

Some Coupled coincidence and common fixed point theorems for hybrid pair of mappings

Rajesh Shrivastava*, S. K. Pandey**, R.K. Bhardwaj ***and R.N. Yadava****
* Department of Mathematics, Govt. Science and Commerce college Benzir Bhopal, (M.P), India
**Department of Mathematics, Millennium Group of Institutions, Bhopal, (M.P.), India
(sudeeppandey1976@gmail.com)
***Department of Mathematics, Truba Institute of Technology and Science, Bhopal, (M.P.), India
****Group Director, Patel Institute of Technology, Bhopal, (M.P.) India

Abstract
In this paper we extend the multi-valued mappings and obtain coupled coincidence points and common coupled fixed point theorems involving hybrid pair of single valued and multi-valued maps satisfying generalized contractive conditions in the frame work of a complete metric space.
Keywords: coupled common fixed point, coupled coincidence point, coupled point of coincidence, w-compatible mappings, F-weakly commuting mappings

1. Introduction and preliminaries

Let (X, d) be a metric space. For $x \in X$ and $A \in X$, we denote $d(x, A) = \inf\{d(x, A): y \in A\}$. The class of all nonempty bounded and closed subsets of X is denoted by CB(X). Let H be the Hausdorff metric induced by the metric d on X, that is,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\},$$

for every $A, B \in CB(X)$.

Lemma. 1.1 [1]Let $A, B \in CB(X)$, and $a >1$. Then, for every $a \in A$, there exists $b \in B$ such that $d(a, b) \leq aH(A, B)$.

Lemma .1.2 [2]Let $A, B \in CB(X)$, then for any $a \in A$, $d(a, B) \leq H(A, B)$.

Definition . 1.3 Let X be a nonempty set, $F : X \times X \to 2^X$ (collection of all nonempty subsets of X) and $g : X \to X$. An element $(x, y) \in X \times X$ is called (i) coupled fixed point of F if $x \in F(x, y)$ and $y \in F(y, x)$ (ii) coupled coincidence point of a hybrid pair {F, g} if $g(x) \in F(x, y)$ and $g(y) \in F(y, x)$ (iii) coupled common fixed point of a hybrid pair {F, g} if $x = g(x) \in F(x, y)$ and $y = g(y) \in F(y, x)$.

We denote the set of coupled coincidence point of mappings F and g by C(F, g). Note that if $(x, y) \in C(F, g)$, then (y, x) is also in C(F, g).

Definition . 1.4 Let $F : X \times X \to 2^X$ be a multi-valued mapping and g be a self map on X. The hybrid pair {F, g} is called w- compatible if $g(F(x, y)) \subseteq F(gx, gy)$ whenever $(x, y) \subseteq C(F, g)$.

Definition . 1.5 Let $F : X \times X \to 2^X$ be a multi-valued mapping and g be a self-mapping on X. The mapping g is called F- weakly commuting at some point $(x, y) \in X \times X$ if $g^2 (x) \in F(gx, gy)$ and $g^2(y) \in F(gy, gx)$.

Bhaskar and Lakshmikantham [3] introduced the concept of coupled fixed point of a mapping F from $X \times X$ to X and established some coupled fixed point theorems in partially ordered sets. As an application, they studied the existence and uniqueness of solution for a periodic boundary value problem associated with a first order ordinary differential equation. Ćirić et al. [4] proved coupled common fixed point theorems for mappings satisfying nonlinear contractive conditions in partially ordered complete metric spaces and generalized the results given in [3]. Sabetghadam et al. [5] employed these concepts to obtain coupled fixed point in the frame work of cone metric spaces. Lakshmikantham and Ćirić [4] introduced the concepts of coupled coincidence and coupled common fixed point for mappings satisfying nonlinear contractive conditions in partially ordered complete metric spaces. The study of fixed points for multi-valued contractions mappings using the Hausdorff metric was initiated by Nadler [1] and Markin [6]. Later, an interesting and rich fixed point theory for such maps was devel-oped which has found applications in control theory, convex optimization, differential inclusion and economics (see [7] and references therein). Klim and Wardowski [8] also obtained existence of fixed point for set-valued contractions in complete metric spaces. Dhage [9,10]established hybrid fixed point theorems and gave some applications (see also [11]). Hong in his recent study [12] proved hybrid fixed point theorems involving multi-valued operators which satisfy weakly generalized contractive conditions in ordered complete metric spaces. The study of coincidence point and common fixed points of hybrid pair of mappings in Banach spaces and metric spaces is interesting and well developed. For applications of hybrid fixed point theory we refer to [13-16]. For a survey of fixed point theory and coincidences of multimaps, their applications and related results, we refer to [16-22].

