Buckling analysis of plate structures in composite materials with a new theory of trigonometric deformation

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Abstract. In this investigation a new trigonometric shear deformation plate theory involving only four unknown functions, as against five functions in case of other shear deformation theories, The theory presented is variationally consistent, does not require shear correction factor, and gives rise to transverse shear stress variation such that the transverse shear stresses vary parabolically across the thickness satisfying shear stress free surface conditions. Governing equations are derived from the virtual work principle. The closed-form solution of a simply supported rectangular plate subjected to in-plane loading has been obtained by using the Navier method. The effectiveness of the theories is brought out through illustrative examples.

1 Introduction

Composite materials have many advantages over conventional materials such as steel and concrete in structural performance with their superior strength-to-weight ratios as well as stiffness-to-weight ratios. For this reason, composite materials, especially in the form of laminated composite plates, are becoming increasingly used in many structural applications for civil infrastructure systems. Therefore, it requires better understanding of their structural behaviour and failure conditions for safer and more economical design.

One of the main failure mechanisms in laminated composite plates is buckling. As in the case of any isotropic homogeneous plates, the presence of in plane loadings may cause buckling in orthotropic laminated plates. The laminates, unlike plates made of conventional materials, are inherently anisotropic and inhomogeneous, thereby making their buckling deformations more complicated. Thus the accurate knowledge of critical buckling loads is essential for reliable and lightweight structural design. Many exact solutions for isotropic and orthotropic plates have been developed, most of them can be found in Timoshenko and Woinowsky-Krieger [1], Timoshenko and Gere [2], Bank and Jin [3], and Kang and Leissa [4]. In company with studies of buckling behaviour of plate, many plate theories have been developed. The simplest one is the classical plate theory (CPT) which neglects the transverse normal and shear stresses. This theory is not appropriate for the thick and orthotropic plate with high degree of modulus ratio. In order to overcome this limitation, the shear deformable theory which takes account of transverse shear effects is recommended. The Reissner [5] and Mindlin [6] theories are known as the first-order shear deform- able theory (FSDT), and account for the transverse shear effects by the way of linear variation of in-plane displacements through the thickness. However, these models do not satisfy the zero traction boundary conditions on the top and bottom faces of the plate, and need to use the shear correction factor to satisfy the constitutive relations for transverse shear stresses and shear strains. For these reasons, many higher-order theories have been developed to improve in FSDT such as Levinson [7] and Reddy [8]. Indeed, Reddy [8] put forward a parabolic shear deformation theory (PSDT) which considers not only the transverse shear strains, but also their parabolic variation across the plate thickness. Recently, Tounsi and his co-workers [9, 10] developed a new simple higher order theories involving only four unknown functions, as against five functions in case of Reissner’s and Mindlin’s theories. The accuracy of this theory has been demonstrated for static bending of plates by Merdaci et al. [9] and Ameur et al. [10], therefore, it seems to be important to extend this theory to the static buckling behaviour of plate.

In this paper, the new trigonometric shear deformation plate theory developed by El meiche et al. [9] and Ameur et al. [10] has been extended to the buckling behaviour of orthotropic plate subjected to the in-plane loading. Using the Navier method, the closed-form solutions have been obtained.Numerical examples involving side-to-thickness ratio and modulus ratio are presented to illustrate the accuracy of the present theory in predicting the critical buckling load of isotropic and orthotropic plates. The results based on the present theory are compared with those obtained by the first-order shear deformation plate theories and classical plate theory. The influences of several parameters are discussed.

2 Present refined shear deformation theory

Unlike the other theories, the number of unknown functions involved in the new trigonometric shear deformation plate theory is only four, as against five in case of other shear deformation theories. The theory presented is variationally consistent, does not require shear correction factor, and gives rise to transverse shear stress variation such that the transverse shear stresses vary parabolically across the thickness satisfying shear stress free surface conditions.

2.1 Assumptions of the present plate theory

Assumptions of the present plate theory are as follows:

(i) The displacements are small in comparison with the plate thickness and, therefore, strains involved are infinitesimal.

