On Steady State Behaviour of a Network Queuing Model with Biserial and Parallel Channels Linked With a Common Server

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Abstract

This paper is an attempt to study the steady state behaviour of a complex Network queuing model in which a common channel is linked in series with each of two systems, one containing two bi-serial channels and other three parallel channels in series. The arrivals of service pattern follow Poisson law. The generating function technique, law of calculus and statistical tools have been used to find the various queue characteristics. The model finds its application in decision making in the process industries, in banking, in networking, in business service and in many administrative setups. A computer programme followed by a numerical example is also given to justify the model.

Keywords: Steady state behaviour, Poisson stream, Biserial channels, Mean queue size, Variance of a queue, Average waiting time etc.

Introduction

Jackson R.R.P[1954] studied the behaviour of a queuing system containing phase type service. Maggu[1970] introduced the concept of bitendom in theory of queues which corresponds to a practical situation arise in production concern. Later on this idea was developed by various authors with different modifications and argumentations. Khodadi Abutaleb[1989] modified the queue system studied by Maggu by assuming the service parameter directly proportional to the queue number. Singh T.P. et. al.[2005] studied the transient behaviour of a queuing network with parallel biseries queue linked with a common channel. Singh T.P. et. al.[2006] studied steady state behaviour of a queue model comprised of two subsystem with biserial channel linked with a common channel. Later Gupta Deepak, Singh T.P. et. al.[2007] studied a network queue model comprised of biserial and parallel channel linked with a common server.

The present queue model differs the study made by [2007] in the sense that in this model the first system consist of biserial channel while the second system consist of three parallel sub channels each of which are linked with the third system in series. The various queue characteristics have been obtained explicitly under the transient behaviour of the system.

1. Practical Situation

Many practical situation of the model arise in industries, administrative setups, banking system, compute networks, office management, super markets and shopping malls etc. For example, in meal department of mall shop consisting of three sections, one is for food, second is for drink and third as common to both for billing. The food section consists of two sub channels and the drink section consists of three sub channels. Suppose in food section, one channel is for Indian food items and second channel is for Chinese food items. Similarly at drink section one channel is for Coffee / Tea, Second channel is for different vegetables soups and Third section is for cold drinks (juice/shakes etc). The customers who arrive to take Indian food items may also take some Chinese food items and then go to the bill counter or may directly go to billing counter. Similarly the customers coming to take Chinese food items may also take some Indian food items and then go to the billing counter or may directly go to the billing counter. Also the customers coming to take some drink items at the second section will join Coffee / Tea sub counter or vegetables soups sub counter or cold drinks sub counter according to their interest and will finally go to the billing counter.

Similarly, in an administrative setup of a particular state, we find the classification of file clearance is made at various phases such as at district level say S.D.M. (Senior Divisional magistrate) / D.C.(Deputy Commissioner) and various ministerial level offices and then finally at secretariats office at C.M. level. Some files for clearance form biserial system at the district level and then for final order pass to the C.M. office, while others enter through a parallel channel of different ministerial level and then finally to the C.M. office.

2. The Problem

The entire queue model is comprised of three service channels S_1, S_2 and S_3. The subsystem S_1 consist of two biserial service channels S_{11} and S_{12}, the subsystem S_2 contain three parallel channels S_{21}, S_{22} and S_{23}. The service channel S_3 is commonly linked in series with each of two servers S_1 and S_2 for competition of final phase service demanded either at a subsystem S_1 or S_2. the service time at S_{ij} ($i=1,2$ and $j=1,2,3$) are distributed exponentially. We assume the service mean rate $\mu_1, \mu_2, \mu_1', \mu_2', \mu_3$ at S_{ij} ($i=1,2$ and $j=1,2,3$) and μ_3 at S_3 respectively. Queues Q_1, Q_2, Q_3, Q_4, Q_5 and Q_6 are said to formed in front of the service channels S_{11}, S_{12}, S_{21}, S_{22} , S_{23} and S_3 respectively, if they are busy. Customers coming at the rate λ_1 after completion of phase service at S_{11} will join S_{12} or S_3 (that is they may either go to the network of servers $S_{11} \to S_{12} \to S_3$ or $S_{11} \to S_3$) with the probabilities p_{12} or p_{13} such that $p_{12} + p_{13} = 1$ and those coming at the rate λ_2 after completion of phase service at S_{12} will join S_{11} or S_3 (that is they may either go to the network of servers $S_{21} \to S_{11} \to S_3$ or $S_{21} \to S_3$) with the probabilities p_{21} or p_{23} such that $p_{21} + p_{23} = 1$. The customers coming at the rate λ_1' go to the network of servers $S_{21} \to S_3$ and those coming at the rate λ_2' go to the network of servers $S_{22} \to S_3$ and those coming at the rate λ_3' go to the network of servers $S_{23} \to S_3$.

