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Common Fixed point theorems for contractive maps of Integral type in modular metric spaces

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Abstract
In this theorem we proves a common fixed point theorem for a pair of ρ-compatible maps of integral type , further generalization is done by the existence of Banach contraction mapping in fixed point theorem in modular metric spaces.

Introduction and Preliminaries

Fixed point theorems in modular spaces, generalised the classical banach fixed point theorem in metric spaces. In [1], Jungck defines the notion of compatible self maps of a metric space (X, d) as a pair of maps as a pair of maps $h, T: X \to X$ such that

for all sequences $\{x_n\}$ in X with $\lim_{n\to\infty} hx_n = \lim_{n\to\infty} Tx_n = x \in X$,

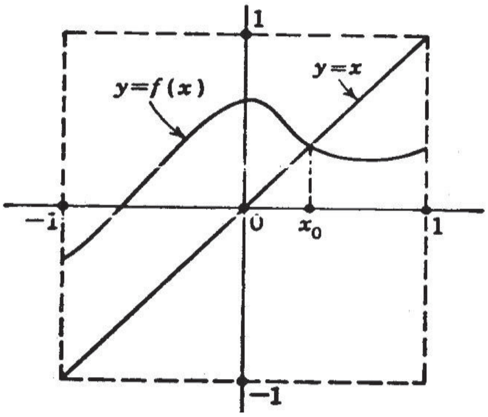
we have $\lim_{n\to\infty} d(hTx_n, Thx_n) = 0$

Then he proves a common fixed point theorem for pairs of compatible maps and a further generalization in[5]. The notion of modulat space,is as a generalization of a metric space,was introduced by NaKano in 1950andredefined and generalized by Musielak and Orlicz in1959. Here our purpose is to define the notion of ρ-compatible mappings in modular spaces for some common fixed point theorems

In the existence of Fixed point theory and a Banach contraction principle occupies a prominent place in the study of metric spaces, it became a most popular tool in solving problems in mathematical analysis. fixed point theory has received much attention in metric spaces endowed with a partial ordering. The study of fixed point of a functions satisfying certain contractive conditions has been at the center of vigorous research activity, because it has a wide range of applications in different areas such as, variational, linear inequalities, optimization and parameterize estimation Problems.

The fixed point theorems in metric spaces are playing a major role to construct methods in mathematics to solve problems in applied mathematics and sciences.

Let f be a continuous mapping of the closed interval [-1, 1] into itself. Figure suggests that the graph of f must touch or cross the indicated diagonal, or more precisely, that there must exist a point x_0 in [-1,1] with the property that $f(x_0) = x_0$.



The proof is easy. We consider the continuous function F defined on [-1,1] by F(x) = f(x) - x, and we observe that F (-1) > 0 and that F(1) £.0. It now follows from the Weierstrass intermediate value theorem that there exists a point x_0 in [-1, 1] such that F $(x_o) = 0$ or $f(x_0) = x_0$.

It is convenient to describe this phenomenon by means of the following terminology. A topological space X is called a fixed point space if every continuous mapping f of X into itself has a fixed point, in the sense that $f(x_0) = x_0$ for x_0 in X. The remarks in the above paragraph show that [-1, 1] is a fixed-point space. Furthermore, the closed disc [(x, y): $x^2 + y^2$£ 1) in the Euclidean plane R^2 is also a fixed-point space.

Example : Let f be a mapping $f: R \to R$ then a function $f(x) = x^2$ for all x has a fixed point on [1,1].i.e.

$f(1) = 1$

$f(0) = 0$

In this section we revised a basic definition and proves a common fixed point theorem for integral type ρ-compatible maps in metric spaces

Definition 1.1 Let X be an arbitrary vector space over $K = (\mathbb{R} \text{ or } \mathbb{C})$.

A) A functional $\rho: X \to [0, \infty]$ is called modular if:

(i) $\rho(x) = 0$ iff $x = 0$.

(ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, for all $x \in X$.

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0, \alpha + \beta = 1$, for all $x, y \in X$

If (iii) is replaced by

(iv) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha, \beta \geq 0, \alpha + \beta = 1$, for all $x, y \in X$

Then the modular ρ is called convex modular.