(ii) The transverse displacement w includes two components of bending w_b, and shear w_s. These components are functions of coordinates x, y only.

$$w(x,y,z)=w_b(x,y)+w_s(x,y) \quad (1)$$

(iii) The transverse normal stress σ_z is negligible in comparison with in-plane stresses σ_x and σ_y.

(iv) The displacements u in *x*-direction and v in *y*-direction consist of extension, bending, and shear components.

$$U=u_0+u_b+u_s, \quad V=v_0+v_b+v_s \quad (2)$$

The bending components u_b and v_b are assumed to be similar to the displacements given by the classical plate theory. Therefore, the expression for u_b and v_b can be given as

$$u_b=-z\frac{\partial w_b}{\partial x}, \quad v_b=-z\frac{\partial w_b}{\partial y} \quad (3)$$

The shear components u_s and v_s give rise, in conjunction with w_s, to the parabolic variations of shear strains γ_{xz}, γ_{yz} and hence to shear stresses τ_{xz}, τ_{yz} through the thickness of the plate in such a way that shear stresses τ_{xz}, τ_{yz} are zero at the top and bottom faces of the plate. Consequently, the expression for u_s and v_s can be given as

$$u_s=-f(z)\frac{\partial w_s}{\partial x}, \quad v_s=-f(z)\frac{\partial w_s}{\partial y}$$

2.2 Displacement Field and Constitutive Equations

In the present analysis, displacement field models satisfying the condition of zero transverse shear stresses on the top and bottom surface of the plate are considered. Based on the assumptions made in preceding section, the displacement field can be obtained using Eqs. (1) – (4) as

$$\begin{aligned} u(x,y,z)&=u_0(x,y)-z\frac{\partial w_b}{\partial x}-f(z)\frac{\partial w_s}{\partial x}\\ v(x,y,z)&=v_0(x,y)-z\frac{\partial w_b}{\partial y}-f(z)\frac{\partial w_s}{\partial y}\\ w(x,y,z)&=w_b(x,y)+w_s(x,y)\end{aligned} \quad (4)$$

Where the function $f(z)$ is chosen in the form

$$f(z)=z-\frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right) \quad (5)$$

The strains associated with the displacements in Eq. (4) are

$$\begin{aligned} \varepsilon_x&=\varepsilon_x^0+z\,k_x^b+f(z)\,k_x^s\\ \varepsilon_y&=\varepsilon_y^0+z\,k_y^b+f(z)\,k_y^s\\ \gamma_{xy}&=\gamma_{xy}^0+z\,k_{xy}^b+f(z)\,k_{xy}^s\\ \gamma_{yz}&=g(z)\,\gamma_{yz}^s\\ \gamma_{xz}&=g(z)\,\gamma_{xz}^s\\ \varepsilon_z&=0\end{aligned} \quad (6)$$

Where

$$\begin{aligned} &\varepsilon_x^0=\frac{\partial u_0}{\partial x},\quad k_x^b=-\frac{\partial^2 w_b}{\partial x^2},\quad k_x^s=-\frac{\partial^2 w_s}{\partial x^2}\\ &\varepsilon_y^0=\frac{\partial v_0}{\partial y},\quad k_y^b=-\frac{\partial^2 w_b}{\partial y^2},\quad k_y^s=-\frac{\partial^2 w_s}{\partial y^2}\\ &\gamma_{xy}^0=\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x},\quad k_{xy}^b=-2\frac{\partial^2 w_b}{\partial x\partial y},\quad k_{xy}^s=-2\frac{\partial^2 w_s}{\partial x\partial y}\\ &\gamma_{yz}^s=\frac{\partial w_s}{\partial y},\quad \gamma_{xz}^s=\frac{\partial w_s}{\partial x},\quad g(z)=1-f'(z)\ \text{and}\\ &f'(z)=\frac{df(z)}{dz}\end{aligned} \quad (7)$$

