The Discrete Quartic Spline Interpolation Over Non Uniform Mesh

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ABSTRACT

The objective of the paper is to investigate precise error estimate concerning deficient discrete quartic spline interpolation.

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1. ITNRODUCTION & PRILIMNARIES

Discrete Splines have been introduced by Mangasarian and Shumaker [5] in connection with certain studied of minimization problems involving differences. Deficient Spline are more useful then usual spline as they require less continuity requirement at the mesh points. Malcolm [4] used discrete spline to compute non-linear spline interactively. Discrete cubic splines which interpolate given functional values at one intermediate point of a uniform mesh have been studied in [1]. These results were generalized by Dikshit and Rana [2] for non-uniform meshes. Rana and Dubey [8] have obtained local behavior of discrete cubic spline interpolation which is some time used to smooth histogram. For some constructive aspect of discrete spline reference may be made to Schumaker [3] and Jia [6].

We have develop a new function for deficient quartic spline interpolation .if we increase the degree of polynomial with boundary condition, then we find that result is better then by comparing with author [1, 2] for smoothness of the function.

In this paper, we have obtained existence, uniqueness and convergence properties of deficient discrete quartic spline interpolation matching the given function at two interior points of interval and first difference at mid points with boundary condition of function.

Let us consider a mesh Δ on [0, 1] which is defined by

$\Delta : 0 = x_0 < x_1 < ...... < x_n = 1$

Such that $x_i - x_{i-1} = P_i$ for i = 1, 2,...., n. Throughout h, will be represent a given positive real number, consider a real continuous function $s(x,h)$ defined over [0,1] which is such that its restriction s_i on $[x_{i-1}, x_i]$ is a polynomial of degree 4 or less for i = 1,2,.....n then $s(x,h)$ defines a discrete deficient quartic spline if

$$D_n^{(j)} s_i(x_i, h\_) = D_n^{(j)} s_{i+1}(x,h), \qquad j = 0,1$$

(1.1)

Where the difference operator D_n are defined as

$$D_n^{(0)} f(x) = f(x), D_n^{(1)} f(x) = \frac{f(x+h) - f(x-h)}{2h}$$

The class of all discrete deficient quartic splines is denoted by $S(4, \Delta, 1, h)$

2. EXISTENCE AND UNIQUENESS

Consider the following conditions -

$$s(\alpha_i) = f(\alpha_i) \tag{2.1}$$

$$s(\beta_i) = f(\beta_i) \tag{2.2}$$

$$D_n^{\{1\}} s(\gamma_i) = D_h^{\{j\}} s(\gamma_i) \tag{2.3}$$

for i = 1,2,.....n

Where $\alpha_i = x_{i-1} + \frac{1}{3} P_i$

$$\beta_i = x_{i-1} + \frac{1}{2} P_i = \gamma_i$$

and boundary conditions

$$s(x_0) = f(x_0) \tag{2.4}$$

$$s(x_n) = f(x_n) \tag{2.5}$$

We shall prove the following.

Problem A : Given $h \geq 0$ for what restriction on P_i does there exist a unique $s(x,h) \in S(4,1,P,h)$ which satisfy the condition (2.1) - (2.3) and boundary condition (2.4)-(2.5).

Proof : Denoting $\frac{x-x_i}{P_i}$ by t, $0 \leq t \leq 1$. Let *P(t)* be a discrete quartic Polynomial on [0,1], then we can show that

$$P(t) = P\left(\frac{1}{3}\right)q_1(t) + P\left(\frac{1}{2}\right)q_2(t) + D_h^{\{1\}}P\left(\frac{1}{2}\right)q_3(t) + P(O)q_4(t) + P(1)q_5(t) \qquad (2.6)$$