B) A modular ρ defines a corresponding modular space ; i.e. the space X_ρ given by:

$X_p = \{x \in X;\ \rho(\alpha x) \to 0 \text{ as } \alpha \to 0\}$

Definition1.2 Let X_p be a modular space.

a) A sequence $(x_n)_{n\in N}$ in X_p is said to be:

i) ρ-convergent to x If $\rho(x_n - x) \to 0$ as $n \to \infty$.

ii) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n \to \infty$.

b) X_p is ρ-Complete if every ρ-Cauchy sequence is ρ-convergent.

c) A subset $B \subset X_p$ is said to be ρ-closed if for any sequence $(x_n)_{n\in N} \subset B$ and $x_n \to x$ we have $x \in B$.

d) A subset $B \subset X_p$ is said to be ρ-bounded if $\delta_\rho(B) = \sup \rho(x-y) < \infty$ for all $x, y \in B$, where $\delta_\rho(B)$ is the ρ-diameter of B.

e) ρ has a Fatou property if

$\rho(x-y) \le \lim\inf \rho(x_n - y_n)$.

Whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

f) ρ is said to satisfy the Δ_2 –condition if $\rho(2x_n) \to 0$, whenever $\rho(x_n) \to 0$ as $n \to \infty$.

Main Section

A common Fixed point Theorem of Integral Type for contractive condition in modular metric space

Definition 2.1. Let X_p be a modular space, where ρ satisfy the Δ_2 -condition. Two self mappings T and h of X_p are called ρ-compatible if $(Thx_n - hTx_n) \to 0$, Whenever $(x_n)_{n\in N}$ is a sequence in X_p such that $hx_n \to z$ and $Tx_n \to z$ for some $z \in X_p$.

Theorem 2.2 Let X_p be a modular space with ρ-compatible , where ρ satisfy the Δ_2 -condition. Suppose $s, r, q \in R^+$,$s > q$ and T , h : $X_p \to X_p$ are two ρ-compatible mappings such that

$T(X_p) \subseteq h(X_p)$ and

$$\int_0^{\rho(s(T_x - T_y))} \psi(t)dt \le \alpha \int_0^{\rho\left(q\left(\frac{h_x - h_y}{h_x h_y}\right)\right)} \psi(t)dt + \beta \int_0^{\rho\left[q\left[\frac{\left(\frac{h_x-h_y}{h_x h_y}\right) + (hx_{n+1} - Tx_n)}{1+\left(\frac{h_x-h_y}{h_x h_y}\right)(hx_{n+1} - Tx_n)}\right]\right]} \psi(t)dt \qquad (1)$$

For some $r \in (0,1)$,where $\psi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable non negative and for all $\epsilon > 0$.

$$\int_0^{\epsilon} \psi(t)dt > 0 \qquad (2)$$

If one of h and T is continuous then there exist a unique common fixed point of h and T.

Proof: Let $\lambda \in R^+$ be the conjugate of $\frac{s}{q}$ i.e. $\frac{s}{q} + \frac{1}{\lambda} = 1$ .Let x be a arbitrary point of X_p and generate

inductively the sequence $\{Tx_n\}$, $n \in N$ as follows $T(X_p) \subseteq h(X_p)$.

For each integer $n \ge 1$, inequality (1) shows that

$$\int_0^{\rho\left(s(Tx_{n+1} - Tx_n)\right)} \psi(t)dt \le \alpha \int_0^{\rho\left(q\left(\frac{hx_{n+1} - hx_n}{hx_{n+1}\, hx_n}\right)\right)} \psi(t)dt$$

$$+ \beta \int_0^{\rho\left[q\left[\frac{\left(\frac{hx_{n+1}-hx_n}{hx_{n+1}\, hx_n}\right) + (hx_{n+1} - Tx_n)}{1+\left(\frac{hx_{n+1}-hx_n}{hx_{n+1}\, hx_n}\right)(hx_{n+1} - Tx_n)}\right]\right]} \psi(t)dt$$

$$\leq \alpha \int_0^{\rho\left(s\left(\frac{T_{x_n}-T_{x_{n-1}}}{T_{x_n}\, T_{x_{n-1}}}\right)\right)} \psi(t)dt + \beta \int_0^{\rho\left(s\left(\frac{T_{x_n}-T_{x_{n-1}}}{T_{x_n}\, T_{x_{n-1}}}\right)\right)+0} \psi(t)dt$$