For an orthotropic plate, the constitutive relations can be written as:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}Q_{11}&Q_{12}&0\\ Q_{12}&Q_{22}&0\\ 0&0&Q_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}\ \text{and} \quad (8)$$

$$\begin{Bmatrix}\tau_{yz}\\ \tau_{zx}\end{Bmatrix}=\begin{bmatrix}Q_{44}&0\\ 0&Q_{55}\end{bmatrix}\begin{Bmatrix}\gamma_{yz}\\ \gamma_{zx}\end{Bmatrix} \quad (4)$$

where (σ_x, σ_y, τ_{xy}, τ_{yz}, τ_{yx}) and (ε_x, ε_y, γ_{xy}, γ_{yz}, γ_{yx}) are the stress and strain components, respectively. Q_{ij} are the plane stress reduced elastic constants in the material axes of the plate, and are defined as

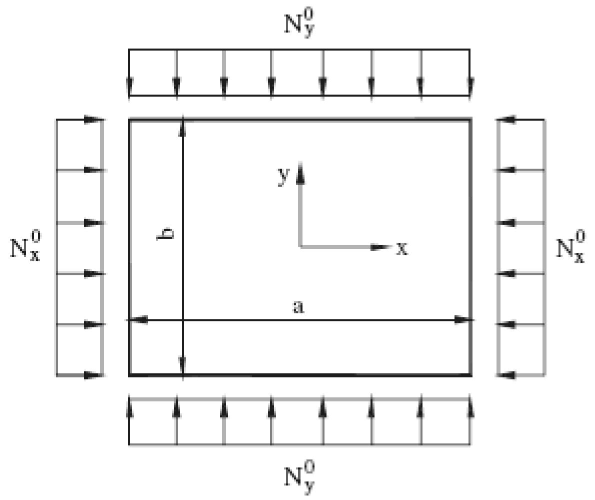
$$Q_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}}, \tag{9}$$
$$Q_{12} = \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{13}$$

in which E_{11}, E_{22} are Young's modulus, G_{12}, G_{23}, G_{13} are shear modulus, and ν_{12}, ν_{21} are Poisson's ratios. For the isotropic plate, these above material properties reduce to $E_{11} = E_{22} = E$, $G_{12} = G_{23} = G_{13} = G$, $\nu_{12} = \nu_{21} = \nu$. The subscripts 1, 2, 3 correspond to x, y, z directions of Cartesian coordinate system, respectively.

3 Governing equations

When a plate is subjected to in-plane compressive forces (Fig. 1), and if the forces are small enough, the equilibrium of the plate is stable and the plate remains flat until a certain load is reached. At that load, called the buckling load, the stable state of the plate is disturbed and plate seeks an alternative equilibrium configuration accompanied by a change in the load-deflection behaviour.

Fig. 1. Rectangular plate under in-plane forces.



The governing equations are derived by using the virtual work principle, which can be written for the plate as

$$\int_V [\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}] dV + \int_A [\bar{N}\delta(w_b + w_s)] dA = 0 \tag{10}$$
$$\bar{N} = \left[N_x^0 \frac{\partial^2 (w_b + w_s)}{\partial x^2} + N_y^0 \frac{\partial^2 (w_b + w_s)}{\partial y^2} + 2N_{xy}^0 \frac{\partial^2 (w_b + w_s)}{\partial x \partial y} \right]$$

The stress resultants N, M, and S are defined by

$$\begin{Bmatrix} N_x, & N_y, & N_{xy} \\ M_x^b, & M_y^b, & M_{xy}^b \\ M_x^s, & M_y^s, & M_{xy}^s \end{Bmatrix} = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) \begin{Bmatrix} 1 \\ z \\ f(z) \end{Bmatrix} dz, \tag{11}$$
$$(S_{xz}^s, S_{yz}^s) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) g(z) dz.$$

