u \ge 0$ and $v \ge 0$ satisfying

a. $u \le \phi(v, v, u, u + v, 0)$ or $u \le \phi(v, u, v, u + v, 0)$

Then we have $u \le h_1 v$.

and if $u \ge 0, v \ge 0$ satisfy

b. $u \le \phi(v, v, u, 0, u + v)$ or $u \le \phi(v, u, v, 0, u + v)$

Then we have $u \le h_2 v$.

c. If $u \ge 0$ is such that

$$u \le \phi(u, 0,0, u, u) \text{ or } u \le \phi(0, u, 0,0, u) \text{ or } u \le \phi(0,0, u, u, 0)$$

Then $u = 0$.

2 Common Fixed Point Theorems for Self Mappings in Banach spaces

In this section we prove some common fixed point results for four self mappings satisfying symmetric rational expression. Our aim of this section is to generalized and extended previous many known results. In fact our first result of this section is as follows,

Theorem 2.1 Let A, B, S, T be continuous self mappings defined on the Banach space X into itself satisfies the following conditions:

2.1(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$

2.1(ii) if one of $A(X)$, $B(X)$, $S(X)$, $T(X)$ is complete subspace of X.

2.1(iii) The pair $\{A, S\}$ and $\{B, T\}$ are weakly compatible.

2.1(iv)
$$\|Ax - By\|^2 \leq \alpha\left[\|Ax - Sy\|^2 + \|By - Ty\|^2\right] + \beta\left[\|Ax - Ty\|^2 + \|By - Sx\|^2\right] + \gamma\left[\|Ax - Sx\|^2 + \|Ax - Ty\|^2\right] + \delta\left[\|By - Sx\|^2 + \|By - Ty\|^2\right] + \eta\left[\|Ax - Sx\|\|By - Ty\| + \|Ax - Ty\|\|By - Sx\|\right]$$

For all $x, y \in X, (x \neq y)$ and for non negative $\alpha, \beta, \gamma, \delta, \eta \in [0,1)$ such that $0 < 2\alpha + 2\beta + \gamma + 4\delta + \eta < 1$. Then A, B, S, T have unique common fixed point in X.

Proof For any arbitrary x_0 in X we define the sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$Ax_{2n} = Tx_{2n+1} = y_{2n} \text{ and } Bx_{2n+1} = Sx_{2n+2} = y_{2n+1} \qquad (2.1\text{ a})$$

for all n= 0, 1, 2, …..

On taking $y_{2n} \neq y_{2n+1}$

$$\|y_{2n}, y_{2n+1}\|^2 = \|Ax_{2n} - Bx_{2n+1}\|^2$$

From 6.2.1(iv) we have

$$\|Ax_{2n} - Bx_{2n+1}\|^2 \leq \alpha\left[\|Ax_{2n} - Sx_{2n}\|^2 + \|Bx_{2n+1} - Tx_{2n+1}\|^2\right] + \beta\left[\|Ax_{2n} - Tx_{2n+1}\|^2 + \|Bx_{2n+1} - Sx_{2n}\|^2\right] + \gamma\left[\|Ax_{2n} - Sx_{2n}\|^2 + \|Ax_{2n} - Tx_{2n+1}\|^2\right] + \delta\left[\|Bx_{2n+1} - Sx_{2n}\|^2 + \|Bx_{2n+1} - Tx_{2n+1}\|^2\right] + \eta\left[\begin{matrix}\|Ax_{2n} - Sx_{2n}\|\,\|Bx_{2n+1} - Tx_{2n+1}\| \\ + \|Ax_{2n} - Tx_{2n+1}\|\|Bx_{2n+1} - Sx_{2n}\|\end{matrix}\right]$$

$$\|y_{2n} - y_{2n+1}\|^2 \leq \alpha\left[\|y_{2n} - y_{2n-1}\|^2 + \|y_{2n+1} - y_{2n}\|^2\right]$$

$$+\beta\left[||y_{2n}-y_{2n}||^2+||y_{2n+1}-y_{2n-1}||^2\right]$$

$$+\gamma\left[||y_{2n}-y_{2n-1}||^2+||y_{2n}-y_{2n}||^2\right]$$

$$+\delta\left[||y_{2n+1}-y_{2n-1}||^2+||y_{2n+1}-y_{2n}||^2\right]$$

$$+\eta\begin{bmatrix}||y_{2n}-y_{2n-1}||\,||y_{2n+1}-y_{2n}|| \\ +||y_{2n}-y_{2n}||\,||y_{2n+1}-y_{2n-1}||\end{bmatrix}$$

$$||y_{2n}-y_{2n+1}|| \le \sqrt{\frac{(\alpha+\beta+2\delta+\gamma)}{(1-\alpha-\beta-2\delta+\eta)}}\;||y_{2n}-y_{2n-1}||$$

Let us denote $\sqrt{\frac{(\alpha+\beta+2\delta+\gamma)}{(1-\alpha-\beta-2\delta+\eta)}} = k,$

since $0 < 2\alpha+2\beta+\gamma+4\delta+\eta < 1$ which gives

$0 < \sqrt{\frac{(\alpha+\beta+2\delta+\gamma)}{(1-\alpha-\beta-2\delta+\eta)}} = k < 1$ and that

$$||y_{2n}-y_{2n+1}|| \le k\,||y_{2n}-y_{2n-1}||$$

Similarly we can show that

$$||y_{2n}-y_{2n-1}|| \le k^2\,||y_{2n-1}-y_{2n-2}||$$

Processing the same way we can write,

$$||y_{2n}-y_{2n-1}|| \le k^n\,||y_0-y_1||$$

for any integer m we have

$$||y_{2n}-y_{2n+m}|| \le ||y_{2n}-y_{2n+1}|| + ||y_{2n+1}-y_{2n+2}|| + \dots\dots\dots\dots\dots + ||y_{2n+m-1}-y_{2n+m}||$$

$$||y_{2n}-y_{2n+m}|| \le k^n.||y_0-y_1|| + k^{n+1}.||y_0-y_1|| + \dots\dots\dots\dots + k^{n+m}.||y_0-y_1||$$

$$||y_{2n}-y_{2n+m}|| \le k^n[1+k+k^2+\cdots\dots\dots\dots+k^m].||y_0-y_1||$$

$$||y_{2n}-y_{2n+m}|| \le \frac{k^n}{1-k}\,||y_0-y_1||$$

as $n \to \infty$ gives that

$$||y_{2n}-y_{2n+m}|| \to 0$$

Thus $\{y_{2n}\}$ is a Cauchy sequence in X. Since $T(X)$ is complete subspace of X then the subsequence $y_{2n} = Tx_{2n+1}$ is Cauchy sequence in $T(X)$ which converges to the some point say u in X. Let $v \in T^{-1}u$ then $Tv = u$. Since $\{y_{2n}\}$ is converges to u and hence $\{y_{2n+1}\}$ also converges to same point u .