$$\leq (\alpha+\beta) \int_0^{\rho\left(s\left(\frac{T_{x_n}-T_{x_{n-1}}}{T_{x_n}\, T_{x_{n-1}}}\right)\right)} \psi(t)dt$$

$$\leq r \int_0^{\rho\left(s\left(\frac{T_{x_n}-T_{x_{n-1}}}{T_{x_n}\, T_{x_{n-1}}}\right)\right)} \psi(t)dt$$

By induction

$$\int_0^{\rho(s(T_{x_{n+1}}-T_{x_n}))} \psi(t)dt \leq r^n \int_0^{\rho\left(q\left(\frac{T_x - x}{T_x\, x}\right)\right)} \psi(t)dt \tag{3}$$

Taking the limit as $n \to \infty$ yields

$$\lim_n \int_0^{\rho(s(T_{x_{n+1}}-T_{x_n}))} \psi(t)dt \leq 0 \tag{4}$$

Now we can show that $\{Tx_n\}$, $n \in N$ is ρ=cauchy. If not , then ,there exist an $\in > 0$ and two sequences of integer $\{\{n(k)\},\{m(k)\}\}$, with $n(k) > m(k) \geq k$ such that

$$d_k = \rho\left(q\left(T_{x_{n(k)}} - T_{x_{m(k)}}\right)\right) \geq \in \qquad \forall\ k = 1,2,3 \text{ ------} \tag{5}$$

We can assume that

$$\rho\left(q\left(T_{x_{n(k)-1}} - T_{x_{m(k)}}\right)\right) < \in \tag{6}$$

In order to show this , suppose n(k) is the smallest number exceeding m(k) for which (5) holds and

$$\textstyle\sum_k = \left\{n \in N : \exists\, m(k) \in N : \rho\left(q\left(T_{x_{n(k)-1}} - T_{x_{m(k)}}\right)\right) \geq \in \text{ and } n > m(k) \geq k\right\}$$

obviously $\sum_k \neq \phi$ and since $\sum_k \subseteq N$ then by the ordering principle , the minimum element of $\sum_k$ *is* denoted by n(k)

and clearly (6) holds .

Now,

$$\int_0^{\rho\left(s\left(T_{x_{m(k)}}-T_{x_{n(k)}}\right)\right)} \psi(t)dt \leq \alpha \int_0^{\rho\left(q\left(\frac{h_{x_{m(k)}}-h_{x_{n(k)}}}{h_{x_{m(k)}}\, h_{x_{n(k)}}}\right)\right)} \psi(t)dt$$

$$+\beta \int_0^{\rho\left[q\left[\frac{\left(\frac{h_{x_{m(k)}}-h_{x_{n(k)}}}{h_{x_{m(k)}}\, h_{x_{n(k)}}}\right)+(hx_{m(k)+1}-Tx_{m(k)})}{1+\left(\frac{h_{x_{m(k)}}-h_{x_{n(k)}}}{h_{x_{m(k)}}\, h_{x_{n(k)}}}\right)(hx_{m(k)+1}-Tx_{m(k)})}\right]\right]} \psi(t)dt$$

$$= \alpha \int_0^{\rho\left(q\left(\frac{T_{x_{m(k)-1}}-T_{x_{n(k)-1}}}{T_{x_{m(k)-1}}\, T_{x_{n(k)-1}}}\right)\right)} \psi(t)dt + \beta \int_0^{\rho\left[q\left(\frac{T_{x_{m(k)-1}}-T_{x_{n(k)-1}}}{T_{x_{m(k)-1}}\, T_{x_{n(k)-1}}}\right)\right]} \psi(t)dt$$

$$= (\alpha+\beta) \int_0^{\rho\left[q\left(\frac{T_{x_{m(k)-1}}-T_{x_{n(k)-1}}}{T_{x_{m(k)-1}}\, T_{x_{n(k)-1}}}\right)\right]} \psi(t)dt$$