Where

$$q_1(t) = t\frac{\left[G\left(\frac{-9}{8},\frac{9}{2}\right) + G\left(\frac{45}{8},\frac{-45}{2}\right)t + G(-9,36)t^2 + G\left(\frac{9}{2},-18\right)t^3\right]}{G\left(\frac{-1}{36},\frac{1}{9}\right)}$$

$$q_2(t) = t\frac{\left[G\left(\frac{8}{9},-\frac{32}{9}\right) + G\left(\frac{-44}{9},\frac{176}{9}\right)t + t^2G(8,-32) + t^3G(-4,16)t^3\right]}{G\left(\frac{-1}{36},\frac{1}{9}\right)}$$

$$q_3(t) = t\frac{\left[G\left(\frac{-1}{9},0\right) + G\left(\frac{2}{3},0\right)t + G\left(\frac{-11}{9},0\right)t^2 + G\left(\frac{2}{3},0\right)t^3\right]}{G\left(\frac{-1}{36},\frac{1}{9}\right)}$$

$$q_4(t) = \left[\frac{1 + \left\{G\left(\frac{2}{9},\frac{-4}{3}\right)t + G\left(\frac{-23}{36},\frac{47}{9}\right)t^2 + G\left(\frac{7}{9},-8\right)t^3 + G\left(\frac{-1}{3},4\right)t^4\right.}{G\left(\frac{-1}{36},\frac{1}{9}\right)}\right]$$

$$q_5(t) = \frac{t\left[G\left(\frac{1}{72},\frac{7}{18}\right) - G\left(\frac{41}{18},\frac{-7}{12}\right)t + G\left(\frac{2}{9},4\right)t^2 - G\left(\frac{1}{6},2\right)t^3\right]}{G\left(\frac{-1}{36},\frac{1}{9}\right)}$$

Where $G(a,b) = a + bh^2, a, b$ are real numbers? We can write (2.6) in the form of the restriction $s_i(x,h)$ of the deficient discrete quartic spline $s(x,h)$ on $[x_i, x_{i+1}]$ as follows

:

$$s_i(x) = f(\alpha_i)q_1(t) + f(\beta_i)q_2(t) + P_i D_h^{\{1\}} f(\gamma_i)q_3(t)$$
$$+ s_i(x)q_4(t) + s_{i+1}(x)q_5(t)$$
(2.7)

Observing that from (2.7), $s_i(x,h)$ is discrete quartic spline on $[x_i, x_{i+1}]$ for i=1,2,...n-1. Satisfying (2.1) - (2.5).

We are set to answer the problem A in the following:

Theorem 2.1 : For any h>0, then there exist a unique deficient discrete quartic spline $s(x,h) \in S(4,1,P,h)$ which satisfies the condition (2.1) - (2.5).

Proof : Now applying continuity condition of first difference of $s(x,h)$ at x_i in (2.7) we get the following system equations.

$$P_i^3\left[G\left(\frac{1}{18},\frac{-10}{9}\right)P_{i-1}^2 + G\left(\frac{89}{9},-8\right)h^2\right]s_{i-1}$$
$$+\left[P_i^3\left\{G\left(\frac{1}{24},\frac{1}{6}\right)P_{i-1}^2 + G\left(\frac{4}{9},4\right)\right\}P_{i-1}^3\left\{G\left(\frac{2}{9},\frac{-4}{3}\right)P_i^2 + G\left(\frac{7}{9},-8\right)\right\}\right]s_i$$
$$+P_{i-1}^3\left\{G\left(\frac{1}{18},\frac{7}{18}\right)P_i^2 + G\left(\frac{2}{9},4\right)\right\}s_{i+1}$$
$$= F_i \quad \text{for i = 1,2,..... n-1} \tag{2.8}$$

Where

$$F_i = P_i^3\left[\left\{G\left(\frac{9}{8},\frac{-9}{2}\right)P_{i-1}^2 + G(9,-36)h^2\right\}f(\alpha_{i-1})\right.$$
$$+\left\{G\left(\frac{-8}{9},\frac{32}{9}\right)P_{i-1}^2 + h^2G(8,32)\right\}f(\beta_{i-1})$$
$$\left.+P_{i-1}\left\{G\left(\frac{2}{9},0\right)P_{i-1}^2 + G\left(\frac{13}{9},0\right)h^2\right\}D_n^{\{1\}}f(\gamma_i)\right] - P_{i-1}^3,$$
$$\left[G\left(\frac{8}{9},\frac{-32}{9}\right)P_i^2 + G\left(\frac{172}{9},-32\right)h^2\right\}f(\beta_i)$$
$$\left.-\left\{G\left(\frac{9}{2},\frac{9}{2}\right)P_i^2 + G(-9,36)h^2\right\}f(\alpha_i)\right]$$

$$+P_{i-1}^{3}P_{i}D_{h}^{\{1\}}f(\gamma_{i})\left\{G\left(\frac{1}{9},0\right)P_{i}^{2}+G\left(\frac{11}{9},0\right)h^{2}\right\}$$