The aim of this article is to obtain coupled coincidence point and common fixed point theorems for a pair of multi-valued and single valued mappings which satisfy generalized contractive condition in complete metric spaces. It is to be noted that to find coupled coincidence points, we do not employ the condition of continuity of any mapping involved therein. Our results unify, extend, and generalize various known comparable results in the literature.

2. Main results

In the following theorem we obtain coupled coincidence and common fixed point for hybrid pair of mappings satisfying a generalized contractive condition.

Theorem . 2.1 Let (X, d) be a metric space, $F : X \times X \to CB(X)$ and $g : X \to X$ be map-pings satisfying

$$H(F(x, y), F(u, v)) \leq a_1 d(F(x, y), gx) + a_2 d(F(x, y), gu) + a_3 d(F(u, v), gx) + a_4 d(F(u, v), gu) \quad (2.1)$$

for all $x, y, u, v \in X$, where $a_i = a_i(x, y, u, v)$, $i = 1, 2, ..., 4$, are nonnegative real numbers such that

$$a_1 + a_2 + a_3 + a_4 \leq h<1 \quad (2.2)$$

where h is a fixed number. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is complete subset of X, then F and g have coupled coincidence point. Moreover F and g have coupled common fixed point if one of the following conditions holds.

(a) F and g are w- compatible, $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x, y) \in C(F, g)$, $u, v \in X$ and g is continuous at u and v.

(b) g is F- weakly commuting for some $(x, y) \in C(g, F)$, $g^2x = gx$ and $g^2y = gy$.

(c) g is continuous at x, y for some $(x, y) \in C(g, F)$ and for some $u, v \in X$, $\lim_{n\to\infty} g^n v = y$ and $\lim_{n\to\infty} g^n v = y$

(d) $g(C(g, F))$ is singleton subset of $C(g, F)$.

Proof. Let $x_0, y_0 \in X$ be arbitrary. Then $F(x_0, y_0)$ and $F(y_0, x_0)$ are well defined. Choose $gx_1 \in F(x_0, y_0)$ and $gy_1 \in F(y_0, x_0)$. This can be done because $F(X \times X) \subseteq g(X)$. If $a_1 = a_2 = a_3 = a_4 = 0$, then

$d(gx_1, F(x_1, y_1)) \leq H(F(x_0, y_0), F(x_1, y_1)) = 0.$

Hence $d(gx_1, F(x_1, y_1)) = 0$. Since $F(x_1, y_1)$ is closed, $gx_1 \in F(x_1, y_1)$. Similarly $gy_1 \in F(y_1, x_1)$. Thus (x_1, y_1) is a coupled coincidence point of $\{F, g\}$ and so we finish the proof. Now assume that $a_i >0$, for some $i = 1, ..., 4$. Then $h>0$ and so there exist $z_1 \in F(x_1, y_1)$ and $z_2 \in F(y_1, x_1)$ such that

$$d(gx_1, z_1) \leq H(F(x_0, y_0), F(x_1, y_1)) + \frac{h^n}{2}$$

$$d(gy_1, z_2) \leq H(F(y_0, x_0), F(y_1, x_1)) + \frac{h^n}{2}$$

Since $F(X \times X) \subseteq g(X)$, there exist x_2 and y_2 in X such that $z_1 = gx_2$ and $z_2 = gy_2$.

