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Groundwater Flows in the Vicinity of Two Well Systems with Finite Element Method using FEniCs Software

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Abstract
Groundwater is available in usable quantities only in aquifers. An aquifer is a geological formation which contains water and permits significant amounts of water to move through it under ordinary field conditions. Aquifer can be categorized depending on the hydraulic conductivity as: isotropic vs. anisotropic, homogeneous vs. nonhomogeneous, etc. In this paper effort is made to see the flow of ground water in aquifer in the vicinity of two well systems, pumping and recharging wells, in some rectangular shaped domain in two dimensions where the pumping well is located at point $p_0 = (-x_1, 0)$ while the recharging well is located at point $p_1 = (x_1, 0),\ x_1 > 0.$ No flow condition is assumed on the boundaries of the domain, while Dirichlet boundary condition is imposed on the boundary of the pumping well and inhomogeneous normal Neumann boundary condition is imposed on the boundary of the recharging well. Numerical experiment is made at both homogeneous and inhomogeneous isotropic aquifer cases. And in each of the aquifer cases, stationary and non-stationary cases are also considered. Finite Element Method is used for the purpose of analysis, where finite element mesh is generated using an external free 3D finite element mesh generator called **Gmsh**. Numerical experiment is performed using free software package called **FEniCs**. Based on the results of the numerical experiment all the cases exhibit the same phenomena. Meaning that, the draw-down in the water level is higher near the pumping well and decrease radially outward creating a feature called the cone of depression. This happens because of a pattern of radially converging flow to the well from the surrounding aquifer which causes the lowering of the water level (the piezometric surface) extending outward from the well. And the build-up in the water level is higher near the recharging well but decrease radially inward creating a feature called the cone of impression. This happens due to a pattern of radially diverging flow from the recharging well made to produce a buildup in the water level (or the piezometric surface).
Keywords: Finite Element Methods, Groundwater, Well, FEniCs, Gmsh

1. Introduction to Groundwater Flow

The term groundwater refers to water found beneath the surface of the ground. Groundwater occurs in two separated zones, the saturated zone and the unsaturated zone. The two zones are separated by surface known as water table. The saturated zone is the one which is below the water table while the unsaturated zone is found above the water table. In the saturated zone, water completely fills the void space of the considered porous medium.

Groundwater is found everywhere. But it is available in usable quantities only in aquifers. An aquifer is a geological (or a group of) formation which contains water and permits significant amounts of water to move through it under ordinary field conditions. Aquifers can be classified as confined or unconfined depending on the absence or presence of a water table. Confined aquifers are bounded between two layers of impervious formation (lower permeability materials) while unconfined aquifer is one in which water table (or phreatic surface) serves as its upper boundary. It is impossible to determine the exact direction of groundwater flow based on surface features alone. However, we know that water in the aquifer near a pumping well will flow toward the well to result in a draw-down near the well. Pumping well is a well through which water is extracted from an aquifer. And the flow of ground water near the recharging well (a recharging well is a well through which water is added to an aquifer) is the reverse of pumping well. The groundwater flows away from the well resulting build-up near the well. In this paper the nature of the flow of ground water near the two well(pumping and recharging well) system in a confined aquifer will be observed using finite element method by using FEniCs software by considering different cases in two dimensional situation.

2. Groundwater Flow in a Confined Aquifer

The fundamental continuity equation for the flow through a confined aquifer of thickness B as shown in the figure (1) is given by:

$$\nabla.(T.\nabla h) + N = S\frac{\partial h}{\partial t} \qquad (1)$$

, where h = piezometric head; S = storativity coefficient; N=volumetric flux per unit volume representing source/sink terms; t = time; T = BK, transmissivity; B= the aquifer thickness; K= hydraulic conductivity