we set $x = x_{2n}$ *and* $y = v$ in $2.1(iv)$

$$||Ax_{2n} - Bv||^2 \le \alpha \left[||Ax_{2n} - Sx_{2n}||^2 + ||Bv - Tv||^2 \right]$$
$$+ \beta \left[||Ax_{2n} - Tv||^2 + ||Bv - Sx_{2n}||^2 \right]$$
$$+ \gamma \left[||Ax_{2n} - Sx_{2n}||^2 + ||Ax_{2n} - Tv||^2 \right]$$
$$+ \delta \left[||Bv - Sx_{2n}||^2 + ||Bx_{2n+1} - Tv||^2 \right]$$
$$+ \eta \begin{bmatrix} ||Ax_{2n} - Sx_{2n}||\,||Bv - Tv|| \\ + ||Ax_{2n} - Tv||\,||Bv - Sx_{2n}|| \end{bmatrix}$$

$as\ n \to \infty \quad ||u - Bv||^2 \le (\alpha + \beta + \gamma + \delta + \eta)\, ||u - Bv||^2$

This is a contradiction, implies that $Bv = u$ also $B(X) \subset S(X)$ *so* $Bv = u$ implies that $u \in S(X)$.

Let $w \in S^{-1}(X)$ then $w = u$ setting $x = w$ *and* $y = x_{2n+1}$ *in* $6.2.1(iv)$ we get

$$||Aw - Bx_{2n+1}||^2 \le \alpha \left[||Aw - Sw||^2 + ||Bx_{2n+1} - Tx_{2n+1}||^2 \right]$$
$$+ \beta \left[||Aw - Tx_{2n+1}||^2 + ||Bx_{2n+1} - Sw||^2 \right]$$
$$+ \gamma \left[||Aw - Sw||^2 + ||Aw - Tx_{2n+1}||^2 \right]$$
$$+ \delta \left[||Bx_{2n+1} - Sw||^2 + ||Bx_{2n+1} - Tx_{2n+1}||^2 \right]$$
$$+ \eta \begin{bmatrix} ||Aw - Sw||\,||Bx_{2n+1} - Tx_{2n+1}|| \\ + ||Aw - Tx_{2n+1}||\,||Bx_{2n+1} - Sw|| \end{bmatrix}$$

$as\ n \to \infty \quad ||Aw - u||^2 \le (\beta + \gamma + \delta + \eta)\, ||Aw - u||^2$

This is a contradiction, implies that, $Aw = u$ this means $Aw = Sw = Bv = Tv = u$.

since $Bv = Tv = u$ so by weak compatibility of (B, T) it follows that, $BTv = TBv$ *and* so we get

$$Bu = BTv = TBv = Tu\,.$$

Since $Aw = Sw = u$ so by weak compatibility of (A, S) it follows that $SAw = ASw$ *and* So we get

$$Au = ASw = SAw = Su\,.$$

Thus from $6.2.1(iv)$ we have

$$||Aw - Bu||^2 \le \alpha \left[||Aw - Sw||^2 + ||ABu - Tu||^2 \right]$$
$$+ \beta \left[||Aw - Tu||^2 + ||Bu - Sw||^2 \right]$$
$$+ \gamma \left[||Aw - Sw||^2 + ||Aw - Tu||^2 \right]$$

$$+\delta\left[||Bu-Sw||^2+||Bu-Tu||^2\right]$$

$$+\eta\begin{bmatrix}||Aw-Sw||||Bu-Tu||\\+||Aw-Tu||||Bu-Sw||\end{bmatrix}$$

$$||u-Bu||^2 \le (\alpha+\beta+\delta)||u-Bu||^2$$

which contradiction

implies that $Bu = u$.

Similarly we can show $Au = u$ by using $6.2.1(iv)$. Therefore

$$u = Au = Bu = Su = Tu.$$

Hence the point u is common fixed point of A, B, S, T.

If we assume that $S(X)$ is complete then the argument analogue to the previous completeness argument proves the theorem. If $A(X)$ is complete then $u \in A(X) \subset T(X)$. similarly if $B(X)$ is complete then $u \in B(X) \subset S(X)$. This complete prove of the theorem.

Uniqueness Let us assume that z is another fixed point of A, B, S, T in X different from u. $i.e.\ u \neq z$ then

$$||u-z||^2 = ||Au-Bz||^2$$

$from\ 2.1(iv)\ we\ get$

$$||u-z||^2 \le (\beta+\gamma+\delta+\eta)||u-z||^2$$

which contradiction the hypothesis .

Hence u is unique common fixed point of A, B, S, T in X.

Before giving our second result of this section we Let R^+ denote the set of non negative real numbers and F a family of all mappings $\phi : (R^+)^5 \to R^+$ such that ϕ is upper semi continuous, non decreasing in each coordinate variable and, for any $\phi(t_1, t_2, t_3, t_4, t_5) < kt$.

***Theorem* 2.2** Let A, B, S, T be continuous self mappings defined on the Banach space X into itself satisfies the following conditions:

$2.2(i)$ $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$

$2.2(ii)$ The pair $\{A, S\}$ and $\{B, T\}$ are weakly compatible.

$2.2(iii)$ $$d^2(Ax, By) \le \phi\begin{pmatrix}||Ax-Sy||^2+||By-Ty||^2,\\ ||Ax-Ty||^2+||By-Sx||^2,\\ ||Ax-Sx||^2+||Ax-Ty||^2,\\ ||By-Sx||^2+||By-Ty||^2,\\ ||Sx-Ty||^2\end{pmatrix}$$

For all $x, y \in X$, $(x \neq y)$ and $\phi \in \Phi$. Then A, B, S, T have unique common fixed point in X.

Proof For any arbitrary x_0 in X we define the sequence $\{x_n\}$ *and* $\{y_n\}$ in X such that

$$Ax_{2n} = Tx_{2n+1} = y_{2n} \text{ and } Bx_{2n+1} = Sx_{2n+2} = y_{2n+1} \qquad (6.2.2\ a)$$

for all n= 0, 1, 2, …..