$$= r\int_0^{\rho\left[q\left(\frac{T_{x_{m(k)-1}}-T_{x_{n(k)-1}}}{T_{x_{m(k)-1}}\,T_{x_{n(k)-1}}}\right)\right]}\psi(t)\,dt$$

condition and (4), then ,

$$\rho\left(q\left(\frac{T_{x_{m(k)-1}}-T_{x_{n(k)-1}}}{T_{x_{m(k)-1}}\,T_{x_{n(k)-1}}}\right)\right)=\rho\left(q\left(\frac{T_{x_{m(k)-1}}-T_{x_{m(k)}}+T_{x_{m(k)}}-T_{x_{n(k)-1}}}{T_{x_{m(k)-1}}\,T_{x_{n(k)-1}}}\right)\right)$$

$$=\rho\left(\left(\frac{\gamma\frac{q}{\gamma}\left(T_{x_{m(k)-1}}-T_{x_{m(k)}}\right)+\frac{qs}{s}\left(T_{x_{m(k)}}-T_{x_{n(k)-1}}\right)}{T_{x_{m(k)-1}}\,T_{x_{n(k)-1}}}\right)\right)$$

$$=\rho\left(\frac{\gamma q\left(T_{x_{m(k)-1}}-T_{x_{m(k)}}\right)}{T_{x_{m(k)-1}}\,T_{x_{n(k)-1}}}\right)+\rho\left(\frac{s\left(T_{x_{m(k)}}-T_{x_{n(k)-1}}\right)}{T_{x_{m(k)-1}}\,T_{x_{n(k)-1}}}\right)$$

Using ∇_2- condition and (4), then ,

$$\lim_{k\to\infty}\rho\left[\gamma q\left(\frac{Tx_{m(k)-1}-Tx_{m(k)}}{Tx_{m(k)-1}\,Tx_{m(k)}}\right)\right]=0$$

Therefore $\quad lim_k\int_0^{\rho\left[q\left(\frac{Tx_{m(k)-1}-Tx_{n(k)-1}}{Tx_{m(k)-1}\,Tx_{n(k)-1}}\right)\right]}\varphi(t)\,dt\le\int_0^{\epsilon}\varphi(t)\,dt \qquad (7)$

Also , by the inequality (5)

We have a contradiction. There fore ,it is a ρ-cauchy sequence since X_ρ *is* ρ-complete, then there exist a $w\in X_\rho$

such that $\rho[s(Tx_n-w)]\to 0$ *as* $n\to\infty$.

If T is continuous, then $T^2x_n\to Tz$ *and* $Thx_n\to Tz$. Since $\rho(hTx_n$-Th $x_n)\to 0$, *then by* ρ-compatibility , hT $x_n\to Tz$.

We now prove that w is a fixed point of T, we have

Taking as $n\to\infty$,

$$\int_0^{\rho[s(Tw-w)]}\psi(t)\,dt\le r\int_0^{\rho\left[q\left(\frac{Tw-w}{Tw\,w}\right)\right]}\psi(t)\,dt$$

Which implies that

$$\int_0^{\rho[s(Tw-w)]}\psi(t)\,dt\le 0$$

Using inequality (2), $\quad \rho[s(Tw-w)]\to 0$ *and* $Tw=w$.

Moreover T(X_ρ)⊆ $h(X_\rho)$ and thus

$\exists$ *a point v such that* $w=Tw=hv$ *then, the inequality*,

as $n\to\infty$,we have

$$\int_0^{\rho[s(Tw-Tv)]}\psi(t)\,dt\le r\int_0^{\rho\left[q\left(\frac{Tw-hv}{Tw\,hv}\right)\right]}\psi(t)\,dt$$

And thus

$$\int_0^{\rho[s(w-Tv)]}\psi(t)\,dt\le r\int_0^{\rho\left[q\left(\frac{w-hv}{w\,hv}\right)\right]}\psi(t)\,dt$$

$$\leq r\int_0^{\rho[q(w-w)]}\psi(t)dt$$

Resulting in $w = Tv = hv$ and $hw = hTv = Thv = Tw = w$, now suppose that w and w_1 are two common fixed point of T and h Then we have,

$$\rho[s(w - w_1)] = 0$$

Hence w = w_1 is a common fixed point of T and h.