3. Mathematical Analysis

Let $P_{n_1,n_2,n_3,n_4,n_5,n_6}$ be the joint probability that there are n_1 units waiting in queue Q_1 in front of S_{11}, n_2 units waiting in queue Q_2 in front of S_{12}, n_3 units waiting in queue Q_3 in front of S_{21}, n_4 units waiting in queue Q_4 in front of S_{22}, n_5 units waiting in queue Q_5 in front of S_{23} and n_6 units waiting in queue Q_6 in front of S_3 (**Figure 1**).In each case the waiting includes a unit in service, if any.

Also, $n_1, n_2, n_3, n_4, n_5, n_6 > 0$.

The standard arguments lead to the following differential difference equations in transient form as

$$P_{n_1,n_2,n_3,n_4,n_5,n_6}(t) = -\left(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' + \mu_1 + \mu_2 + \mu_1' + \mu_2' + \mu_3\right)P_{n_1,n_2,n_3,n_4,n_5,n_6}(t)$$

$$+\lambda_1 P_{n_1-1,n_2,n_3,n_4,n_5,n_6}(t) + \lambda_2 P_{n_1,n_2-1,n_3,n_4,n_5,n_6}(t) + \mu_1\left(n_1+1\right)p_{13}P_{n_1+1,n_2,n_3,n_4,n_5,n_6-1}(t) +$$

$$\mu_1\left(n_1+1\right)p_{12}P_{n_1+1,n_2-1,n_3,n_4,n_5,n_6}(t) + \mu_2\left(n_2+1\right)p_{23}P_{n_1,n_2+1,n_3,n_4,n_5,n_6-1}(t) +$$

$$\mu_2\left(n_2+1\right)p_{21}P_{n_1-1,n_2+1,n_3,n_4,n_5,n_6}(t) + \lambda_1' P_{n_1,n_2,n_3-1,n_4,n_5,n_6}(t) + \lambda_2' P_{n_1,n_2,n_3,n_4-1,n_5,n_6}(t)$$

$$+\lambda_3' P_{n_1,n_2,n_3,n_4,n_5-1,n_6}(t) + \mu_3(n_6+1) P_{n_1,n_2,n_3,n_4,n_5,n_6+1}(t) + \mu_1'(n_3+1) P_{n_1,n_2,n_3+1,n_4,n_5,n_6-1}(t)$$

$$+\mu_2'(n_4+1) P_{n_1,n_2,n_3,n_4+1,n_5,n_6-1}(t) + \mu_3'(n_5+1) P_{n_1,n_2,n_3,n_4,n_5+1,n_6-1}(t).$$

The steady state equation $(t \to \infty)$ governing the model are depicted as

$$\left(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' + \mu_1 + \mu_2 + \mu_1' + \mu_2' + \mu_3\right) P_{n_1,n_2,n_3,n_4,n_5,n_6} =$$

$$\lambda_1 P_{n_1-1,n_2,n_3,n_4,n_5,n_6} + \lambda_2 P_{n_1,n_2-1,n_3,n_4,n_5,n_6} + \mu_1(n_1+1) p_{13} P_{n_1+1,n_2,n_3,n_4,n_5,n_6-1} +$$

$$\mu_1(n_1+1) p_{12} P_{n_1+1,n_2-1,n_3,n_4,n_5,n_6} + \mu_2(n_2+1) p_{23} P_{n_1,n_2+1,n_3,n_4,n_5,n_6-1} +$$

$$\mu_2(n_2+1) p_{21} P_{n_1-1,n_2+1,n_3,n_4,n_5,n_6} + \lambda_1' P_{n_1,n_2,n_3-1,n_4,n_5,n_6} + \lambda_2' P_{n_1,n_2,n_3,n_4-1,n_5,n_6}$$

$$+\lambda_3' P_{n_1,n_2,n_3,n_4,n_5-1,n_6} + \mu_3(n_6+1) P_{n_1,n_2,n_3,n_4,n_5,n_6+1} + \mu_1'(n_3+1) P_{n_1,n_2,n_3+1,n_4,n_5,n_6-1}$$

$$+\mu_2'(n_4+1) P_{n_1,n_2,n_3,n_4+1,n_5,n_6-1} + \mu_3'(n_5+1) P_{n_1,n_2,n_3,n_4,n_5+1,n_6-1}. \quad\text{-----------------------} \qquad (1)$$