On taking $y_{2n} \neq y_{2n+1}$

$$||y_{2n}, y_{2n+1}||^2 = ||Ax_{2n} - Bx_{2n+1}||^2$$

From 2.2(iii) we have

$$||Ax_{2n} - Bx_{2n+1}||^2 \leq \phi\begin{pmatrix} ||Ax_{2n} - Sx_{2n}||^2 + ||Bx_{2n+1} - Tx_{2n+1}||^2, \\ ||Ax_{2n} - Tx_{2n+1}||^2 + ||Bx_{2n+1} - Sx_{2n}||^2, \\ ||Ax_{2n} - Sx_{2n}||^2 + ||Ax_{2n} - Tx_{2n+1}||^2, \\ ||Bx_{2n+1} - Sx_{2n}||^2 + ||Bx_{2n+1} - Tx_{2n+1}||^2, \\ ||Sx_{2n} - Tx_{2n+1}||^2 \end{pmatrix}$$

$$||y_{2n}, y_{2n+1}||^2 \leq \phi\begin{pmatrix} ||y_{2n} - y_{2n-1}||^2 + ||y_{2n+1} - y_{2n}||^2, \\ ||y_{2n} - y_{2n}||^2 + ||y_{2n+1} - y_{2n-1}||^2, \\ ||y_{2n} - y_{2n-1}||^2 + ||y_{2n} - y_{2n}||^2, \\ ||y_{2n+1} - y_{2n-1}||^2 + ||y_{2n+1} - y_{2n}||^2, \\ ||y_{2n} - y_{2n-1}||^2 \end{pmatrix}$$

from the property of ϕ we have

$$||y_{2n} - y_{2n+1}||^2 \leq k\,||y_{2n} - y_{2n-1}||^2$$

similarly we can show that

$$||y_{2n} - y_{2n+1}|| \leq k^2\,||y_{2n-1} - y_{2n-2}|| \leq k^n||y_0 - y_1||$$

processing the same way we can write,

for any integer m we have

$$||y_{2n} - y_{2n+m}|| \leq ||y_{2n} - y_{2n+1}|| + ||y_{2n+1} - y_{2n+2}|| + \ldots\ldots\ldots\ldots\ldots + ||y_{2n+m-1} - y_{2n+m}||$$

$$||y_{2n} - y_{2n+m}|| \leq k^n.||y_0 - y_1|| + k^{n+1}.||y_0 - y_1|| + \ldots\ldots\ldots\ldots + k^{n+m}.||y_0 - y_1||$$

$$||y_{2n} - y_{2n+m}|| \leq k^n[1 + k + k^2 + \cdots\ldots\ldots\ldots + k^m].||y_0 - y_1||$$

$$||y_{2n} - y_{2n+m}|| \leq \frac{k^n}{1-k}\,||y_0 - y_1||$$

as $n \to \infty$ gives that

$$||y_{2n} - y_{2n+m}|| \rightarrow 0$$

Thus $\{y_{2n}\}$ is a Cauchy sequence in X. Since $T(X)$ is complete subspace of X then the subsequence $y_{2n} = Tx_{2n+1}$ is Cauchy sequence in $T(X)$ which converges to the some point say u in X. Let $v \in T^{-1}u$ then $Tv = u$. Since $\{y_{2n}\}$ is converges to u and hence $\{y_{2n+1}\}$ also converges to same point u .

We set $x = x_{2n}$ and $y = v$ in $2.1(iv)$

$$||Ax_{2n} - Bv||^2 \leq \phi \begin{pmatrix} ||Ax_{2n} - Sx_{2n}||^2 + ||Bv - Tv||^2, \\ ||Ax_{2n} - Tv||^2 + ||Bv - Sx_{2n}||^2, \\ ||Ax_{2n} - Sx_{2n}||^2 + ||Ax_{2n} - Tv||^2, \\ ||Bv - Sx_{2n}||^2 + ||Bx_{2n+1} - Tv||^2, \\ ||Sx_{2n} - Tv||^2 \end{pmatrix}$$

$as\ n \rightarrow \infty$

$$||u - Bv||^2 \leq ||u - Bv||^2$$

which contradiction

implies that B$v = u$ also $B(X) \subset S(X)$ so $Bv = u$ implies that $u \in S(X)$.

Let $w \in S^{-1}(X)$ then w = u setting $x = w$ and $y = x_{2n+1}$ in $6.2.2(iii)$ we get

$$||Aw - Bx_{2n+1}||^2 \leq \phi \begin{pmatrix} ||Aw - Sw||^2 + ||Bx_{2n+1} - Tx_{2n+1}||^2, \\ ||Aw - Tx_{2n+1}||^2 + ||Bx_{2n+1} - Sw||^2, \\ ||Aw - Sw||^2 + ||Aw - Tx_{2n+1}||^2, \\ ||Bx_{2n+1} - Sw||^2 + ||Bx_{2n+1} - Tx_{2n+1}||^2, \\ ||Sw - Tx_{2n+1}||^2 \end{pmatrix}$$

$as\ n \rightarrow \infty,$ $||Aw - u|| \leq ||Aw - u||$

This is a contradiction, implies that, $Aw = u$ this means $Aw = Sw = Bv = Tv = u.$

since $Bv = Tv = u$ so by weak compatibility of (B, T) it follows that, $BTv = TBv$ and so we get

$$Bu = BTv = TBv = Tu\,.$$

Since $Aw = Sw = u$ so by weak compatibility of (A, S) it follows that $SAw = ASw$ and So we get

$$Au = ASw = SAw = Su$$

Thus from $2.2(iii)$ we have

$$||Aw - Bu||^2 \leq \phi \begin{pmatrix} ||Aw - Sw||^2 + ||ABu - Tu||^2, \\ ||Aw - Tu||^2 + ||Bu - Sw||^2, \\ ||Aw - Sw||^2 + ||Aw - Tu||^2, \\ ||Bu - Sw||^2 + ||Bu - Tu||^2, \\ ||Sw - Tu||^2 \end{pmatrix}$$

$$||u - Bu|| \leq ||u - Bu||$$

which contradiction

implies that $Bu = u$.

Similarly we can show $Au = u$ by using $6.2.2(iii)$. Therefore

$$u = Au = Bu = Su = Tu.$$

Hence the point u is common fixed point of A, B, S, T.

If we assume that $S(X)$ is complete then the argument analogue to the previous completeness argument proves the theorem. If $A(X)$ is complete then $u \in A(X) \subset T(X)$. similarly if $B(X)$ is complete then $u \in B(X) \subset S(X)$. This complete prove of the theorem.

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$$||u - z||^2 = ||Au - Bz||^2$$

$from$ $2.2(iii)$ $we\ get$

$$||u - z||^2 \leq ||u - z||^2$$

which contradiction the hypothesis . Hence u is unique common fixed point of A, B, S, T in X.