Thus

$$d(gx_1, gx_2) \leq H(F(x_0, y_0), F(x_1, y_1))+ \frac{h^n}{2}$$

$$d(gy_1, gy_2) \leq H(F(y_0,x_0), F(y_1,x_1)) + \frac{h^n}{2}$$

Continuing this process, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$gx_{n+1} \in F(x_n, y_n)$ and $gy_{n+1} \in F(y_n, x_n)$,

$d(gx_n , gx_{n+1}) \leq H(F(x_{n-1} , y_{n-1}), F(x_n , y_n)) + \frac{h^n}{2}$

$d(gy_n , gy_{n+1}) \leq H(F(y_{n-1} ,x_{n-1}), F(y_n ,x_n)) + \frac{h^n}{2}$

From (2.1), we have
$d(gx_n , gx_{n+1})$

$\leq H(F(x_{n-1} , y_{n-1}), F(x_n , y_n)) + \frac{h^n}{2}$

$\leq a_1 d(F(x_{n-1} , y_{n-1}), gx_{n-1}) + a_2 d(F(x_{n-1} , y_{n-1}), gx_n) + a_3 d(F(x_n, y_n), gx_{n-1})$

$+ a_4 d(F(x_n , y_n), gx_n) + \frac{h^n}{2}$

$\leq a_1 d(gx_n ,gx_{n-1}) + a_3 d(gx_{n+1} ,gx_{n-1}) + a_4 d(gx_{n+1} ,gx_n) + \frac{h^n}{2}$

$\leq a_1 d(gx_n ,gx_{n-1}) + a_3 d(gx_{n+1} ,gx_n) + a_3 d(gx_n ,gx_{n-1}) + a_4 d(gx_{n+1}, gx_n) + \frac{h^n}{2}$

$(a_1 + a_3)d(gx_{n-1} , gx_n) + (a_3 + a_4)d(gx_n , gx_{n+1}) + \frac{h^n}{2}$

and it follows that

$$(1 - a_3 - a_4) d(gx_n , gx_{n+1}) \leq (a_1 + a_3) d(gx_{n-1} , gx_n) + \frac{h^n}{2} \qquad (.2.3)$$

Similarly it can be shown that,

$$(1 - a_3 - a_4) d(gy_n , gy_{n+1}) \leq (a_1 + a_3) d(gy_{n-1} , gy_n) + \frac{h^n}{2} \qquad (2.4)$$

Again,
$d(gx_{n+1} , gx_n)$

$\leq H(F(x_n , y_n),F(x_{n-1} , y_{n-1})) + \frac{h^n}{2}$

$\leq a_1 d(F(x_n , y_n), gx_n) + a_2 d(F(x_n , y_n), gx_{n-1}) + a_3 d(F(x_{n-1} , y_{n-1}), gx_n)$

$+ a_4 d(F(x_{n-1} , y_{n-1}), gx_{n-1}) + \frac{h^n}{2}$

$\leq a_1 d(gx_{n+1} ,gx_n) + a_2 d(gx_{n+1} ,gx_{n-1}) + a_4 d(gx_{n+1} ,gx_{n-1}) + \frac{h^n}{2}$

$\leq a_1 d(gx_{n+1} ,gx_n) + a_2 d(gx_{n+1} ,gx_n) + a_2 d(gx_n ,gx_{n-1}) + a_4 d(gx_n ,gx_{n-1}) + \frac{h^n}{2}$

$\leq (a_1 + a_2)d(gx_{n+1} , gx_n) + (a_2 + a_4)d(gx_n , gx_{n-1}) + \frac{h^n}{2}$

Hence

$$(1 - a_1 - a_2) d(gx_{n+1} , gx_n) \leq (a_2 + a_4) d(gx_{n-1} , gx_n) + \frac{h^n}{2} \qquad (2.5)$$

And

$(1 - a_1 - a_2)\, d(gy_{n+1}, gy_n) \le (a_2 + a_4)\, d(gy_{n-1}, gy_n) + \dfrac{h^n}{2}$ (2.6)

Let
$\delta_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})$.
Now, from (2.3) and (2.4), and respectively (2.5) and (2.6), we obtain:

$(1 - a_3 - a_4)\, \delta_n \le (a_1 + a_3)\, \delta_{n-1} + \dfrac{h^n}{2}$ (2.7)