Let us define the generating function as

$$F(X,Y,Z,R,S,T) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty} P_{n_1,n_2,n_3,n_4,n_5,n_6} X^{n_1} y^{n_2} Z^{n_3} R^{n_4} S^{n_5} T^{n_6}$$

$$where\ |X| = |Y| = |Z| = |R| = |S| = |T| = 1.$$

Also we define partial generating functions as

$$F_{n_2,n_3,n_4,n_5,n_6}(X) = \sum_{n_1=0}^{\infty} P_{n_1,n_2,n_3,n_4,n_5,n_6} X^{n_1}$$

$$F_{n_3,n_4,n_5,n_6}(X,Y) = \sum_{n_2=0}^{\infty} P_{n_2,n_3,n_4,n_5,n_6}(X) Y^{n_2}$$

$$F_{n_4,n_5,n_6}(X,Y,Z) = \sum_{n_3=0}^{\infty} P_{n_3,n_4,n_5,n_6}(X,Y) Z^{n_3}$$

$$F_{n_5,n_6}(X,Y,Z,R) = \sum_{n_4=0}^{\infty} P_{n_4,n_5,n_6}(X,Y,Z) R^{n_4}$$

$$F_{n_6}(X,Y,Z,R,S) = \sum_{n_5=0}^{\infty} P_{n_5,n_6}(X,Y,Z,R) S^{n_5}$$

$$F(X,Y,Z,R,S,T) = \sum_{n_6=0}^{\infty} P_{n_6}(X,Y,Z,R,S) T^{n_6}$$

Now, on taking $n_1, n_2, n_3, n_4, n_5, n_6$ equal to zero one by one and then taking two of them pairwise, three of them at a time, four of them at a time, five of them at a time and all of them; we get 63 equations. Now proceeding on the lines of Maggu and Singh T.P. et.al. and following the standard technique, which after manipulation gives the final reduced result as -

$$F(X,Y,Z,R,S,T) = \frac{\begin{array}{c}\mu_1\left(1-\dfrac{T}{X}p_{13}-\dfrac{Y}{X}p_{12}\right)F(Y,Z,R,S,T)+\mu_2\left(1-\dfrac{T}{Y}p_{23}-\dfrac{X}{Y}p_{21}\right)F(X,Z,R,S,T)+\\[2mm] \mu_1'\left(1-\dfrac{T}{Z}\right)F(X,Y,R,S,T)+\mu_2'\left(1-\dfrac{T}{R}\right)F(X,Y,Z,S,T)+\mu_3'\left(1-\dfrac{T}{S}\right)F(X,Y,Z,S,T)\\[2mm] +\mu_3\left(1-\dfrac{1}{T}\right)F(X,Y,Z,R,S)\end{array}}{\begin{array}{c}\lambda_1(1-X)+\lambda_2(1-Y)+\mu_1\left(1-\dfrac{Y}{X}p_{12}-\dfrac{T}{X}p_{13}\right)+\mu_2\left(1-\dfrac{T}{Y}p_{23}-\dfrac{X}{Y}p_{21}\right)+\lambda_1'(1-Z)+\\[2mm] \lambda_2'(1-R)+\lambda_3'(1-S)+\mu_1'\left(1-\dfrac{T}{Z}\right)+\mu_2'\left(1-\dfrac{T}{R}\right)+\mu_3'\left(1-\dfrac{T}{S}\right)+\mu_3\left(1-\dfrac{1}{T}\right)\end{array}}$$

-------------------- **(2)**

For convenience, let us denote

$F(Y,Z,R,S,T) = F_1$

$F(X,Z,R,S,T) = F_2$

$F(X,Y,R,S,T) = F_3$

$F(X,Y,Z,S,T) = F_4$

$F(X,Y,Z,R,T) = F_5$

$F(X,Y,Z,R,S) = F_6$

Also *F(1,1,1,1,1,1)=1* , the total probability.

On taking *X =1* as $Y,Z,R,S,T \rightarrow 1$, *F(X,Y,Z,R,S,T)* is of $\dfrac{0}{0}$ indeterminate form.