Theorem 2.3 Let A, B, S, T be continuous self mappings defined on the Banach space X into itself satisfies the following conditions:

$2.3(i)$ $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$

$2.3(ii)$ $if\ one\ of$ $A(X)$, $B(X)$, $S, (X)$, $T(X)$ is complete subspace of X.

$2.3(iii)$ The pair $\{A, S\}$ and $\{B, T\}$ are weakly compatible.

$$2.3(iv)\quad ||Ax - By||^2 \leq \alpha \max \begin{Bmatrix} ||Ax - Sy||^2 + ||By - Ty||^2, \\ ||Ax - Ty||^2 + ||By - Sx||^2, \\ ||Ax - Sx||^2 + ||Ax - Ty||^2, \\ ||By - Sx||^2 + ||By - Ty||^2 \end{Bmatrix}$$

For all $x, y \in X$, $(x \neq y)$ and for non negative $\alpha \in [0,1)$. Then A, B, S, T have unique common fixed point in X.

Proof For any arbitrary x_0 in X we define the sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$Ax_{2n} = Tx_{2n+1} = y_{2n}\ and\ Bx_{2n+1} = Sx_{2n+2} = y_{2n+1} \qquad (2.3\ a)$$

for all n= 0, 1, 2, …..

On taking $y_{2n} \neq y_{2n+1}$

$$||y_{2n}, y_{2n+1}||^2 = ||Ax_{2n} - Bx_{2n+1}||^2$$

From 2.3(iv) we have

$$||Ax_{2n} - Bx_{2n+1}||^2 \leq \alpha\, max \begin{Bmatrix} ||Ax_{2n} - Sx_{2n}||^2 + ||Bx_{2n+1} - Tx_{2n+1}||^2, \\ ||Ax_{2n} - Tx_{2n+1}||^2 + ||Bx_{2n+1} - Sx_{2n}||^2, \\ ||Ax_{2n} - Sx_{2n}||^2 + ||Ax_{2n} - Tx_{2n+1}||^2, \\ ||Bx_{2n+1} - Sx_{2n}||^2 + ||Bx_{2n+1} - Tx_{2n+1}||^2 \end{Bmatrix}$$

$$||y_{2n} - y_{2n+1}||^2 \leq \alpha\, max \begin{Bmatrix} ||y_{2n} - y_{2n-1}||^2 + ||y_{2n+1} - y_{2n}||^2, \\ ||y_{2n} - y_{2n}||^2 + ||y_{2n+1} - y_{2n-1}||^2, \\ ||y_{2n} - y_{2n-1}||^2 + ||y_{2n} - y_{2n}||^2, \\ ||y_{2n+1} - y_{2n-1}||^2 + ||y_{2n+1} - y_{2n}||^2 \end{Bmatrix}$$

$$(1-\alpha)||y_{2n} - y_{2n+1}||^2 \leq \alpha\, ||y_{2n} - y_{2n-1}||^2$$

$$||y_{2n} - y_{2n+1}|| \leq \sqrt{\frac{\alpha}{1-\alpha}}\, ||y_{2n} - y_{2n-1}||$$

Let us denote $\sqrt{\frac{\alpha}{1-\alpha}} = k,$

$$||y_{2n} - y_{2n+1}||^2 \leq k\, ||y_{2n} - y_{2n-1}||^2$$

Similarly we can show that

$$||y_{2n} - y_{2n-1}|| \leq k^2\, ||y_{2n-2} - y_{2n-1}||$$

Processing the same way we can write,

$$||y_{2n} - y_{2n-1}|| \leq k^n\, ||y_0 - y_1||$$

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$$||y_{2n} - y_{2n+m}|| \leq k^n . ||y_0 - y_1|| + k^{n+1} . ||y_0 - y_1|| + \dots\dots\dots\dots + k^{n+m} . ||y_0 - y_1||$$

$$||y_{2n} - y_{2n+m}|| \leq k^n [1 + k + k^2 + \cdots \dots\dots\dots + k^m] . ||y_0 - y_1||$$

$$||y_{2n} - y_{2n+m}|| \leq \frac{k^n}{1-k}\, ||y_0 - y_1||$$

as $n \to \infty$ gives that

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Thus $\{y_{2n}\}$ is a Cauchy sequence in X. Since $T(X)$ is complete subspace of X then the subsequence $y_{2n} = Tx_{2n+1}$ is Cauchy sequence in $T(X)$ which converges to the some point say u in X. Let $v \in T^{-1}u$ then $Tv = u$. Since $\{y_{2n}\}$ is converges to u and hence $\{y_{2n+1}\}$ also converges to same point u .

we set $x = x_{2n}$ *and* y $= v$ in $2.3(iv)$

$$||Ax_{2n} - Bv||^2 \le \alpha \max \begin{Bmatrix} ||Ax_{2n} - Sx_{2n}||^2 + ||Bv - Tv||^2, \\ ||Ax_{2n} - Tv||^2 + ||Bv - Sx_{2n}||^2, \\ ||Ax_{2n} - Sx_{2n}||^2 + ||Ax_{2n} - Tv||^2, \\ ||Bv - Sx_{2n}||^2 + ||Bx_{2n+1} - Tv||^2 \end{Bmatrix}$$

as $n \to \infty$

$$||u - Bv|| \le \alpha||u - Bv||$$

which contradiction

implies that B$v = u$ also $B(X) \subset S(X)$ *so* $Bv = u$ implies that $u \in S(X)$.

Let $w \in S^{-1}(X)$ then w $= u$ setting $x = w$ *and* $y = x_{2n+1}$ *in* $2.3(iv)$ we get

$$||Aw - Bx_{2n+1}||^2 \le \alpha \max \begin{Bmatrix} ||Aw - Sw||^2 + ||Bx_{2n+1} - Tx_{2n+1}||^2, \\ ||Aw - Tx_{2n+1}||^2 + ||Bx_{2n+1} - Sw||^2, \\ ||Aw - Sw||^2 + ||Aw - Tx_{2n+1}||^2, \\ ||Bx_{2n+1} - Sw||^2 + ||Bx_{2n+1} - Tx_{2n+1}||^2 \end{Bmatrix}$$

as $n \to \infty$

$$||Aw - u|| \le \alpha||Aw - u||$$

which contradiction

implies that, $Aw =$ u this means $Aw = Sw = Bv = Tv = u.$

since $Bv = Tv = u$ so by weak compatibility of (B, T) it follows that, $BTv = TBv$ *and* so we get

$$Bu = BTv = TBv = Tu\,.$$

Since $Aw = Sw = u$ so by weak compatibility of (A, S) it follows that $SAw = ASw$ *and* So we get

$$Au = ASw = SAw = Su$$

Thus from $2.2.3(iv)$ we have

$$||Aw - Bu||^2 \quad \leq \quad \alpha \max \begin{Bmatrix} ||Aw - Sw||^2 + ||ABu - Tu||^2, \\ ||Aw - Tu||^2 + ||Bu - Sw||^2, \\ ||Aw - Sw||^2 + ||Aw - Tu||^2, \\ ||Bu - Sw||^2 + ||Bu - Tu||^2 \end{Bmatrix}$$

$$||u - Bu||^2 \quad \leq \alpha ||u - Bu||^2$$

which contradiction

implies that $Bu = u$.