$(1 - a_1 - a_2)\, \delta_n \le (a_2 + a_4)\, \delta_{n-1} + \dfrac{h^n}{2}$ (2.8)

Adding (2.7) and (2.8) we get
$(2 - a_1 - a_2 - a_3 - a_4)\, \delta_n \le (a_1 + a_2 + a_3 + a_4)\, \delta_{n-1} + h^n$. (2.9)
Since by (2.2), $a_1 + a_2 + a_3 + a_4 \le h < 1$, so we have

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= 2(a_1 + a_2 + a_3 + a_4) - a_1 - a_2 - a_3 - a_4 \\ &\le 2h - (a_1 + a_2 + a_3 + a_4) \\ &\le 2h - h(a_1 + a_2 + a_3 + a_4) \\ &= h\,(2 - a_1 - a_2 - a_3 - a_4). \end{aligned}$$

Thus from (2.9) we get
$(2 - a_1 - a_2 - a_3 - a_4)\, \delta_n \le h\,(2 - a_1 - a_2 - a_3 - a_4)\, \delta_{n-1} + h^n$.
Hence, as $1/(2 - a_1 - a_2 - a_3 - a_4) < 1$,
$\delta_n \le h\delta_{n-1} + h^n$.
Thus we have
$\delta_n \le h(h\delta_{n-2} + h^{n-1}) + h^n = h^2\, \delta_{n-2} + 2h^n$.
Continuing this process we obtain
$\delta_n \le h^n\, \delta_0 + nh^n$ (2.10)
By the triangle inequality and (2.10), for m, n $\in$ N with m > n, we have
$d(gx_n, gx_{m+n}) + d(gy_n, gy_{m+n})$
$\le d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{n+m-1}, gx_{m+n}) + d(gy_n, gy_{n+1})$
$+ d(gy_{n+1}, gy_{n+2}) \ldots + d(gy_{n+m-1}, gy_{m+n})$
$\le (h^n\delta_0 + nh^n) + (h^{n+1}\delta_0 + (n+1)h^{n+1}) + \cdots + (h^{n+m-1}\delta_0 + (n+m-1)h^{n+m-1}) + (h^n\delta_0 + nh^n) + (h^{n+1}\delta_0 + (n+1)h^{n+1}) + \cdots + (h^{n+m-1}\delta_0$
$+ (n+m-1)h^{n+m-1})$.
Thus

$$d(gx_n, gx_{m+n}) + d(gy_n, gy_{m+n}) \le \sum_{i=n}^{n+m-1} \delta_0 h^i + \sum_{i=n}^{n+m-1} i h^i$$

Since h <1, we conclude that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in g(X). Since g(X) is complete, there exist x, y $\in$ X such that $gx_n \to gx$ and $gy_n \to gy$. Then from (2.1), we obtain

$$\begin{aligned} d(F(x, y), gx) &\le d(F(x, y), gx_{n+1}) + d(gx_{n+1}, gx) \\ &\le H(F(x, y), F(x_n, y_n)) + d(gx_{n+1}, gx) \\ &\le a_1 d(F(x, y), gx) + a_2\, d(F(x, y), gx_n) \\ &\quad + a_3\, d(F(x_n, y_n), gx) + a_4 d(F(x_n, y_n), gx_n) + d(gx_{n+1}, gx) \\ &\le a_1 d(F(x, y), gx) + a_2\, d(F(x, y), gx_n) \\ &\quad + a_3\, d(gx_{n+1}, gx) + a_4 d(gx_{n+1}, gx_n) + d(gx_{n+1}, gx) \end{aligned}$$

On taking limit as $n \to \infty$, we have
$d(F(x, y), gx) \le (a_1 + a_2) d(F(x, y), gx)$,
which implies that $d(F(x, y), gx) = 0$ and hence $F(x, y) = gx$. Similarly, $F(y, x) = gy$. Hence (x, y) is coupled coincidence point of the mappings F and g. Suppose now that (a) holds. Then for some (x, y) $\in C(F, g)$, $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$, where u, v $\in$ X. Since g is continuous at u and v, so we have that u and v are fixed points of g. As F and g are w- compatible, $g^n x \in C(F, g)$ for all $n \ge 1$ and $g^n x \in F(g^{n-1}x, g^{n-1}y)$.