Remark 2.3 If s = q or s = q = 1, then theorem is not true .

Next The following theorem is another version of Theorem 2.2 when q = s by summing the restrictions that $T, h: B \to B$, where B is a ρ-closed and ρ-bounded subset of X_ρ.

Theorem 2.4. Let X_p be a ρ-complete modular space, where ρ satisfy the Δ_2 -condition and B is a ρ-closed and ρ-bounded subset of X_ρ. Suppose $T, h: B \to B$ are two ρ-compatible mappings such that T(X_ρ)⊆ $h(X_\rho)$ and

$$\int_0^{\rho(s(T_x-T_y))}\psi(t)dt \leq \alpha\int_0^{\rho\left(s\left(\frac{h_x-h_y}{h_xh_y}\right)\right)}\psi(t)dt + \beta\int_0^{\rho\left[s\left[\frac{\left(\frac{h_x-h_y}{h_xh_y}\right)+(hx_{n+1}-Tx_n)}{1+\left(\frac{h_x-h_y}{h_xh_y}\right)(hx_{n+1}-Tx_n)}\right]\right]}\psi(t)dt$$

(1)

For all $x, y \in B$, where $s, r \in R^+$ with r ∈ (0,1) $and\ \psi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable , non negative and

$$\int_0^{\epsilon}\psi(t)dt > 0 \quad \text{for all} \quad \epsilon > 0 \qquad (2)$$

If one of h and T is continuous then there exist a unique common fixed point of h and T.

Proof : Let $x \in B,\ and\ m, n \in N\ ,\ then$

$$\int_0^{\rho\left(s\left(T_{x_{n+m}}-T_{x_m}\right)\right)}\psi(t)dt \leq \alpha\int_0^{\rho\left(s\left(\frac{h_{x_{n+m}}-h_{x_m}}{h_{x_{n+m}}h_{x_m}}\right)\right)}\psi(t)dt$$

$$+\beta\int_0^{\rho\left[s\left[\frac{\left(\frac{h_{x_{n+m}}-h_{x_m}}{h_{x_{n+m}}h_{x_m}}\right)+(hx_{2n+m+1}-Tx_{2n+m})}{1+\left(\frac{h_{x_{n+m}}-h_{x_m}}{h_{x_{n+m}}h_{x_m}}\right)(hx_{2n+m+1}-Tx_{2n+m})}\right]\right]}\psi(t)dt$$

[7]

$$\leq \alpha\int_0^{\rho\left(s\left(\frac{T_{x_{n+m-1}}-T_{x_{m-1}}}{T_{x_{n+m-1}}T_{x_{m-1}}}\right)\right)}\psi(t)dt + \beta\int_0^{\rho\left[s\left[\frac{\left(\frac{T_{x_{n+m-1}}-T_{x_{m-1}}}{T_{x_{n+m-1}}T_{x_{m-1}}}\right)+(Tx_{2n+m}-Tx_{2n+m})}{1+\left(\frac{T_{x_{n+m-1}}-T_{x_{m-1}}}{T_{x_{n+m-1}}T_{x_{m-1}}}\right)(Tx_{2n+m}-Tx_{2n+m})}\right]\right]}\psi(t)dt$$

$$\leq (\alpha+\beta)\int_0^{\rho\left(s\left(\frac{T_{x_{n+m-1}}-T_{x_{m-1}}}{T_{x_{n+m-1}}T_{x_{m-1}}}\right)\right)}\psi(t)dt$$

Further proceeding as we get,

$$\int_0^{\rho\left(s\left(T_{x_{n+m}}-T_{x_m}\right)\right)} \psi(t)dt \le r^n \int_0^{\rho(s(Tx_n-x))} \psi(t)dt$$
$$\le r^n \int_0^{\mu_\rho(B)} \psi(t)dt$$

Since B is ρ-bounded then,

$$\lim_{n,m\to\infty} \int_0^{\rho\left(s(T_{x_{n+m}}-T_{x_m})\right)} \psi(t)dt \le 0$$

Which implies that $\lim_{n,m\to\infty} \rho\left(s\left(T_{x_{n+m}} - T_{x_m}\right)\right) \to 0$