Writing $s(x_i,h)=m_i(h)=m_i$ (Say) for all *i* we can easily see that excess of the absolute value of the coefficient of m_i dominants the sum of the absolute value of the coefficient of m_{i-1} and m_{i+1} in (2.8) under the condition theorem 2.1 and is given by

$$T_i(h)=\left[G\left(\frac{-7}{12},\frac{23}{18}\right)P_{i-1}^{2}+G(-10,24)h^{2}+G\left(\frac{31}{18},\frac{-1}{6}\right)P_{i}^{2}\right]$$

Therefore, the coefficient matrix of the system of equation (2.6) is diagonally dominant and hence invertible. Thus the system of equation has unique solution. This complete proof of theorem 2.1.

3. ERROR BOUNDS :

Now system of equation (2.6) may be written as

$$A(h),M(h)=F \tag{3.1}$$

Where A(h) is coefficient matrix and $M(h)=m_i(h)$. However as already shown in the proof of theorem 2.1 A(h) is invertible. Denoting the inverse of $A(h)$ by $A^{-1}(h)$ we note that max norm $A^{-1}(h)$ satisfies the following inequality.

$$\|A^{-1}(h)\|\le J(h) \tag{3.2}$$

Where $J(h)=\max\{T_1(h)\}^{-1}$, for convenience we assume in this section that 1 = Nh where N is positive integer, it is also assume that the mesh points $\{x_i\}$ are such that

$x_i\in[0,1]_h$, for i = 0,1,..., n

Where discrete interval $[0,1]_h$ is the set of points $\{0,h,.....Nh\}$ for a function *f* and two disjoint points x_1 and x_2 in its domain the first divided difference is defined by

$$[x_1,x_2]_f=\frac{f(x_1)-f(x_2)}{x_1-x_2} \tag{3.3}$$

For convenience we write $f^{\{1\}}$ for $D_n^{\{1\}}f$ and $w(f,P)$ for modulus of continuity of *f*, the discrete norm of the function *f* over the interval [0, 1] is defined by

$$\|f\|=\max_{x\in[0,1]}|f(x)| \tag{3.4}$$

We shall obtain the following the bound of error function $e(x)=s(x,h)-f(x)$ over the discrete interval $[0,1]_h$.

Theorem 3.1 : Suppose $s(x,h)$ is the discrete quartic splines of theorem 2.1 then

$$\|e(x)\| \le K(P,h)\, w\left(f^{\{1\}},P\right) \qquad (3.5)$$

$$\|e(x_i)\| \le J(h)\, K'(P,h)\, w\left(f^{\{1\}},P\right) \qquad (3.6)$$

$$\|e'(x)\| \le K''(P,h)\, w\left(f^{\{1\}},h\right) \qquad (3.7)$$

Where $K(P,h), K^1(P,h)$ and $K''(P,h)$ are positive function of P and h.

Proof : Equation (3.1) may be written as

$$A(h).e(x_i)=F_i(h)-A(h)\, f_i = L_i(f) \qquad \text{(Say)}$$

When $e(x_i)=s(x_i,h)-f_i$ (3.8)

We need following Lemma due to Lyche [9,10] to estimate inequality (3.3).

Lemma 3.1 : Let $\{a_i\}_{i=1}^m$ and $\{b_j\}_{j=1}^n$ be given sequence of non-negative real numbers such that $\sum a_i = \sum b_j$ then for any real value function *f* defined on discrete interval $[0,1]_h$, we have

$$\left| \sum_{x=1}^{m} a_1 \left[x_{i_0}, x_{i_1} \ldots . x_{i_k}\right]_f - \sum_{j=1}^{h} b_j \left[y_{j_0}, y_{j_1}, \ldots y_{j_k}\right]_f \right| \le w\left(f^{\{1\}} |1-Kh|\right) \sum \frac{a_i}{K!}$$

(3.9)

