Differential Approach to Cardioid Distribution

Dattatreya Rao AV
Department of Statistics, Acharya Nagarjuna University
Guntur, India
E-mail: avdrao@gmail.com

Girija SVS (Corresponding Author)
Department of Mathematics, Hindu College
Guntur, India
E-mail: svs.girija@gmail.com

Phani Y
Department of Mathematics, Swarnandhra College of Engineering and Technology
Seetharampuram – 534 280, Narasapur, India
E-mail: phaniyedlapalli23@gmail.com

Received: 2011-10-23
Accepted: 2011-10-29
Published:2011-11-04

Abstract

Jeffreys (1961) introduced Cardioid distribution and used it to modeling directional spectra of ocean waves. Here an attempt is made to derive pdf of cardioid model as a solution of a second order non homogeneous linear differential equation having constant coefficients with certain initial conditions. We also arrive at new unimodal and symmetric distribution on real line from Cardioid model induced by Mobius transformation called "Cauchy type models".

Keywords: Circular model, Mobius transformation, Cardioid and Uniform distributions, Cauchy type models.

1. Introduction

Jeffreys (1961) introduced Cardioid distribution and used it to modeling directional spectra of ocean waves. Following Fejer's theorem in Fernandez (2006) we may define a family of circular distributions by

$$f(\theta;n)=\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{n}\left\{a_k\cos(k\theta)+b_k\sin(k\theta)\right\} \tag{1.1}$$

When n = 1

$f(\theta\ ;1)=\frac{1}{2\pi}+\frac{1}{\pi}\left(a_1\cos\theta+b_1\sin\theta\right)$ represents the Cardioid distribution.

The probability density and distribution functions of Cardioid distribution are respectively given by

$$f(\theta\,;\mu,\rho)=\frac{1}{2\pi}\big(1+2\rho\cos(\theta-\mu)\big) \tag{1.2}$$

where $\theta,\ \mu\in[-\pi,\pi)$ and $-\frac{1}{2}<\rho<\frac{1}{2}$

$$F(\theta)=\frac{1}{2\pi}\big(\theta+2\rho\sin(\theta-\mu)+2\rho\sin\mu\big) \tag{1.3}$$

In Section 2, we make certain assumptions on arbitrary constants in the general solution of a linear differential equation to get Cardioid distribution and Section 3 deals with the generation of Cauchy type distributions from Cardioid model induced by Mobius transformation/stereographic projection.

2. Cardioid distribution through a Differential equation

By making use of certain assumptions on arbitrary constants in the general solution of a differential equation we construct the pdf of Cardioid model.

Theorem 2.1:

The solution of the initial value problem $\frac{d^2y}{d\theta^2}+y=\frac{1}{2\pi},\ y(0)=\frac{1+2\rho\cos\mu}{2\pi},\ y'(0)=\frac{\rho\sin\mu}{\pi}$ admits

i) the particular integral which is pdf of Uniform distribution on Unit Circle and

ii) $y(\theta)=\frac{1}{2\pi}\big(1+2\rho\cos(\theta-\mu)\big)$ which is probability density function of Cardioid distribution

where $-\pi\le\theta,\ \mu<\pi$ and $|\rho|<0.5$.

Proof: Consider a non homogeneous second order linear differential equation with constant coefficients

$$\frac{d^2y}{d\theta^2}+y=\frac{1}{2\pi}. \tag{2.1}$$

The Particular Integral of (2.1) admits pdf of circular uniform distribution

$$y_p=\frac{1}{2\pi}. \tag{2.2}$$

General solution of the above differential equation is

$$y(\theta)=C_1\cos\theta\ +C_2\sin\theta\ +\frac{1}{2\pi}, \tag{2.3}$$

where C_1 and C_2 are arbitrary constants.

Under the following initial conditions the above solution (2.3) also admits probability density function of Cardioid distribution for

$$y(0)=\frac{1+2\rho\cos\mu}{2\pi},\quad y'(0)=\frac{\rho\sin\mu}{\pi}. \tag{2.4}$$

From (2.3),

$$C_1=\frac{\rho\cos\mu}{\pi}\quad \text{and}\quad C_2=\frac{\rho\sin\mu}{\pi}. \tag{2.5}$$

Hence

$$\begin{aligned} y(\theta)&=\frac{1}{2\pi}(1+2\rho\cos\mu\cos\theta+2\rho\sin\mu\sin\theta)\\ &=\frac{1}{2\pi}(1+2\rho\cos(\theta-\mu)). \end{aligned} \tag{2.6}$$

Conveniently we write this equation as $f(\theta)=\frac{1}{2\pi}(1+2\rho\cos(\theta-\mu))$.

3. Cauchy Type Distributions Using Mobius Transformation On Cardioid Distribution

Ahlfors (1966) defined **Mobius transformation** as follows

"The transformation of the form $w=T(z)=\frac{az+b}{cz+d}$, where a, b, c and d are complex constants such that $ad-bc\neq 0$ is known as Bilinear transformation or Linear fractional transformation or Mobius transformation or stereographic projection".

Minh and Farnum (2003) imposed certain restrictions on parameters a, b, c and d in T(Z) and arrived at the following

$$T(z)=\frac{Cz+\bar{C}}{z+1},\ \text{with Im(C)}\ \neq 0. \tag{3.1}$$

Where C = u –i v and $\bar{C}$ = u + i v.