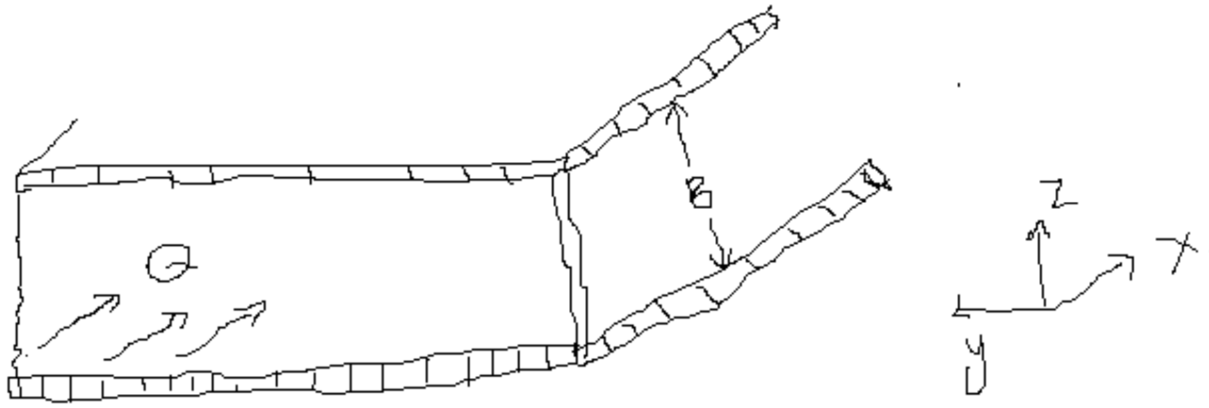


Figure 1: Groundwater Flow in a confined Aquifer

The two terms on the left-hand side express excess inflow over outflow per unit area and unit time by groundwater flow, and by net replenishment respectively. The term on the right-hand side expresses the resulting change in storage (volume per unit area and per unit time).

It is important to note that T and S are aquifers properties. A storativity S for a confined aquifer is defined as the volume of water released from storage (or added to it) per unit horizontal area of aquifer and per unit decline (or rise) of piezometric head h.

The product BK , denoted by T appears whenever the flow through the entire thickness of the aquifer is being considered, is called transmissivity. It is an aquifer characteristic which is defined by the rate of flow per unit width through the entire thickness of an aquifer per unit hydraulic gradient. The concept is valid only in two-dimensional flow, or aquifer-type flow. For the sake of simplicity in this paper N is considered to be zero.

3. Isotropic Versus Anisotropic Aquifer

In isotropic aquifers the hydraulic conductivity (which is denoted by K) is equal for flows in all directions, while in anisotropic aquifer it is differs, notably in horizontal and vertical sense. Meaning that, if the hydraulic conductivity K at the considered point is independent of direction, the medium is said to be isotropic at that point, otherwise it is called anisotropic aquifer. Isotropic aquifer is composed of only isotropic materials; hence the hydraulic conductivity K and transmissivity T are scalar.

3.1. Homogeneous Isotropic Aquifer

A porous medium domain is said to be homogeneous if its hydraulic conductivity K is the same at all its points. In homogeneous isotropic aquifer the hydraulic conductivity K and transmissivity T are scalar and constant. In that case equation (1) above reduces to:

$$T\nabla.(\nabla h) = S\frac{\partial h}{\partial t} \quad (2)$$

Taking the notation $\nabla.(\nabla h) = \Delta h$. Hence, the hydraulic head h can be obtained by solving the equation:

$$T\Delta h = S\frac{\partial h}{\partial t} \quad (3)$$

Assuming a negligible S , under two dimensional cases, equation (3) becomes:

$$\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad (4)$$

3.2. Heterogeneous anisotropic Aquifer

A porous medium domain is said to be heterogeneous if its hydraulic conductivity K is not the same at all its points. The hydraulic head h in a heterogeneous anisotropic medium in two dimensional cases is given by:

$$\nabla.(T.\nabla h) = S\frac{\partial h}{\partial t} \quad (5)$$

, where, $T = Bk$, with $K = \begin{pmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{pmatrix}$. The hydraulic conductivity tensor is symmetric matrix, meaning that $K_{xy} = K_{yx.}$

3.3. Hydraulics of Pumping and Recharging Wells

Water well is a hole that is used for the purpose of extracting (or pumping out or discharging) groundwater from the subsurface or recharging water into the ground. It is impossible to determine the exact direction of groundwater flow based on surface features alone. However, we know that water in the aquifer near a pumping well will flow toward the well. Hence, water wells affect the overall flow pattern of water in the aquifers. In a confined aquifer, the discharge of water through well is provided by the release of water stored in the aquifer due to compressibility of water.

As water is extracted from a well the water level within the well drops, which produces a pattern of a radially converging flow to the well from the surrounding aquifers and this causes lowering of the water level (or the piezometric surface) extending outward from the well. The drop-dawn in the water level is high near the well and decreases radially outward creating a feature called the cone of depression. The drop-dawn is the vertical distance between the initial piezometric surface and the piezometric surface at some later time t at the same point. Water can also be recharged from the ground surface through a well. A recharging well produces a pattern of radially diverging flow from the well to produce a pattern of radially diverging flow from the well and this causes buildup of the water level (or the piezometric surface) near the recharging well. Figure 2 illustrate the above explanation clearly.