Now, on differentiating numerator and denominator of **(2)** separately w.r.t X, we have

$$F_6 = 1 -- 1 = \frac{\mu_1(p_{13}+p_{12})F_1 + \mu_2(-p_{21})F_2}{-\lambda_1 + \mu_1(p_{12}+p_{13}) + \mu_2(-p_{21})}$$

$$\Rightarrow \mu_1 F_1 - \mu_2 p_{12} F_2 = -\lambda_1 + \mu_1 - \mu_2 p_{21} \ (\because p_{12}+p_{13}=1)$$ -------------------- **(3)**

Similarly ,on Diff. numerator and denominator of **(2)** separately w.r.t Y by taking *Y=1* and $X,Z,R,S,T \rightarrow 1$ we have

$$1 = \frac{\mu_1(-p_{12})F_1 + \mu_2(p_{23}+p_{21})F_2}{-\lambda_2 + \mu_1(-p_{12}) + \mu_2(p_{23}+p_{21})}$$

$$\Rightarrow \mu_1 p_{12} F_1 - \mu_2 F_2 = -\lambda_2 - p_{12}\mu_1 + \mu_2 \ (\because p_{23}+p_{21}=1)$$ -------------------- **(4)**

Again ,on Diff. numerator and denominator of **(2)** separately w.r.t Z by taking *Z=1* and $X,Y,R,S,T \rightarrow 1$ we have

$$1 = \frac{\mu_1' F_3}{-\lambda_1' + \mu_1'} \Rightarrow \mu_1' F_3 = -\lambda_1' + \mu_1'$$ -------------------- **(5)**

Again ,on Diff. numerator and denominator of (2) separately w.r.t R by taking R=1 $and\ X, Y, Z, S, T \to 1$ we have

$$1 = \frac{\mu_2' F_4}{-\lambda_2' + \mu_2'} \Rightarrow \mu_2' F_4 = -\lambda_2' + \mu_2'$$ ---------------------- (6)

Again ,on Diff. numerator and denominator of (2) separately w.r.t S by taking S=1 $and\ X, Y, Z, R, T \to 1$ we have

$$1 = \frac{\mu_3' F_5}{-\lambda_3' + \mu_3'} \Rightarrow \mu_3' F_5 = -\lambda_3' + \mu_3'$$ ---------------------- (7)

Again ,on Diff. numerator and denominator of (2) separately w.r.t T by taking T=1 $and\ X, Y, Z, R, S \to 1$ we have

$$1 = \frac{-\mu_1 p_{12} F_1 - \mu_2 p_{23} F_2 + \mu_1'(-F_3) + \mu_2'(-F_4) + \mu_3'(-F_5) + \mu_3 F_6}{\mu_1(-p_{13}) + \mu_2(-p_{23}) + \mu_1'(-1) + \mu_2'(-1) + \mu_3'(-1) + \mu_3}$$

$$\Rightarrow -\mu_1 P_{13} F_1 - \mu_2 P_{23} F_2 - \mu_1' F_3 - \mu_2' F_4 - \mu_3' F_5 + \mu_3 F_6 = -p_{13}\mu_1 - p_{23}\mu_2 - \mu_1' - \mu_2' - \mu_3' - \mu_3$$

--------------------- (8)

On multiplying (4)with p_{21} and adding to (3), we get

$$F_1(\mu_1 - \mu_1 p_{12} p_{21}) = -\lambda_1 + \mu_1(1 - p_{12} p_{21}) - \lambda_2 p_{21}$$

$$\Rightarrow F_1 = 1 - \frac{\lambda_1 + \lambda_2 p_{21}}{(1 - p_{12} p_{21})\mu_1}$$ --------------------- (9)

$$F_3 = 1 - \frac{\lambda_1'}{\mu_1'} \quad \textbf{(using (5))}$$ ------------------- (10)

$$F_4 = 1 - \frac{\lambda_2'}{\mu_2'} \quad \textbf{(using (6))}$$ ------------------- (11)

$$F_5 = 1 - \frac{\lambda_3'}{\mu_3'} \quad \textbf{(using (7))}$$ ------------------- (12)

On multiplying (3)with p_{21} and adding to (4), we get

$$\mu_2(1 - p_{12} p_{21})F_2 = -\lambda_2 - \lambda_1 p_{12} + \mu_2(p_{12} - p_{21}) + \mu_1(1 - p_{21} p_{12})$$

$$F_2 = 1 - \frac{\lambda_2 + \lambda_1 p_{12}}{(1 - p_{12} p_{21})\mu_2}$$ ----------------- (13)