Similarly we can show $Au = u$ by using $2.3(iv)$. Therefore

$$u = Au = Bu = Su = Tu.$$

Hence the point u is common fixed point of A, B, S, T.

If we assume that $S(X)$ is complete then the argument analogue to the previous completeness argument proves the theorem. If $A(X)$ is complete then $u \in A(X) \subset T(X)$. similarly if $B(X)$ is complete then $u \in B(X) \subset S(X)$. This complete prove of the theorem.

Uniqueness Let us assume that z is another fixed point of A, B, S, T in X different from u. $i.e.\ u \neq z$ then

$$||u - z||^2 = ||Au - Bz||^2$$

$from$ $6.2.3(iv)$ $we\ get$

$$||u - z||^2 \leq \alpha ||u - z||^2$$

which contradiction the hypothesis . Hence u is unique common fixed point of A, B, S, T in X.

3 FIXED POINT THEOREM SATISFYING RATIONAL TYPE CONTRACTION CONDITION IN PARTIAL ORDERED BANACH SPACES

Fixed point for multivalued functions is a vast chapter of functional analysis. In particular, the function $\delta(A, B)$ has been used in many works in this area. Some of these works are noted in Choudhury [2], Fisher [5] and Fisher and Iseki [6].

We will use the following relation between two non empty subsets of a partially ordered set.

Let (X, d) be a metric space. We denote the class of non empty and bounded subsets of X by $B(X)$. For $A, B \in B(X)$, function $D(A, B)$ and $\delta(A, B)$ are defined as follows:

$$||A - B||_D = inf \{ ||a - b|| : a \in A, b \in B \}$$
$$||A - B||_\delta = sup \{ ||a - b|| : a \in A, b \in B \}$$

If $A = \{ a \}$ then we write $||A - B||_D = ||a - B||_D$ and $||A - B||_\delta = ||a - B||_\delta$. Also in addition, if $B = \{ b \}$, then $||A - B||_D = ||a - b||$ and $||A - B||_\delta = ||a - b||$. Obviously, $||A - B||_D \leq ||A - B||_\delta$. For all $A, B, C \in B(X)$, the definition of $||A - B||_\delta$ yields the following:

$$||A - B||_\delta = ||B - A||_\delta$$
$$||A - B||_\delta \leq ||A - C||_\delta + ||C - B||_\delta$$
$$||A - B||_\delta = 0 \text{ iff } A = B = \{ a \}$$
$$||A - B||_\delta = diam\ A \quad .$$

$\boldsymbol{Definition}\ \mathbf{3(a)}$ Let A and B be two non empty subsets of a partially ordered set $(X, \preccurlyeq)$. The relation between A and B is denoted and defined as follows:

$A \preccurlyeq B$, if for every $a \in A$ there exists $b \in B$ such that $a \preccurlyeq b$.

In 1984, M.S. Khan, M. Swalech and S. Sessa [7] expanded the research of the metric fixed point theory to a new category by introducing a control function which they called an altering distance function.

We will utilize the following control function which is also referred to a Altering distance function.

$\boldsymbol{Definition}\,.\mathbf{3(b)}$ A function $\psi : [0, \infty) \to [0, \infty)$ is called an Altering distance function if the following properties are satisfied:

i. ψ is monotone increasing and continuous,

ii. $\psi(t) = 0$ if and only if $t = 0$.

The above control function has been utilized in a large number of works in metric fixed point theory. Some recent references are Choudhury [2], Doric [4], Dutta and Choudhury [3], Naidu [8] and Sastry and Babu [9]. This control function has also been extended and applied to fixed point problems in probabilistic metric spaces, and fuzzy metric spaces.

The purpose of this chapter is to establish the existence of fixed point if multivalued mappings in partially ordered metric spaces. The mappings are assumed to satisfy certain inequalities which involved the above mentioned control functions. Further we have established that in the corresponding singlevalued cases of partial ordered condition of the metric space can be omitted if the function is continuous.

$\boldsymbol{Theorem}\ \mathbf{3.1}$ Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a norm $||.||$ in X such that (X, d) is a Banach space. Let $T : X \to B(X)$ be a multivalued mapping such that the following conditions are satisfied;

3.1(i) there exists $x_0 \in X$ such that $\{x_0\} \preccurlyeq Tx_0$,

3.1(ii) for $x, y \in X, x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$,

3.1(iii) if $x_n \to x$ is a non decreasing sequence in X, then $x_n \preccurlyeq x$ for all n,

3.1(iv)
$$\psi\left(||Tx - Ty||_{\delta}\right) \le \alpha\, \psi\left(max\left\{\frac{||x-Tx||_D \cdot ||y-Ty||_D}{1+||x-y||}, \frac{||x-Ty||_D \cdot ||y-Tx||_D}{1+||x-y||}\right\}\right) + \beta\, \psi\left(max\left\{\begin{matrix}||x-Tx||_D, ||y-Ty||_D \\ ||x-Ty||_D, ||y-Tx||_D\end{matrix}\right\}\right) + \gamma\, \psi\left(||x-y||\right)$$

For all comparable $x, y \in X$ where $\alpha, \beta, \gamma \in (0,1)$ such that $0 < \alpha + 2\beta + \gamma < 1$ and ψ is an altering distance function. Then T has a fixed point.

$\boldsymbol{Proof}$ By the assumption 3.1(i) there exists $x_1 \in Tx_0$ such that $x_0 \preccurlyeq x_1$. By the assumption 3.1(ii), $Tx_0 \preccurlyeq Tx_1$. Then there $exists\ x_2 \in Tx_1$ such that $x_1 \preccurlyeq x_2$. Continuing the process we construct a monotone increasing sequence $\{x_n\}$ in X such $that\ x_{n+1} \in Tx_n$ for all $n \ge 0$. Thus we have

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq x_3 \preccurlyeq \ldots\ldots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \ldots..$$

If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T. Hence we shall assume that $x_n \ne x_{n+1}$ for all $n \ge 0$.