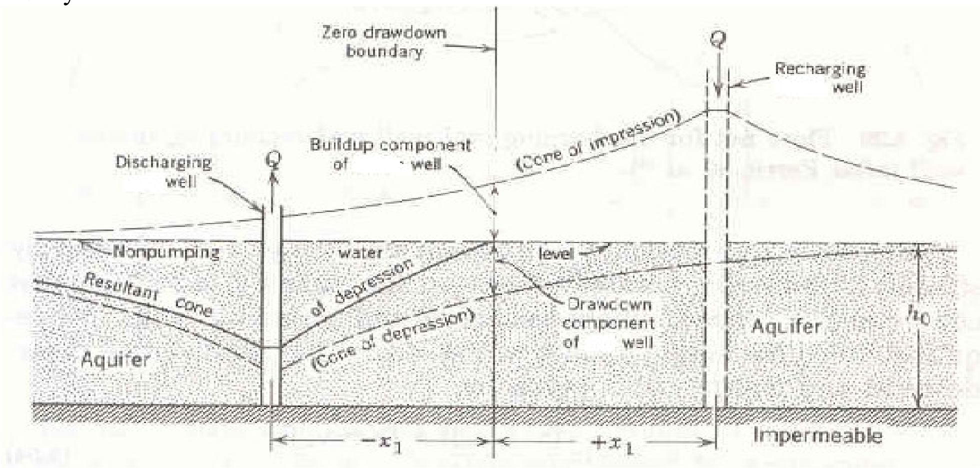


Figure 2. Pumping and Recharging Wells(taken from free source)

3.3.1 Variation Formulation and Numerical Experiment of Ground water flow near the Vicinity of Two well systems: Pumping and Recharging Wells

In this section numerical experiment is made using Finite element method by using FEniCs software in order to see and verify the theoretical explanation of the flow of ground water in a confined aquifer near a well system. For the sake of simplicity two well systems (one pumping and one recharging well) are considered. Rectangular-shaped domain in two dimensions are chosen, where the pumping well is located at point $p_0 = (-x_1, 0)$ while the recharging well is located at point $p_1 = (x_1, 0)$, $x_1 > 0$.

It is assumed that no flow condition is considered on the boundary of the rectangular domain which is denoted by Γ_2 . That is the normal derivative is assumed to be zero on the boundary Γ_2 . And a Dirichlet boundary condition is set on the boundary of the pumping well which is denoted by Γ_0 and inhomogeneous Neumann boundary condition is set on the boundary of the recharging well which is denoted by denoted Γ_1. That is:

$$\begin{aligned} h &= p \;\; on \;\; \Gamma_0 \\ \frac{\partial h}{\partial n} &= q \;\; on \;\; \Gamma_1 \\ \frac{\partial h}{\partial n} &= 0 \;\; on \;\; \Gamma_2 \end{aligned} \tag{6}$$

3.3.2. Homogeneous Isotropic Aquifer

Since the hydraulic conductivity K and transmissivity T are the same at all points in the homogeneous isotropic aquifer case, the hydraulic head h can be given by the equation:

$$T\nabla h = S\frac{\partial h}{\partial t} \tag{7}$$

So, in the following we will see the homogeneous Isotropic aquifer in two cases, stationary and non-stationary cases, by taking an appropriate boundary and initial values. In either case, we consider wells with very small radius so that the piezometric head does not vary over them.

Case 1. Stationary Case

$$\Delta h = 0 \; in \;\; \Omega \tag{8}$$

Where, $\Omega = [-x_1, x_1] \times [-x_1, x_1] = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ the domain and $\partial\Omega$ is the boundary of the domain. Equation (8) together with equation (6) becomes the complete boundary-value problem in stationary homogeneous isotropic aquifer case.