Now on putting the values of F_1, F_2, F_3, F_4, F_5 in (8) , **we get**

$$F_6 = 1 - \left[\frac{\lambda_1' + \lambda_2' + \lambda_3'}{\mu_3} + \frac{(\lambda_1 + \lambda_2 p_{12}) p_{13} + (\lambda_2 + \lambda_1 p_{21}) p_{23}}{\mu_3(1 - p_{12} p_{21})} \right].$$ ----------------- (14)

On using the values of F_1, F_2, F_3, F_4, F_5 and F_6 the joint probability is given by

$$P_{n_1, n_2, n_3, n_4, n_5, n_6} = \rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3} \rho_4^{n_4} \rho_5^{n_5} \rho_6^{n_6} (1 - \rho_1)(1 - \rho_2)(1 - \rho_3)(1 - \rho_4)(1 - \rho_5)(1 - \rho_6)$$

Where $\rho_1 = 1 - F_1, \rho_2 = 1 - F_2$, $\rho_3 = 1 - F_3$, $\rho_4 = 1 - F_4$, $\rho_5 = 1 - F_5$, $\rho_6 = 1 - F_6$.

Further the solution in a steady state condition exist if $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6 < 1$.

6.1.1 Mean Queue Length

Average number of the customer (L)

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_1+n_2+n_3+n_4+n_5+n_6)P_{n_1,n_2,n_3,n_4,n_5,n_6}$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}n_1 P_{n_1,n_2,n_3,n_4,n_5,n_6} + \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}n_2 P_{n_1,n_2,n_3,n_4,n_5,n_6} + - - - - - -$$

$$+ \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}n_6 P_{n_1,n_2,n_3,n_4,n_5,n_6}$$

$$L = L_1 + L_2 + L_3 + L_4 + L_5 + L_6$$

Where

$$L_1 = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}n_1 P_{n_1,n_2,n_3,n_4,n_5,n_6}$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}n_1\rho_1^{n_1}\rho_2^{n_2}\rho_3^{n_3}\rho_4^{n_4}\rho_5^{n_5}\rho_6^{n_6}(1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4)(1-\rho_5)(1-\rho_6)$$

$$= (1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4)(1-\rho_5)(1-\rho_6)\sum_{n_1=0}^{\infty}n_1\rho_1^{n_1}\sum_{n_2=0}^{\infty}\rho_2^{n_2}\sum_{n_3=0}^{\infty}\rho_3^{n_3}\sum_{n_4=0}^{\infty}\rho_4^{n_4}\sum_{n_5=0}^{\infty}\rho_5^{n_5}\sum_{n_6=0}^{\infty}\rho_6^{n_6}$$

$$= \frac{\rho_1}{1-\rho_1}.$$

Similarly $L_2 = \dfrac{\rho_2}{1-\rho_2}, L_3 = \dfrac{\rho_3}{1-\rho_3}, L_4 = \dfrac{\rho_4}{1-\rho_4}, L_5 = \dfrac{\rho_5}{1-\rho_5}, L_6 = \dfrac{\rho_6}{1-\rho_6}.$

6.1.2. Variance of Queue

$$V(n_1+n_2+n_3+n_4+n_5+n_6) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_1+n_2+n_3+n_4+n_5+n_6)^2 P_{n_1,n_2,n_3,n_4,n_5,n_6} - L^2$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_1)^2 P_{n_1,n_2,n_3,n_4,n_5,n_6} + \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_2)^2 P_{n_1,n_2,n_3,n_4,n_5,n_6} + - - - - - +$$

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_6)^2 P_{n_1,n_2,n_3,n_4,n_5,n_6} + 2\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_1 n_2) P_{n_1,n_2,n_3,n_4,n_5,n_6} + - - - - - +$$

$$2\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_5 n_6) P_{n_1,n_2,n_3,n_4,n_5,n_6} - L^2$$

$$V = \frac{\rho_1}{(1-\rho_1)^2}+\frac{\rho_2}{(1-\rho_2)^2}+\frac{\rho_3}{(1-\rho_3)^2}+\frac{\rho_4}{(1-\rho_4)^2}+\frac{\rho_5}{(1-\rho_5)^2}+\frac{\rho_6}{(1-\rho_6)^2}.$$

7. Algorithm

The following algorithm provides the procedure to determine the joint probability and various queues characteristics of above discussed queueing model:

Step 1: Obtain the number of customers $n_1, n_2, n_3, n_4, n_5, n_6$.