Using (6.2.1), we obtain,

$d(gu, F(u, v)) \leq d(gu, g^n x) + d(g^n x, F(u, v))$
$\leq d(gu, g^n x) + H(F(g^{n-1} x, g^{n-1} y), F(u, v))$
$\leq d(gu, g^n x) + a_1 d(F(g^{n-1} x, g^{n-1} y), g^n x) + a_2 d(g^n x, gu) + a_3 d(F(u, v), g^n x) + a_4 d(F(u, v), gu).$

On taking limit as $n \to \infty$, we have

$d(gu, F(u, v)) \leq (a_3 + a_4)d(gu, F(u, v)),$

which implies $d(gu, F(u, v)) = 0$ and hence $gu \in F(u, v)$. Similarly, $gv \in F(v, u)$. Con-sequently $u = gu \in F(u, v)$ and $v = gv \in F(v, u)$. Hence (u, v) is a coupled fixed point of Fand g. Suppose now that (b) holds.
If for some $(x, y) \in C(F, g)$, g is F- commuting, $g^2x = gx$ and $g^2y = gy$, then $gx = g^2x \in F(gx, gy)$ and $gy = g^2y \in F(gy, gx)$. Hence (gx, gy) is a coupled fixed point of F and g. Suppose now that (c) holds and assume that for some $(x, y) \in C(g, F)$ and for some $u, v \in X$, $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$. By the continuity of g at x and y, we get $x = gx \in F(x, y)$ and $y = gy \in F(y, x)$. Hence (x, y) is coupled fixed point of F and g. Finally, suppose that (d) holds. Let $g(C(F, g)) = \{(x, x)$. Then $\{x\} = \{gx\} = F(x, x)$. Hence (x, x) is coupled fixed point of F and g. ■

Now we present following example to support our Theorem(2.1).

Exam ple 3. Let $X = [1, 5]$ and $F : X \times X \to CB(X)$, $g : X \to X$ be defined as follows:

$F(x, y) = [2, 3]$ for all $x, y \in X$,

$g(x) = 5 - \frac{3}{5}x$, for all $x \in X$.

Then $H(F(x, y), F(u, v)) = 0$ for all $x, y, u, v \in X$. Therefore, F and g satisfy (2.1) for any $a_i \in [0, 1)$, $i = 1, 2, ..., 4$. Also $(2,4) \in X \times X$ is a coupled coincidence point of hybrid pair $\{F, g\}$. Note that F and g do not satisfy anyone of the conditions from (a)-
(d) of Theorem (2.1) and do not have a coupled common fixed point.

If in Theorem (2.1) $g = I$ (I = the identity mapping), then we have the following result.

Corollary.4. Let (X, d) be a complete metric space, $F : X \times X \to CB(X)$ be a map-ping satisfying

$$H(F(x, y), F(u, v)) \leq a_1 d(F(x, y), x) + a_2 d(F(x, y), u) + a_3 d(F(u, v), x) + a_4 d(F(u, v), u)$$

for all $x, y, u, v \in X$, where $a_i = a_i(x, y, u, v)$, $i = 1, 2, ..., 4$, satisfy (2.2). Then F has a coupled fixed point.

Corollary 5. Let (X, d) be a metric space, $F : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying

$$H((F(x, y), F(u, v)) \leq \frac{k}{2} [d(gx, gu) + d(gy, gv)] \tag{2.11}$$

for all $x, y, u, v \in X$, where $k \in [0, 1)$. If $F(X \times X) \in g(X)$ and $g(X)$ is complete subset of X, then F and g have a coupled coincidence point in X. Moreover, F and g have a coupled common fixed point if anyone of the conditions (a)-(d) of Theorem 8 holds.

Corollary 6. Let (X, d) be a complete metric space, $F : X \times X \to CB(X)$ be a map-ping satisfying

$$H((F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)]$$

for all $x, y, u, v \in X$, where $k \in [0, 1)$, then F has a coupled fixed point in X.

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