For instance let us set $p = 10$ and $q = -10$. These values for the variables have no physical relevance or meaning, they are an arbitrary chosen values. The variational formulation of the complete boundary value problem is then stated as:

Find $h \in V \equiv \{v \in H^1(\Omega)|v(0) = 0\}$ such that:

$$\int_\Omega v \,\Delta\, h dx = 0 \,,\ \forall\ v \in \mathrm{V} \tag{9}$$

Using Green's formulas as usual for the second derivative term, equation (9) becomes:

$$-\int_\Omega \nabla h . \nabla v dx + \int_{\partial\Omega} v \frac{\partial h}{\partial n} ds = 0 \tag{10}$$

Then, split the boundary integral in to parts over Γ_0, Γ_1 and Γ_2 since we have different boundary conditions at those parts. But the boundary integral vanishes on Γ_0 and Γ_2 to give us

$$\int_{\partial\Omega} v \frac{\partial h}{\partial n} ds = \int_{\Gamma_1} v \frac{\partial h}{\partial n} ds \tag{11}$$

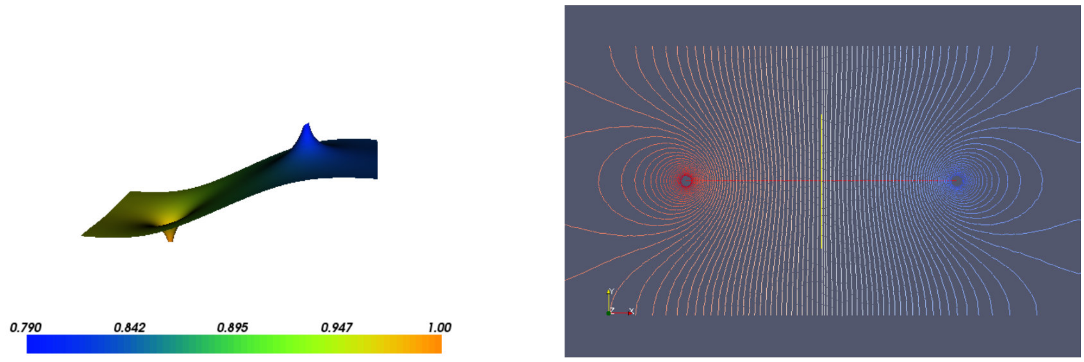
Therefore equation (10) reduces to:

$$-\int_\Omega \nabla h . \nabla v dx + \int_{\Gamma_1} qvds = 0 \tag{12}$$

The variational form or the weak form of our problem then becomes:

$$a(u, v) = \int_\Omega \nabla h . \nabla v dx$$
$$L(v) = \int_{\Gamma_1} qvds \tag{13}$$

After appropriate implementation of the weak form and all relevant information in FEniCs the following solution graph (figure (3a)) and the contour plot (figure (3b)) are obtained.



(a): Draw down and build-up in two Well systems (b). The contour plot

Figure 3. Stationary case for homogeneous isotropic aquifer

The solution graph (figure (3a)) shows the nature of the groundwater flow in the vicinity of the two well systems (pumping and recharging wells). As can be seen from graph, the one that looks like a valley shows the drop in the water level while the one that looks like the peak shows the buildup in the water level. The drop in the water level is high near the pumping well and decreases radially outward creating a feature called the cone of depression. This is happened because of a pattern of radially converging flow to the well from the surrounding aquifer which causes the lowering of the water level (or the piezometric surface) extending outward from the well. And a pattern of radially diverging flow from the recharging well made to produce a build-up in the water level (or the piezometric surface). Again the buildups in the water level also high near the recharging well. The contour plot shows how the flow line looks like around the wells.

Case 2: Non-Stationary (Time dependent) Case

Considering equation (3) with an initial condition $h = h_0 = 0$ at $t = 0$ together with equation (6) above to have the complete time dependent homogeneous isotropic initial and boundary problems:

$$\Delta h = S \frac{\partial h}{\partial t} \text{ in } \Omega$$
$$h = h_0 = 0 \text{ at } t = 0 \tag{14}$$

$$\frac{\partial h}{\partial n} = q \text{ on } \Gamma_1$$
$$\frac{\partial h}{\partial n} = q \text{ on } \Gamma_2$$

The variational form of the equation (14) and using Backward Euler formula to discretize the time derivative to have:

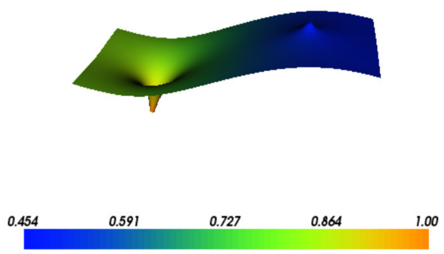
$$\int_{\Omega} \frac{S}{T}\frac{\partial h}{\partial t}\Big|_{t=t_k} v dx = -\int_{\Omega} \nabla h^k . \nabla v dx + \int_{\Gamma_1} q v ds \tag{15}$$

$$\int_{\Omega} \frac{S}{T}\frac{h^k - h^{k-1}}{\Delta t} v dx = -\int_{\Omega} \nabla h^k . \nabla v dx + \int_{\Gamma_1} q v ds \tag{16}$$