Using the monotone property $of\ \psi$ and the condition 3.1 (iv), we have for all $n \ge 0$,

$$\psi\left(||x_{n+1} - x_{n+2}||\right) \le \psi\left(||Tx_n - Tx_{n+1}||_{\delta}\right)$$

$$\psi\left(||Tx_n - Tx_{n+1}||_{\delta}\right) \le \alpha\, \psi\left(max\left\{\begin{matrix}\frac{||x_n-Tx_n||_D \cdot ||x_{n+1}-Tx_{n+1}||_D}{1+||x_n-x_{n+1}||}, \\ \frac{||x_n-Tx_{n+1}||_D \cdot ||x_{n+1}-Tx_n||_D}{1+||x_n-x_{n+1}||}\end{matrix}\right\}\right) + \beta\, \psi\left(max\left\{\begin{matrix}||x_n - Tx_n||_D, ||x_{n+1} - Tx_{n+1}||_D, \\ ||x_n - Tx_{n+1}||_D, ||x_{n+1} - Tx_n||_D\end{matrix}\right\}\right) + \gamma\psi\left(||x_n - x_{n+1}||\right)$$

$$\psi\left(||x_{n+1} - x_{n+2}||\right) \le \alpha\, \psi\left(max\left\{\begin{matrix}\frac{||x_n-x_{n+1}|| \cdot ||x_{n+1}-x_{n+2}||}{1+||x_n-x_{n+1}||}, \\ \frac{||x_n-x_{n+2}|| \cdot ||x_{n+1}-x_{n+1}||}{1+||x_n-x_{n+1}||}\end{matrix}\right\}\right)$$

$$+\beta\,\psi\left(max\left\{\begin{matrix}||x_n - x_{n+1}||, ||x_{n+1} - x_{n+2}||,\\ ||x_n - x_{n+2}||, ||x_{n+1} - x_{n+1}||\end{matrix}\right\}\right)$$
$$+\gamma\,\psi\left(||x_n - x_{n+1}||\right)$$

$$\psi\left(d(x_{n+1}, x_{n+2})\right) \le \alpha\,\psi\left(max\left\{||x_{n+1} - x_{n+2}||, 0\right\}\right)$$
$$+\beta\,\psi\left(max\left\{\begin{matrix}||x_n - x_{n+1}||, ||x_{n+1} - x_{n+2}||,\\ ||x_n - x_{n+2}||, 0\end{matrix}\right\}\right)$$
$$+\gamma\,\psi\left(||x_n - x_{n+1}||\right)$$

There arise two four cases.

Case – 1 if we take $max\left\{\begin{matrix}||x_n - x_{n+1}||, ||x_{n+1} - x_{n+2}||,\\ ||x_n - x_{n+2}||, 0\end{matrix}\right\} = ||x_n - x_{n+1}||$

then,

$$\psi\left(||x_{n+1} - x_{n+2}||\right) \le \frac{\gamma}{1-\alpha-\beta}\ \psi\left(||x_n - x_{n+1}||\right)$$

Case - 2, if we take $max\left\{\begin{matrix}||x_n - x_{n+1}||, ||x_{n+1} - x_{n+2}||,\\ ||x_n - x_{n+2}||, 0\end{matrix}\right\} = ||x_{n+1} - x_{n+2}||$

then,

$$\psi\left(||x_{n+1} - x_{n+2}||\right) \le \frac{\beta+\gamma}{1-\alpha-\beta}\ \psi\left(||x_n - x_{n+1}||\right)$$

Case - 3, if we take $max\left\{\begin{matrix}||x_n - x_{n+1}||, ||x_{n+1} - x_{n+2}||,\\ ||x_n - x_{n+2}||, 0\end{matrix}\right\} = ||x_n - x_{n+2}||$

$$\psi\left(||x_{n+1} - x_{n+2}||\right) \le \frac{\beta+\gamma}{1-\alpha-\beta}\ \psi\left(||x_n - x_{n+1}||\right)$$

Case - 4, if we take $max\left\{\begin{matrix}||x_n - x_{n+1}||, ||x_{n+1} - x_{n+2}||,\\ ||x_n - x_{n+2}||, 0\end{matrix}\right\} = 0$ then,

$$\psi\left(||x_{n+1} - x_{n+2}||\right) \le \frac{\gamma}{1-\alpha}\ \psi\left(||x_n - x_{n+1}||\right)$$

Since $0 < \alpha + 2\beta + \gamma < 1$ in both cases, this implies

$$\psi\left(||x_{n+1} - x_{n+2}||\right) \le k\ \psi\left(||x_n - x_{n+1}||\right) \tag{2.1}$$

where $k = max\left\{\frac{\gamma}{1-\alpha-\beta}, \frac{\beta+\gamma}{1-\alpha-\beta}, \frac{\beta+\gamma}{1-\alpha-\beta}, \frac{\gamma}{1-\alpha}\right\}$.

Therefore, $||x_{n+1} - x_{n+2}|| < ||x_n - x_{n+1}||$ for all $n \ge 0$ and $\{||x_n - x_{n+1}||\}$ is monotone decreasing sequence of non negative real numbers. Hence there exists an $r \ge 0$ such that,

$$||x_n - x_{n+1}|| \to r \quad as \quad n \to \infty. \qquad 3.1(a)$$

Taking the limit as $n \to \infty$ in $6.3.1(iv)$and using the continuity of ψ, e have

$$\psi\,(r) \le k\,\psi\,(r)$$

which is a contradiction unless $r = 0$.

Hence,

$$lim_{n\to\infty}\,||x_n - x_{n+1}|| = 0 \qquad 3.1(b)$$

Next we show that $\{x_n\}$ is a Cauchy sequence. If otherwise, there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $\left||x_{m(k)} - x_{n(k)}|\right| \ge \epsilon$.