The Mobius transformation defined by (3.1) is a real-valued for any z on the Unit Circle

$$x=T(\theta)=u+v\left(\frac{\sin\theta}{1+\cos\theta}\right)=u+v\tan\left(\frac{\theta}{2}\right),\ \text{which is real.} \tag{3.2}$$

Hence the Mobius transformation defined by $T(z)=\frac{Cz+\bar{C}}{z+1}$, maps every point on the Unit circle onto the real line.

Form (3.2), we have

$$x=u+v\tan\left(\frac{\theta}{2}\right)$$

$$\Rightarrow\frac{x-u}{v}=\tan\left(\frac{\theta}{2}\right)$$

$$\Rightarrow T^{-1}(x)=\theta=2\tan^{-1}\left(\frac{x-u}{v}\right).$$

(3.3)

which maps every point on the real line onto the Unit Circle and the mapping is a bijection .

$$f(x) = g(\theta(x)) \frac{2}{v\left(1+\left(\frac{x-u}{v}\right)^2\right)}, \quad \text{when } v > 0 \tag{3.4}$$

Theorem 3.1 :

If θ follows Cardiod Distribution in [-π, π), then $T(\theta) = x = u + v\tan\left(\frac{\theta}{2}\right)$

has a 2-parameter linear distribution on the real line given by

$$f(x) = \frac{1}{v\pi\left(1+\left(\frac{x-u}{v}\right)^2\right)}\left[1+2\rho\left(\frac{1-\left(\frac{x-u}{v}\right)^2}{1+\left(\frac{x-u}{v}\right)^2}\right)\right]$$

Proof : If θ follows Cardioid Distribution with $\mu = 0$ in [-π, π), then pdf $g(\theta)$ is

$$g(\theta) = \frac{1}{2\pi}(1+2\rho\cos\theta).$$

By applying theorem of Minh and Farnum (2003), we have

$$f(x) = g(\theta(x)) \frac{2}{v\left(1+\left(\frac{x-u}{v}\right)^2\right)}, v > 0$$

$$= \frac{2}{v\left(1+\left(\frac{x-u}{v}\right)^2\right)} \cdot \frac{1}{2\pi}\left(1+2\rho\cos\left(2\tan^{-1}\left(\frac{x-u}{v}\right)\right)\right)$$

$$= \frac{1}{\pi v\left(1+\left(\frac{x-u}{v}\right)^2\right)}\left[1+2\rho\left[\frac{1-\left(\frac{x-u}{v}\right)^2}{1+\left(\frac{x-u}{v}\right)^2}\right]\right]. \tag{3.5}$$

and

1) $f(x) \geq 0 \ \forall \ -\infty < x < \infty$ (since $v > 0$)

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

is a family of distributions on the real line and are named by us as **Cauchy type distributions** obtained from the circular model called Cardioid distribution induced by Mobious Transformation.

When $\rho = 0$ in (3.5), we get

$$f(x) = \frac{1}{\pi v\left[1+\left(\frac{x-u}{v}\right)^2\right]}. \tag{3.6}$$

which is the density function of the 2 – parameter Cauchy's distribution with location parameter u and scale parameter v .When u = 0 and v = 1, we get $f(x) = \frac{1}{\pi\left[1+x^2\right]}$, which is standard Cauchy distribution.

4 Graph

We observe that the probability distribution on real line generated by using Mobius Transformation on Cardioid Model is also Unimodal and symmetric for $0 < \rho < 0.5$.

Acknowledgements:

We acknowledge Prof. (Retd.) I. Ramabhadra Sarma, Dept. of Mathematics, Acharya Nagarjuna University for his suggestions which have helped in improving the presentation of this paper.

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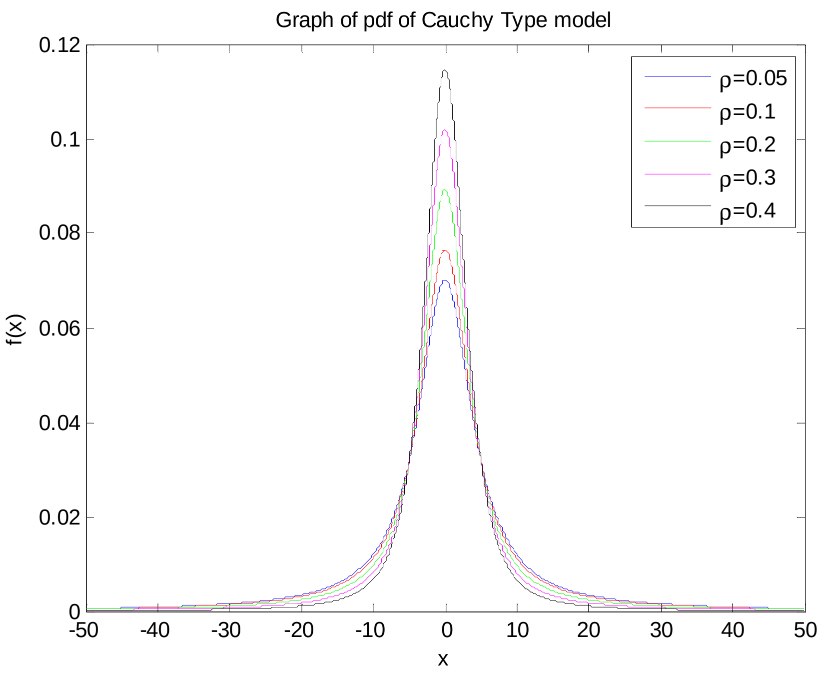

Figure – 1Graph of pdf of Cauchy Type Model