Substituting Equation (8) into Equation (13) and integrating through the thickness of the plate, the stress resultants are given as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix},$$
$$\begin{Bmatrix} M_x^b \\ M_y^b \\ M_{xy}^b \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix}, \tag{12}$$
$$\begin{Bmatrix} M_x^s \\ M_y^s \\ M_{xy}^s \end{Bmatrix} = \begin{bmatrix} H_{11}^s & H_{12}^s & 0 \\ H_{12}^s & H_{22}^s & 0 \\ 0 & 0 & H_{66}^s \end{bmatrix} \begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix},$$
$$\begin{Bmatrix} S_{xz}^s \\ S_{yz}^s \end{Bmatrix} = \begin{bmatrix} A_{44}^s & 0 \\ 0 & A_{55}^s \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

and stiffness components are given as:

$$(A_{ij}, D_{ij}, H_{ij}^s) = \int_{-h/2}^{h/2} Q_{ij} (1, z^2, f^2(z)) dz \qquad (i, j = 1,2,6)$$
$$A_{ij}^s = \int_{-h/2}^{h/2} Q_{ij} g^2(z) dz \qquad (i, j = 4,5) \tag{13}$$

Collecting the coefficients of δu_0, δv_0, δw_b and δw_s in Equation (12), the governing equations are obtained as:

$$\delta u_0 : \; N_{x,x} + N_{xy,y} = 0$$
$$\delta v_0 : \; N_{xy,x} + N_{y,y} = 0 \tag{14}$$
$$\delta w_b : \; M_{x,xx}^b + 2M_{xy,xy}^b + M_{y,yy}^b + \bar{N} = 0$$
$$\delta w_s : \; M_{x,xx}^s + 2M_{xy,xy}^s + M_{y,yy}^s + S_{xz,x}^s + S_{yz,y}^s + \bar{N} = 0$$

The displacement functions that satisfy the equations of boundary conditions (18) are selected as the following Fourier series:

$$\begin{Bmatrix} w_b \\ w_s \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} W_{bmn} \sin(\lambda x)\sin(\mu y) \\ W_{smn} \sin(\lambda x)\sin(\mu y) \end{Bmatrix} \quad (15)$$

4 Numerical results and discussion

In this study, buckling analysis of simply supported rectangular plates by the new trigonometric shear deformation plate theory (with four unknown functions) is suggested for investigation. Plate subjected to the loading conditions, as shown in Fig. 2, is considered to illustrate the accuracy of the present models in predicting the buckling behavior of the plate. Comparisons are made with solutions obtained using the first and parabolic shear deformation theory (with five unknown functions). In order to investigate the effects of side-to-thickness ratio and modulus ratio, the present example is applied for isotropic and orthotropic square plates. The following engineering constants are used [11]

$$E_1 / E_2 \text{ Varied, } G_{12} / E_2 = G_{13} / E_2 = 0.5 \quad (16)$$
$$G_{23} / E_2 = 0.2 \,,\; \nu_{12} = 0.25$$