Assume that $n(k)$ is the smallest such positive integer we get, $n(k) > m(k) > k$

$$\left||x_{m(k)} - x_{n(k)}|\right| \ge \epsilon$$

and

$$\left||x_{m(k)} - x_{n(k)-1}|\right| < \epsilon.$$

Now,

$$\epsilon \le \left||x_{m(k)} - x_{n(k)}|\right|$$
$$\le \left||x_{m(k)} - x_{n(k)-1}|\right| + \left||x_{n(k)-1} - x_{n(k)}|\right|$$

that is,

$$\epsilon \leq \left\|x_{m(k)} - x_{n(k)}\right\| \leq \epsilon + \left\|x_{n(k)-1} - x_{n(k)}\right\|$$

Taking the limit as $k \to \infty$ in the above inequality and 6.3.1(b)we have

$$lim_{n\to\infty} \left\|x_{m(k)} - x_{n(k)}\right\| = \epsilon \qquad 3.1(c)$$

Again,

$$\left\|x_{m(k)} - x_{n(k)}\right\| \leq \left\|x_{m(k)} - x_{m(k)+1}\right\| + \left\|x_{m(k)+1} - x_{n(k)+1}\right\| + \left\|x_{n(k)+1} - x_{n(k)}\right\|$$

and,

$$\left\|x_{m(k)+1} - x_{n(k)+1}\right\| \leq \left\|x_{m(k)+1} - x_{m(k)}\right\| + \left\|x_{m(k)} - x_{n(k)}\right\| + \left\|x_{n(k)} - x_{n(k)+1}\right\|$$

Taking the limit as $k \to \infty$ in the above inequality and 6.3.1(b) and 6.3.1(c) we have,

$$lim_{n\to\infty} \left\|x_{m(k)+1} - x_{n(k)+1}\right\| = \epsilon \qquad 3.1(d)$$

Again,

$$\left\|x_{m(k)} - x_{n(k)}\right\| \leq \left\|x_{m(k)} - x_{n(k)+1}\right\| + \left\|x_{n(k)+1} - x_{n(k)}\right\|$$

and,

$$\left\|x_{m(k)} - x_{n(k)+1}\right\| \leq \left\|x_{m(k)} - x_{n(k)}\right\| + \left\|x_{n(k)} - x_{n(k)+1}\right\|$$

Taking the limit as $k \to \infty$ in the above inequality and 3.1$(b)$and 3.1(c) we have,

$$lim_{n\to\infty} \left\|x_{m(k)} - x_{n(k)+1}\right\| = \epsilon \qquad 3.1(e)$$

Similarly we have that

$$lim_{n\to\infty} \left\|x_{n(k)} - x_{m(k)+1}\right\| = \epsilon \qquad 3.1(f)$$

For each positive integer k, $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ are comparable. Then using the monotone property of ψ and the condition (iv), we have

$$\psi\left(\left\|x_{m(k)+1} - x_{n(k)+1}\right\|\right) \leq \psi\left(\left\|Tx_{m(k)} - Tx_{n(k)}\right\|_{\delta}\right)$$

$$\psi\left(\left\|Tx_{m(k)} - Tx_{n(k)}\right\|_{\delta}\right) \leq \alpha\,\psi\left(max\left\{\frac{\left\|x_{m(k)}-Tx_{m(k)}\right\|_{D}\cdot\left\|x_{n(k)}-Tx_{n(k)}\right\|_{D}}{1+\left\|x_{m(k)}-x_{n(k)}\right\|}, \frac{\left\|x_{m(k)}-Tx_{n(k)}\right\|_{D}\cdot\left\|x_{n(k)}-Tx_{m(k)}\right\|_{D}}{1+\left\|x_{m(k)}-x_{n(k)}\right\|}\right\}\right) + \beta\,\psi\left(max\left\{\begin{array}{l}\left\|x_{m(k)} - Tx_{m(k)}\right\|_{D}, \left\|x_{n(k)} - Tx_{n(k)}\right\|_{D}, \\ \left\|x_{m(k)} - Tx_{n(k)}\right\|_{D}, \left\|x_{n(k)} - Tx_{m(k)}\right\|_{D}\end{array}\right\}\right) + \gamma\,\psi\left(\left\|x_{m(k)} - x_{n(k)}\right\|\right)$$

By using (iv) and on taking limit as $k \to \infty$ in the above inequality and 6.3.1(b) and using the continuity of ψ we have,

$$\psi(\epsilon) \leq k\,\psi(\epsilon)$$

which is contradiction by virtue of a property of ψ.

Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists a $z \in X$ such that

$$x_n \to z \quad as \quad n \to \infty \qquad 3.1(g)$$

By the assumption (iii), $x_n \preccurlyeq z$, for all n.

Then by the monotone property of ψ and the condition (iv), we have

$$\psi\left(\|x_{n+1} - Tz\|\right) \leq \psi\left(\|Tx_n - T(z)\|_{\delta}\right)$$

By using (iv) and on taking limit as $k \to \infty$ in the above inequality from 6.3.1$(b)$and 6.3.1(f) and using the continuity of ψ we have,

$$\psi\left(\left\|z-Tz\right\|_{\delta}\right)\leq\ k\,\psi\left(\left\|z-Tz\right\|_{D}\right)\leq\ k\,\psi\ \left(\left\|z-Tz\right\|_{\delta}\right),$$

which implies that, $\left\|z-Tz\right\|_{\delta}\ =\ 0$ or that $\{\,z\,\}=\ Tz$. Moreover, z is a fixed point of T.

***Corollary* 3.2** Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there exists a norm $||.||$ in X such that (X,d) is a Banach space. Let $T:\ X\rightarrow B(X)$ be a multivalued mapping such that the following conditions are satisfied;

3.2(i) there exists $x_0\in X$ such that $\{\,x_0\}\preccurlyeq\ Tx_0$,

3.2(ii) for $x,y\in\ X,x\preccurlyeq\ y$ implies $Tx\preccurlyeq Ty$,

3.2(iii) if $x_n\rightarrow\ x$ is a non decreasing sequence in X, then $x_n\preccurlyeq\ x$ for all n,

3.2(iv) $$\left\|Tx-Ty\right\|_{\delta}\leq\alpha\,max\left\{\frac{\left\|x-Tx\right\|_{D}.\left\|y-Ty\right\|_{D}}{1+\left\|x-y\right\|},\frac{\left\|x-Ty\right\|_{D}.\left\|y-Tx\right\|_{D}}{1+\left\|x-y\right\|}\right\} +\beta\,max\left\{\begin{matrix}\left\|x-Tx\right\|_{D},\left\|y-Ty\right\|_{D}\\ \left\|x-Ty\right\|_{D},\left\|y-Tx\right\|_{D}\end{matrix}\right\} +\gamma\left\|x-y\right\|$$

For all comparable $x,y\in\ X$ where $\alpha\,,\beta\,,\gamma\in\ (0,1)$ such that $0\ <\alpha\ +\ 2\beta+\gamma\ <\ 1$. Then T has a fixed point.

Proof On taking ψ be an identity function in Theorem 6.3.1, and then the above result is true and noting to prove.

The following corollary is a special case of Theorem 3.1 when T is a singlevalued mapping.