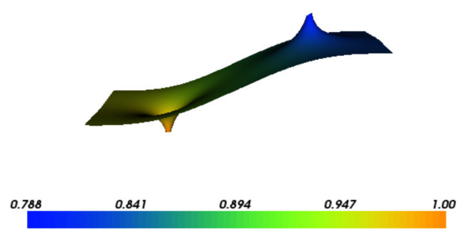
Where, h^k denotes the piezometric head at time t_k and h^{k-1} at timet_{k-1}.
Introducing the symbol h for h^k(which is natural in the programming too), the resulting weak forms can be written as

$$a(u,v) = S\int_{\Omega} hvdx + \Delta tT\int_{\Omega} \nabla h.\nabla vdx$$
$$L(v) = \int h^{k-1}vdx + \Delta tT\int_{\Gamma_1} qvds \tag{17}$$

Again after appropriately implementing the variational formulation and the initial and boundary data to get the result and the contour plot as displayed in the figure (4).

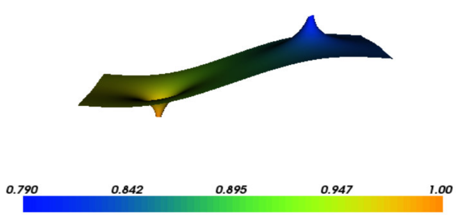


(a). Draw dawn and buildup $t = 2$.

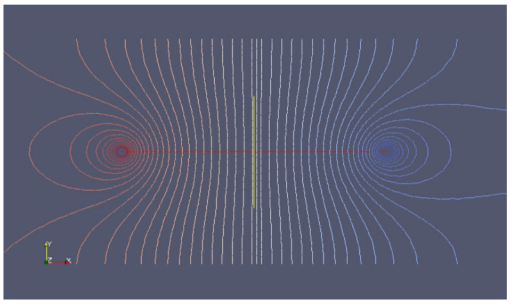


(b). Draw dawn and build-up $t = 10$

,



(c). Draw dawn and buildup Systems, $t = 100$



(d). Contour Plot

Figure (4). Time dependent case for homogeneous isotropic Aquifer

3.3.3 Inhomogeneous Anisotropic Aquifer

In this case hydraulic conductivity is not a scalar rather a matrix. Taking equation (1) above and all the relevant information. And suppose $K_{xy} = K_{yx} = 0.1$, $K_{xx} = 0.3$ and $K_{yy} = 0.2$

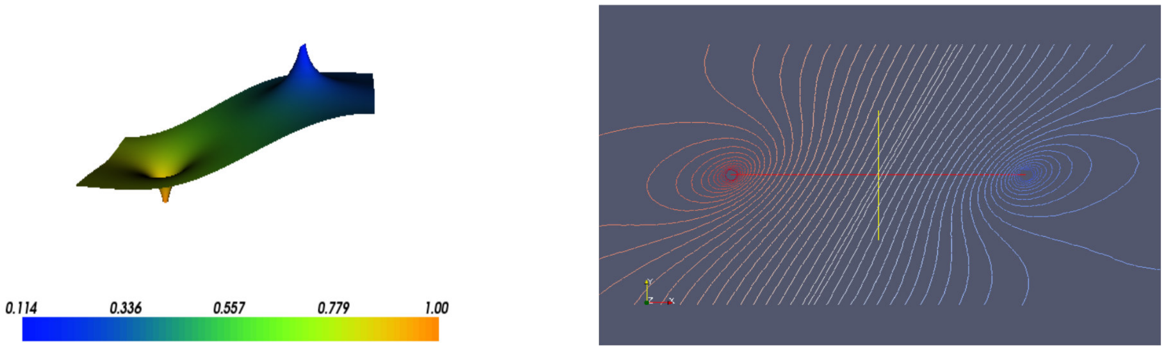
Case 1. Stationary Case

Writing the variational formulation of the problem and get the bilinear and linear form as

$$a(u,v) = \int_{\Omega} \nabla v . K \nabla h dx$$

$$L(v) = \int_{\Gamma_1} qvds \qquad (18)$$

So, the solution graphs look like at different time as follows.