Step 2: Obtain the values of mean service rate $\mu_1, \mu_2, \mu_1', \mu_2', \mu_3', \mu_3$.

Step 3: Obtain the values of mean arrival rate $\lambda_1, \lambda_2, \lambda_1', \lambda_2', \lambda_3'$.

Step 4: Obtain the values of the probabilities $p_{11}, p_{13}, p_{21}, p_{23}$.

Step 5: Calculate the values of

i. $F_1 = 1 - \dfrac{\lambda_1 + \lambda_2 p_{21}}{(1 - p_{12} p_{21}) \mu_1}$.

ii. $F_2 = 1 - \dfrac{\lambda_2 + \lambda_1 p_{12}}{(1 - p_{12} p_{21}) \mu_2}$

iii. $F_3 = 1 - \dfrac{\lambda_1'}{\mu_1'}$.

iv. $F_4 = 1 - \dfrac{\lambda_2'}{\mu_2'}$.

v. $F_5 = 1 - \dfrac{\lambda_3'}{\mu_3'}$.

vi. $F_6 = 1 - \left[\dfrac{\lambda_1' + \lambda_2' + \lambda_3'}{\mu_3} + \dfrac{(\lambda_1 + \lambda_2 p_{12}) p_{13} + (\lambda_2 + \lambda_1 p_{21}) p_{23}}{\mu_3 (1 - p_{12} p_{21})} \right]$

vii. $\lambda = \lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' + \lambda_3'$.

Step 6: Calculate:

i. $\rho_1 = 1 - F_1$.

ii. $\rho_2 = 1 - F_2$.

iii. $\rho_3 = 1 - F_3$.

iv. $\rho_4 = 1 - F_4$.

v. $\rho_5 = 1 - F_5$.

vi. $\rho_6 = 1 - F_6$.

Step 7: Check: $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6 < 1$, if so then go to step no 8, else steady state condition does not holds good.

Step 8: The joint probability

$$P_{n_1, n_2, n_3, n_4, n_5, n_6} = \rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3} \rho_4^{n_4} \rho_5^{n_5} \rho_6^{n_6} (1 - \rho_1)(1 - \rho_2)(1 - \rho_3)(1 - \rho_4)(1 - \rho_5)(1 - \rho_6)$$

Step 9: Calculate average no. of customers (Mean Queue Length)

$$L = \frac{\rho_1}{1 - \rho_1} + \frac{\rho_2}{1 - \rho_2} + \frac{\rho_3}{1 - \rho_3} + \frac{\rho_4}{1 - \rho_4} + \frac{\rho_5}{1 - \rho_5} + \frac{\rho_6}{1 - \rho_6}.$$

Step 10: Calculate Variance of queue:

$$V = \frac{\rho_1}{(1 - \rho_1)^2} + \frac{\rho_2}{(1 - \rho_2)^2} + \frac{\rho_3}{(1 - \rho_3)^2} + \frac{\rho_4}{(1 - \rho_4)^2} + \frac{\rho_5}{(1 - \rho_5)^2} + \frac{\rho_6}{(1 - \rho_6)^2}.$$

Step 11: Calculate average waiting time for customer $E(w) = \dfrac{L}{\lambda}$.