***Corollary* 3.3** Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there exists a norm $||.||$ in X such that (X,d) is a Banach space. Let $T:\ X\rightarrow\ X$ be a mapping such that the following conditions are satisfied;

3.3(i) there exists $x_0\in\ X$ such that $\{\,x_0\}\preccurlyeq Tx_0$,

3.3(ii) for $x,y\in\ X,x\preccurlyeq\ y\ implies\ Tx\preccurlyeq\ Ty,$

3.3(iii) if $x_n\rightarrow\ x$ is a non decreasing sequence in X, then $x_n\preccurlyeq\ x$ for all n,

3.3(iv) $$\psi\left(\left\|Tx-Ty\right\|_{\delta}\right)\leq\ \psi\left(max\left\{\begin{matrix}\frac{\left\|x-Tx\right\|_{D}.\left\|y-Ty\right\|_{D}}{1+\left\|x-y\right\|},\\ \frac{\left\|x-Ty\right\|_{D}.\left\|y-Tx\right\|_{D}}{1+\left\|x-y\right\|}\end{matrix}\right\}\right) +\beta\,\psi\left(max\left\{\begin{matrix}\left\|x-Tx\right\|_{D},\left\|y-Ty\right\|_{D}\\ \left\|x-Ty\right\|_{D},\left\|y-Tx\right\|_{D}\end{matrix}\right\}\right) +\gamma\,\psi\left(\left\|x-y\right\|\right)$$

For all comparable $x,y\in X$ where $\alpha,\beta,\gamma\in(0,1)$ such that $0<\alpha+2\beta+\gamma<1$ and ψ is an altering distance function. Then T has a fixed point.

In the following theorem we replace condition 2.3(iii) of the above corollary by requiring T to be continuous.

Theorem 3.4 Let $(\mathrm{X},\preccurlyeq)$ be a partially ordered set and suppose that there exists a norm $||.||$ in X such that (X, d) is a Banach space. Let $\mathrm{T}:\ \mathrm{X}\rightarrow\ \mathrm{X}$ be a mapping such that the following conditions are satisfied;

3.4(i) there exists $\mathrm{x}_0\in\ \mathrm{X}$ such that $\{\,\mathrm{x}_0\}\preccurlyeq\ $ Tx_0 ,

3.4(ii) for $\mathrm{x},\mathrm{y}\in\ \mathrm{X},\mathrm{x}\preccurlyeq\ \mathrm{y}$ implies $\mathrm{Tx}\preccurlyeq\ \mathrm{Ty}$,

3.4(iii) $$\psi\left(\left\|\mathrm{Tx}-\mathrm{Ty}\right\|_{\delta}\right)\leq\alpha\,\psi\left(\max\left\{\begin{matrix}\frac{\left\|\mathrm{x}-\mathrm{Tx}\right\|_{\mathrm{D}}.\left\|\mathrm{y}-\mathrm{Ty}\right\|_{\mathrm{D}}}{1+\left\|\mathrm{x}-\mathrm{y}\right\|},\\ \frac{\left\|\mathrm{x}-\mathrm{Ty}\right\|_{\mathrm{D}}.\left\|\mathrm{y}-\mathrm{Tx}\right\|_{\mathrm{D}}}{1+\left\|\mathrm{x}-\mathrm{y}\right\|}\end{matrix}\right\}\right) +\beta\,\psi\left(\max\left\{\begin{matrix}\left\|\mathrm{x}-\mathrm{Tx}\right\|_{\mathrm{D}},\left\|\mathrm{y}-\mathrm{Ty}\right\|_{\mathrm{D}}\\ \left\|\mathrm{x}-\mathrm{Ty}\right\|_{\mathrm{D}},\left\|\mathrm{y}-\mathrm{Tx}\right\|_{\mathrm{D}}\end{matrix}\right\}\right) +\gamma\,\psi\left(\left\|\mathrm{x}-\mathrm{y}\right\|\right)$$

For all comparable $\mathrm{x},\mathrm{y}\in\ \mathrm{X}$ where $\alpha\,,\beta\,,\gamma\in\ (0,1)$ such that $0\ <\alpha\ +\ 2\beta\ +\gamma\ <\ 1$ and ψ is an altering distance function. Then T has a fixed point.

Proof: We can treat T as a multivalued mapping in which case Tx is a singleton set for every $\mathrm{x}\in\ \mathrm{X}$. Then we consider the same sequence $\{\mathrm{x}_{\mathrm{n}}\}$ as in the proof of Theorem 3.1, Arguing exactly as in the proof of Theorem 3.1, we have that $\{\,\mathrm{x}_{\mathrm{n}}\}$ is a Cauchy sequence and $\lim_{\mathrm{n}\rightarrow\infty}\ (\mathrm{x}_{\mathrm{n}})\ =\ \mathrm{z}$. Then the continuity of T implies that,

$$\mathrm{z}\ =\ \lim_{\mathrm{n}\rightarrow\infty}(\mathrm{x}_{\mathrm{n}+1})\ =\ \lim_{\mathrm{n}\rightarrow\infty}\mathrm{T}(\mathrm{x}_{\mathrm{n}})\ =\ \mathrm{Tz}$$

and this proves that z is a fixed point of T.

REFERENCES

1. Browder F. E. " Non expansive non linear operators in Banach Spaces," proc. Nat. Acad. Sci. U. S. A., 54 (1965) 1041-1044.
2. Choudhury B.S. , `Certain common fixed point theorems in complete metric spaces', Soochow J Math. 22(3)(1996) 427-34.
3. Datta P.N. and Choudhury B.S. , `A generalisation of contaction principle in metric spaces', Fixed Point Theory and Appl. (2008) article ID 406368.
4. Doric D., `Common fixed point for generalized (ψ ,ϕ) - weak contractions.', Appl. Math. Lett. 22 (2009) 1896-900.
5. Fisher B., `Common fixed point of mappings and setvalued mappings', Rostock Math Colloq 18(1981) 69-77.
6. Fisher B. and Ise'ki K., `Fixed points for set valued mappings on complete and compact metric spaces', Math. Japonica 28(1983) 639-46.
7. Khan M.S., Swaleh M. and Sessa S., "Fixed point theorems by altering distances between the points." Bull. Austral. Math. Soc., 30 (1984) 1- 9.
8. Naidu S.V.R., `some fixed point theorems in metric spaces by altering distances', Cezechoslovak Math. J.53(1) (2003) 205-12.
9. Sastry K.P.R. and Babu G.V.R., `Some fixed point theorems by altering distances between the points.', Ind. J Pure Appl. Math. 30(6) (1999) 641-7.