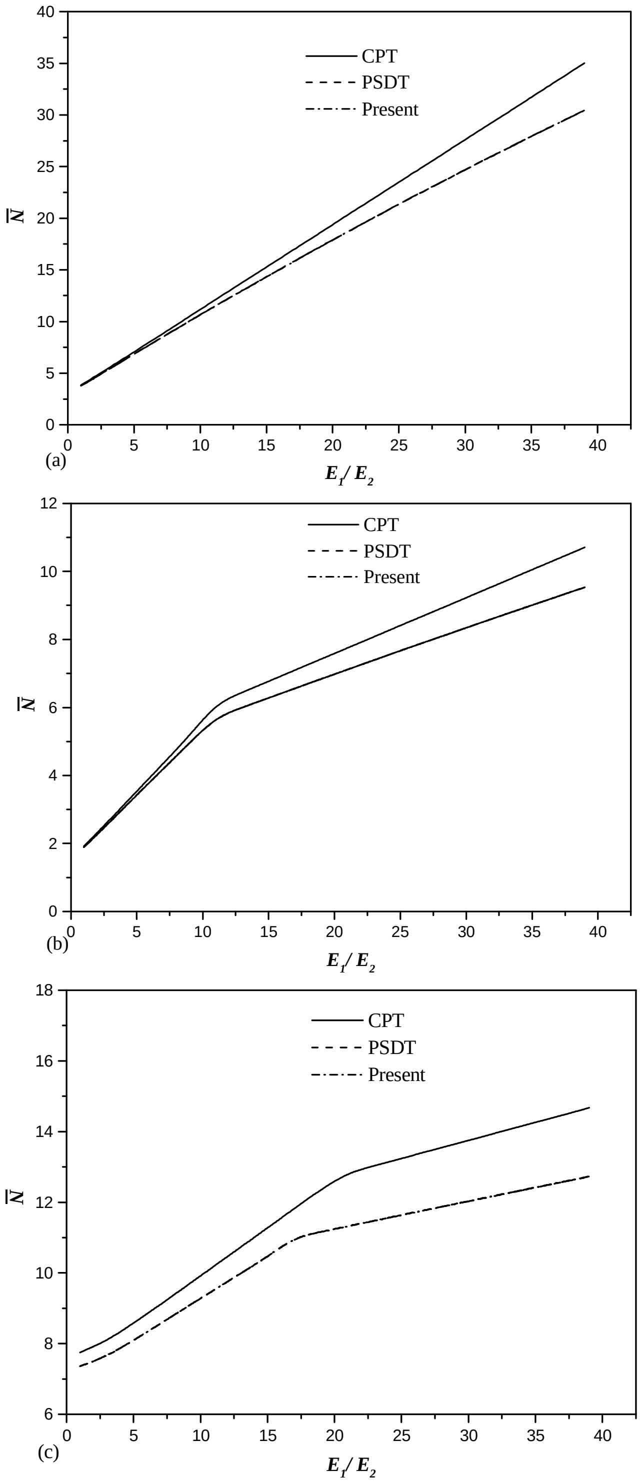
For simplicity, the non-dimensional critical buckling parameter is defined as

$$\overline{N} = \frac{N_{cr} a^2}{E_2 h^3} \quad (17)$$

Where a the length of the square is plate and h is the thickness of the plate.

Fig 2 shows that when the orthotropic plate is used, the difference between the present model and CPT will increase with the increase of modulus ratio E_1 / E_2 . Furthermore, the present model (with only four unknown functions) gives identical results to those obtained with PSDT (with five unknown functions) for all values of E_1 / E_2 .

Fig. 2. The effect of modulus ratios on the critical buckling load of square plate with $a / h = 20$: (a) plate subjected to uniaxial compression; (b) plate subjected to biaxial compression; (c) subjected to tension in the *x* direction and compression in the *y* direction.



Provide critical buckling loads for simply supported square plate using various plate theories. the results of the present theory are close to the results of the first shear deformation theory (FSDT). Hence, the present new trigonometric shear deformation plate theory (with four unknown functions) gives comparable results to those obtained with FSDT (with five unknown functions). As presented in Fig 2 and Fig 3, the differences between the present model and FSDT with the shear correction factor 5/6, and the present model and FSDT with the shear correction factor 1 are 16.80 % and 2.82 %, respectively, for the same case of square plate (a= b= 5h and $\boldsymbol{E}_1 / \boldsymbol{E}_2 = 40$). It can be seen from figures 2–3 that the difference of critical buckling load between the present model and FSDT depends on not only the side-to-thickness and modulus ratios, but also the in-plane loading conditions

5. Conclusions

The buckling analysis of isotropic and orthotropic plates using a new trigonometric shear deformation plate theory is presented. The number of primary variables in this theory is even less than that of first- and higher-order shear deformation plate theories. The theory takes account of transverse shear effects and parabolic distribution of the transverse shear strains through the thickness of the plate, hence it is unnecessary to use shear correction factors. It can be concluded that the present new plate theory can accurately predict the critical buckling loads of the isotropic and orthotropic plates.

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