Therefore by the Δ_2 -condition ,
(T x_n)n$\in N$ is a ρ-cauchy, since X_ρ is a ρ-complete and B is ρ-closed, then there exist a $w \in B$ such that $\lim_{n\to\infty} \rho(s(Tx_n - w)) \to 0$
If T is continuous ,then $T^2x_n \to Tw \; and \; Thx_n \to Tw$, since $\rho(\,hTx_n - Thx_n) \to 0 then\ by$ $\rho\ compatiblity$, $hTx_n \to Tw$
Therefore we proves w is a fixed point of T as hw = Tw = w .
Finally , let w and v are two arbitrary common fixed points of T and h. then

$$\int_0^{\rho(s(w-v))} \psi(t)dt = \int_0^{\rho\left(s\left(\frac{(Tw-Tv)}{Tw\ Tv}\right)\right)} \psi(t)dt$$
$$\le r \int_0^{\rho\left(s\left(\frac{(w-v)}{w\ v}\right)\right)} \psi(t)dt$$

Which implies that $\rho(s(w-v)) = 0 \;\; and\ hence\ \; w = v.$

In the next section , the existence of a common fixed point of integral type for a quasi-contraction map in modular spaces is presented.

3. A common fixed point theorem of integral type for quasi-contraction maps

In this section we study for quasi-contraction maps of integral type for this By definition

Definition 3.1. Two self mappings $T, h : X_\rho \to X_\rho$ of a modular space X_ρ are (s, q, t) generalized contraction of integral type. If there exists $0 < q < 1$ and s, q$\in R^+$ with $s > q$ such that

$$\int_0^{\rho(s(T_x-T_y))} \psi(t)dt \le t \int_0^{m(x,y)} \psi(t)dt \quad \text{for all } x, y \in X_\rho,$$

Where,

$$m(x,y) = max\left\{\rho\big(l(hx-hy)\big),\ \rho\big(l(hx-Tx)\big), \rho\big(l(hy-Ty)\big), \frac{[\rho(l(hx-Ty))+\rho(l(hy-Tx))]}{2}\right\}$$

And $\psi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and

$\int_0^{\epsilon} \psi(t)dt > 0$ for all $\epsilon > 0$.

We now shows in the main theorem in this section.

Theorem 3.2. Let X_ρ be a ρ-complete modular space , where ρ satisfies the Δ_2 –condition. Suppose T and H are (s, q, t)-generalized contraction of integral type self maps of X_ρ and T(X_ρ) $\subseteq h(X_\rho)$. If one of h and T is continuous, then there exists a unique common fixed point of hand T.

Proof. Let us choose $s > 2q$ $and\ also$ $\lambda \in R^+$ be the conjugate of $\frac{s}{q}$; i.e.

$\frac{q}{s} + \frac{1}{\lambda} = 1.$ $Then$, $s > 2q$

Implies that $\lambda q < s$.

Let x be an arbitrary point of X_ρ and generate inductively the sequence (T x_n) n∈ Nas follows :

T $x_n = hx_{n+1}$ and $T(X_\rho)) \subseteq h(X_\rho)$. Thus we have,

$$\int_0^{\rho\left(s\left(T_{x_{n+1}} - T_{x_n}\right)\right)} \psi(t)dt \leq r \int_0^{a(x_{n+1} - x_n)} \psi(t)dt$$

Where,

$$a(x_{n+1} - x_n) = max \left\{ \begin{array}{c} \rho\left(q(hx_{n+1} - hx_n)\right), \rho\left(q(Tx_n - hx_n)\right), \rho\left(q(hx_{n+1} - Tx_{n+1})\right), \\ \dfrac{\left[\rho\left(q(hx_{n+1} - Tx_n)\right) + \rho\left(q(hx_n - Tx_{n+1})\right)\right]}{2}, \\ \dfrac{\left[\rho\left(q(Tx_n - hx_n)\right) + \rho\left(q(hx_{n+1} - Tx_n)\right)\right]}{1 + \left[\rho\left(q(Tx_n - hx_n)\right)\right]\left[\rho\left(q(hx_{n+1} - Tx_n)\right)\right]} \end{array} \right\}$$