8. Programme

```
#include<iostream.h>
#include<stdio.h>
#include<conio.h>
#include<process.h>
#include<math.h>
int n[6],u[6],L[5];
float p[4],r[6];
float a1,b1,a2,b2,a3,b3,c1,c2,c3,P,Q,V,W,M;
void main()
{
    clrscr();
    cout<<"Enter the number of customers and values of mean service rate";
    for(int i=1;i<=6;i++)
    {
            cout<<"\nEnter the number of customers "<<i<<":";
            cin>>n[i];
            cout<<"\nEnter the value of mean service rate"<<i<<":";
            cin>>u[i];
    }
    cout<<"Enter the values of mean Arrival rate";
    for(int j=1;j<=5;j++)
    {
            cout<<"\nEnter the value of mean arrival rate"<<j<<":";
            cin>>L[j];
    }
    cout<<"Enter the values of probabilities";
    for(int k=1;k<=4;k++)
    {
            cout<<"\nEnter the value of probability"<<k<<":";
            cin>>p[k];
    }
    a1=L[1]+L[2]*p[3];
    b1=(1-p[1]*p[3])*u[1];
    r[1]=a1/b1;
    a2=L[2]+L[1]*p[1];
    b2=(1-p[1]*p[3])*u[2];
    r[2]=a2/b2;
```

```
        r[3]=L[3]/u[3];

        r[4]=L[4]/u[4];

        r[5]=L[5]/u[5];

        a3=L[3]+L[4]+L[5];

        c1=a3/u[6];

        b3=(L[1]+L[2]*p[3])*p[2]+(L[2]+L[1]*p[4]);

        c2=u[6]*(1-p[1]*p[3]);

        c3=b3/c2;

        r[6]=c1+c3;

        M=L[1]+L[2]+L[3]+L[4]+L[5];

        if(r[1],r[2],r[3],r[4],r[5],r[6]>1)

        {

                cout<<"Steady state condition does not holds good...\nExitting";

                getch();

                exit(0);

        }

        P=(pow(r[1],n[1]))*(pow(r[2],n[2]))*(pow(r[3],n[3]))*(pow(r[4],n[4]))*(pow(r[5],n[5]))*(pow(r[6
],n[6]))*(1-r[1])*(1-r[2])*(1-r[3])*(1-r[4])*(1-r[5])*(1-r[6]);

        cout<<"\nThe joint probability is:"<<P<<"\n";

        Q=(r[1]/(1-r[1]))+(r[2]/(1-r[2]))+(r[3]/(1-r[3]))+(r[4]/(1-r[4]))+(r[5]/(1-r[5]))+(r[6]/(1-r[6]));

        cout<<"\nThe mean queue length is :"<<Q<<"\n";

        V=(r[1]/pow((1-r[1]),2))+(r[2]/pow((1-r[2]),2))+(r[3]/pow((1-r[3]),2))+(r[4]/pow((1-
r[4]),2))+(r[5]/pow((1-r[5]),2))+(r[6]/pow((1-r[6]),2));

        cout<<"\n The Variance of Queue is :"<<V<<"\n";

        W=Q/M;

        cout<<"\nAverage waiting time for the customer is:"<<W<<"\n";

        getch();

        }
```

9. Numerical Illustration

Give customers coming to three servers out of which one server consist two biserial channels and other consist of three parallel service channels and further these two service channels are linked with common server. The number of customers, mean service rate, mean arrival rate and associated probabilities are given as follows:

S.No.	No. of Customers	Mean Service Rate	Mean arrival rate	Probabilities
1	$n_1 = 5$	$\mu_1 = 10$	$\lambda_1 = 4$	$p_{12} = 0.4$
2	$n_2 = 8$	$\mu_2 = 9$	$\lambda_2 = 5$	$p_{13} = 0.6$
3	$n_3 = 6$	$\mu_1' = 7$	$\lambda_1' = 3$	$p_{21} = 0.5$
4	$n_4 = 3$	$\mu_2' = 6$	$\lambda_2' = 5$	$p_{23} = 0.5$
5	$n_5 = 4$	$\mu_3' = 5$	$\lambda_3' = 4$	

6	$n_6 = 20$	$\mu_3 = 22$		

Find the joint probability, mean queue length, variance of queue and average waiting time for customer.

Solution:- We have

$$\rho_1 = \frac{\lambda_1 + \lambda_2 p_{21}}{(1 - p_{12} p_{21}) \mu_1} = \frac{4 + 5 \times 0.5}{(1 - 0.4 \times 0.5)0.1} = \frac{6.5}{8} = 0.8125$$

$$\rho_2 = \frac{\lambda_2 + \lambda_1 p_{12}}{(1 - p_{12} p_{21}) \mu_2} = \frac{5 + 4 \times 0.4}{(1 - 0.4 \times 0.5)9} = \frac{6.6}{7.2} = 0.9166$$

$$\rho_3 = \frac{\lambda_1'}{\mu_1'} = \frac{3}{7} = 0.4285$$

$$\rho_4 = \frac{\lambda_2'}{\mu_2'} = \frac{5}{6} = 0.8333$$

$$\rho_5 = \frac{\lambda_3'}{\mu_3'} = \frac{4}{5} = 0.8$$

$$\rho_6 = \left[\frac{\lambda_1' + \lambda_2' + \lambda_3'}{\mu_3} + \frac{(\lambda_1 + \lambda_2 p_{12}) p_{13} + (\lambda_2 + \lambda_1 p_{21}) p_{23}}{\mu_3 (1 - p_{12} p_{21})} \right] = \frac{12}{22} + \frac{(4 + 5 \times 0.5)0.6 + (5 + 4 \times 0.4)0.5}{22(1 - 0.4 \times 0.5)} = 0.9545$$