(a). Draw dawn and buildup (b). The Contour Plot

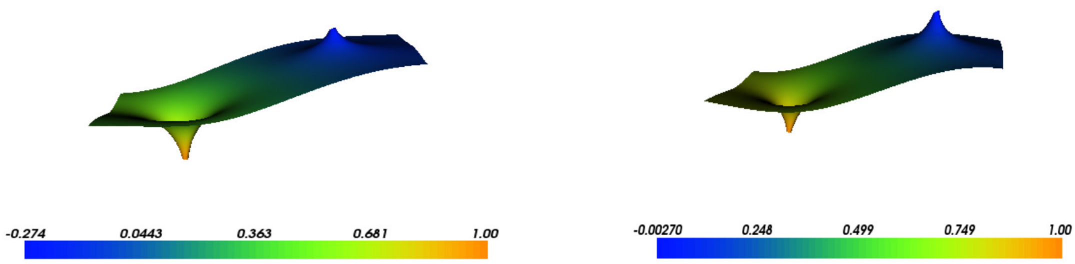
Figure 5. Inhomogeneous Anisotropic Aquifer: Stationary Case

Case 2. Time dependent Case

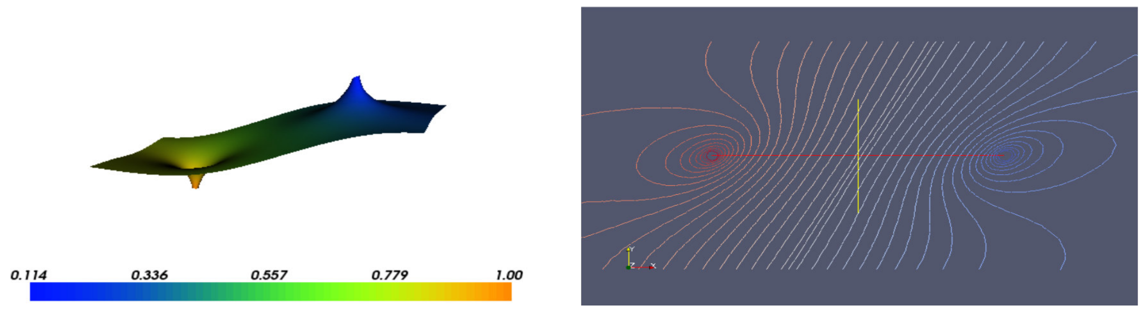
Once again including the time derivative term together with the stationary case we get time dependent problem. And again use Backward Euler method for the time derivative term and add some initial condition, $h = h_0 = 0$ at $t = 0$. Doing and writing the weak form of the problem as we did many times so far we can have

$$a(h,v) = S\int_{\Omega} hvdx + \Delta tT\int_{\Omega} \nabla v . \nabla hdx$$

$$L(v) = \int_{\Omega} h^{k-1}vdx + \Delta tT\int_{\Gamma_1} qvds \qquad (19)$$



(a). Draw dawn and build $t = 2$ (b). Draw dawn and bild-up $t = 10$



(c). Draw dawn and buildup, $t = 100$ (d). The Contour Plot

Figure 6. Time dependent case for inhomogeneous anisotropic Aquifer

4. Conclusion

Finite Element Method is used to solve the model problems the flow of groundwater in aquifer in the vicinity of two well systems: pumping and recharging wells. Finite element mesh is generated using an external free $3D$ finite element mesh generator called Gmsh while numerical experiment is performed using a free software package called FEniCs.

Rectangular shaped domain in two dimensions are considered where the pumping well is located at point $p_0 = (-x_1, 0)$ while the recharging well is located at pint $p_1 = (x_1, 0),\ x_1 > 0.$ No flow condition is assumed on the boundaries of the domain while Dirichlet boundary condition is imposed on the boundary of the pumping well and inhomogeneous normal Neumann boundary condition is imposed on the boundary of the recharging well. The numerical experiment considered the case of both homogeneous isotropic and inhomogeneous isotropic aquifer where in both cases stationary and non-stationary cases are considered. In all the cases the result shows/exhibits the same phenomena. That is, the draw-down in the water level is higher near the pumping well and decrease radially outward creating a feature called the cone of depression. This happens because of a pattern of radially converging flow to the well from the surrounding aquifer which causes the lowering of the water level (the piezometric surface) extending outward from the well. And the build-up in the water level is higher near the recharging well but decrease radially inward creating a feature called the cone of impression. This happens because a pattern of radially diverging flow from the recharging well made to produce a buildup in the water level (or the piezometric surface).

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