Then

[9]

$$a(x_{n+1} - x_n) = max \left\{ \begin{array}{c} \rho\left(q(hx_{n+1} - hx_n)\right), \rho\left(q(hx_{n+1} - Tx_{n+1})\right), \\ \dfrac{\left[0 + \rho\left(q(hx_n - Tx_{n+1})\right)\right]}{2}, \\ \dfrac{\left[\rho\left(q(hx_{n+1} - hx_n)\right) + 0\right]}{1 + 0} \end{array} \right\}$$

$$a(x_{n+1} - x_n) = max \left\{ \rho\left(q(hx_{n+1} - hx_n)\right), \rho\left(q(hx_{n+1} - Tx_{n+1})\right), \frac{\left[\rho\left(q(hx_n - Tx_{n+1})\right)\right]}{2} \right\}$$

Moreover, by

$$\begin{aligned} \rho\left(q(hx_n - Tx_{n+1})\right) &= \rho\left(q(Tx_{n-1} - Tx_{n+1})\right) \\ &= \rho\left(\lambda \frac{q}{\lambda}\left(T_{x_{n+1}} - T_{x_n}\right) + \frac{qs}{s}(Tx_n - Tx_{n-1})\right) \\ &= \rho\left(\lambda q\left(T_{x_{n+1}} - T_{x_n}\right) + s(Tx_n - Tx_{n-1})\right) \\ &\leq \rho\left(s\left(T_{x_{n+1}} - T_{x_n}\right) + s(Tx_n - Tx_{n-1})\right) \end{aligned}$$

Then, $a\left(x_{n+1} - x_n\right) \leq \rho\left(s\left(Tx_n - Tx_{n-1}\right)\right)$

And

$$\int_0^{\rho\left(s\left(T_{x_{n+1}} - T_{x_n}\right)\right)} \psi(t)dt \leq r \int_0^{\rho(s(Tx_n - Tx_{n-1}))} \psi(t)dt$$

Continuing this process , we have,

$$\int_0^{\rho\left(s\left(T_{x_{n+1}} - T_{x_n}\right)\right)} \psi(t)dt \leq r^n \int_0^{\rho(s(Tx - x))} \psi(t)dt$$

Taking the limit as n→ ∞ results in lim n $\rho\left(s\left(Tx_n - Tx_{n+1}\right)\right) = 0$.

Now, suppose $q < s' < 2l$. since ρ is an increasing function , then it may be written as $\rho\left(s'\left(Tx_n - Tx_{n+1}\right)\right) \leq \rho\left(s\left(Tx_n - Tx_{n+1}\right)\right)$, whenever $s' < 2l < s$.Taking the limit from both the sides of this inequality shows that lim n

$\rho\left(s'\left(Tx_n - Tx_{n+1}\right)\right) = 0, for\ l < s' < 2l$. thus we have

lim n $\rho\left(s\left(Tx_n - Tx_{n+1}\right)\right) = 0\ for\ any\ s > q$.

We now show that (T x_n) n∈ N is a ρ –Cauchy. If not, then using the same argument as in the proof of theorem 2.2 there exists an $\in > 0$ and two subsequences {m(k)} and {n(k)} and {n(k)} >{m(k)}≥ k

Such that $\rho\left(s\left(Tx_{m(k)} - Tx_{n(k)}\right)\right) \geq \in$

As preceeding theorem 2.2 Using the Δ_2 –condition (T x_n) n∈ N is a ρ –Cauchy. Since X_ρ is a ρ-complete, then there exists a $w \in X_\rho$ such that

$\rho\left(s\left(Tx_n - z\right)\right) \to 0$ as n→ ∞. Also taking a limit as n→ ∞, w is a fixed point of T.

Moreover, If h is continuous instead of T, by a similar proof as above , $hw = Tw = w$. Now, for uniqueness let w and v be two arbitrary fixed points of T and h. then,

$$m(w, v) = max\left\{\rho\big(q(w - v)\big), 0, 0, \frac{\left[\rho\big(q(w - v)\big) + \rho\big(q(v - w)\big)\right]}{2}\right\}$$

Therefore

$$\int_0^{\rho(s(w-v))} \psi(t)dt \ \leq\ r \int_0^{\rho(q(w-v))} \psi(t)dt$$

Which implies that $w = v$.

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