The joint probability is

$$P_{n_1,n_2,n_3,n_4,n_5,n_6} = \rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3} \rho_4^{n_4} \rho_5^{n_5} \rho_6^{n_6} (1 - \rho_1)(1 - \rho_2)(1 - \rho_3)(1 - \rho_4)(1 - \rho_5)(1 - \rho_6)$$

$$= \left(\frac{6.5}{8}\right)^5 \left(\frac{66}{72}\right)^8 \left(\frac{3}{7}\right)^6 \left(\frac{5}{6}\right)^3 \left(\frac{4}{5}\right)^4 (0.9545)^{20} \left(1 - \frac{6.5}{8}\right)\left(1 - \frac{3}{7}\right)\left(1 - \frac{5}{6}\right)\left(1 - \frac{4}{5}\right)(1 - 0.9545)$$

$$= 1.3833 \times 10^{-9}.$$

The Mean queue length (Average no. of customers)

$$L = \frac{\rho_1}{1 - \rho_1} + \frac{\rho_2}{1 - \rho_2} + \frac{\rho_3}{1 - \rho_3} + \frac{\rho_4}{1 - \rho_4} + \frac{\rho_5}{1 - \rho_5} + \frac{\rho_6}{1 - \rho_6}$$

$$= \frac{0.8125}{1 - 0.8125} + \frac{0.9166}{1 - 0.9166} + \frac{0.4285}{1 - 0.4285} + \frac{0.8333}{1 - 0.8333} + \frac{0.8}{1 - 0.8} + \frac{0.9545}{1 - 0.9545}$$

$$= 46.0833$$

Variance of queue

$$V = \frac{\rho_1}{(1 - \rho_1)^2} + \frac{\rho_2}{(1 - \rho_2)^2} + \frac{\rho_3}{(1 - \rho_3)^2} + \frac{\rho_4}{(1 - \rho_4)^2} + \frac{\rho_5}{(1 - \rho_5)^2} + \frac{\rho_6}{(1 - \rho_6)^2}.$$

$$= \frac{0.8125}{(1 - 0.8125)^2} + \frac{0.9166}{(1 - 0.9166)^2} + \frac{0.4285}{(1 - 0.4285)^2} + \frac{0.8333}{(1 - 0.8333)^2} + \frac{0.8}{(1 - 0.8)^2} + \frac{0.9545}{(1 - 0.9545)^2}$$

$$= 668.4245.$$

Average waiting time for customer

$$E(w) = \frac{L}{\lambda} = \frac{46.0833}{21} = 2.1944 \ .$$

10. Concluding Remarks

1. If only two parallel sub channels in the second channel are considered, then proceeding in samemanner the results resemble with Gupta Deepak et. al,[8].

2. If only biserial subsystem at first phase is considered and transient behaviour of a queue model is analysed, then results tally with the work of T.P.Singh[6].

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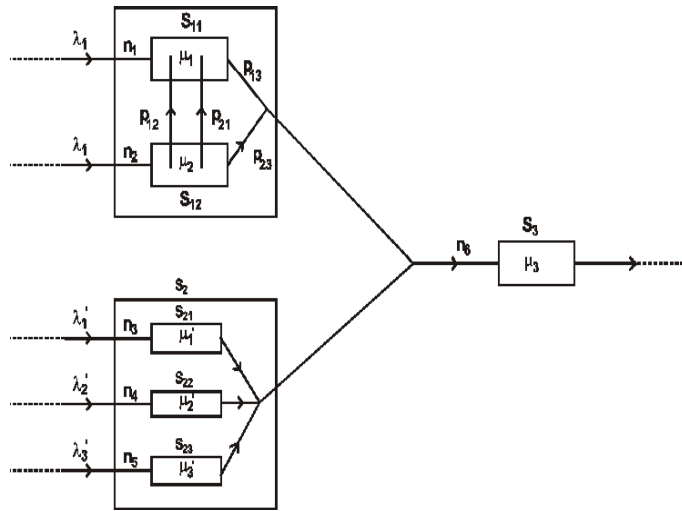
Figure I :Network Queueing Model