Where $x_{i_K}, y_{j_k} \in [0,1]_h$, for relevant values of i , j and k. We can write the equation (2.8) is of the form of error function as follows:

$$P_i^3\left[G\left(\frac{1}{18},\frac{-10}{9}\right)P_{i-1}^2 + G\left(\frac{89}{9},-8\right)h^2\right]e_{i-1}$$

$$\left[P_i^3\left\{G\left(\frac{-1}{24},\frac{1}{8}\right)P_{i-1}^2 + G\left(\frac{4}{9},4\right)h^2\right\} + P_{i-1}^3\right.$$

$$\left.\left\{G\left(\frac{2}{9},\frac{-4}{3}\right)P_i^2 + G\left(\frac{7}{9},-8\right)h^2\right\}\right]e_i + P_{i-1}^3\left\{G\left(\frac{1}{18},\frac{7}{18}\right)P_i^2 + G\left(\frac{2}{9},4\right)h^2\right\}P_{i+1} = F_i$$

$$-P_i^3\left\{G\left(\frac{1}{8},\frac{-10}{9}\right)P_{i-1}^2 + G\left(\frac{89}{9},-8\right)h^2\right\} f_{i-1}$$

$$-\left[P_i^3\left\{\left(-\frac{1}{24},\frac{1}{6}\right)P_{i-1}^2+G\left(\frac{4}{9},4\right)h^2\right\}\right.$$

$$\left.+P_{i-1}^3\left\{G\left(\frac{2}{9},\frac{-4}{3}\right)P_i^2+G\left(\frac{7}{9},-8\right)h^2\right\}\right]$$

$$f_i-P_{i-1}^3\left\{G\left(\frac{1}{18},\frac{7}{18}\right)P_1^2+G\left(\frac{2}{9},4\right)h^2\right\}f_{i+1}\qquad =L_i(f)\qquad \text{(Say)}\qquad (3.10)$$

First we write $L_i(f)$ is in the form of divided difference and using Lemma of Lyche [9, 10] , we get

$$|L_i(f)|\le w\left(f^{\{1\}},1-P\right)\sum_{i=1}^{5}a_i=\sum_{j=1}^{5}b_j \qquad (3.11)$$

Where $\sum_{i=1}^{5}a_i=\sum_{j=1}^{5}b_j=\left[P_1^3P_{i-1}^3\,G\left(\frac{1}{432},\frac{-13}{36}h^2\right)+P_1^3P_{i-1}G\left(\frac{3}{2},-6\right)h^2+\right.$

$$\left.P_{i-1}^3P_iG\left(\frac{-40}{27},\frac{8}{3}\right)h^2\right]$$

$$a_1=P_1^3P_{i-1}\left[G\left(\frac{-17}{144},\frac{17}{36}\right)P_{i-1}^2-h^2G\left(\frac{1}{2},-2\right)\right]$$

$$a_2=P_1^3P_{i-1}\left[G\left(\frac{1}{18},\frac{-10}{9}\right)P_{i-1}^2+G\left(\frac{5}{9},-8\right)h^2\right]$$

$$a_3=P_{i-1}^3P_i\left[G\left(\frac{-4}{27},\frac{16}{27}\right)P_i^2+G\left(\frac{-4}{3},\frac{10}{3}\right)\right]$$

$$a_4=P_{i-1}^3P_i\left[G\left(\frac{-1}{108},\frac{-7}{27}\right)P_i^2-G\left(\frac{4}{27},\frac{8}{3}\right)h^2\right]$$

$$a_5=P_i^3P_{i-1}\left[G\left(\frac{2}{9},0\right)P_{i-1}^2+G\left(\frac{13}{9},0\right)h^2\right]$$

$$b_1=P_i^3P_{i-1}\left[G\left(\frac{3}{16},\frac{-3}{4}\right)P_{i-1}^2+G\left(\frac{3}{2},-6\right)h^2\right]$$

$$b_2=P_{i-1}^3P_i^3\left[G\left(\frac{1}{36},\frac{1}{9}\right)\right]$$

$$b_3=P_i^3P_{i-1}^3G\left(\frac{1}{12},\frac{1}{9}\right)$$

$$b_4=\frac{-1}{9}P_i^3P_{i-1}^3$$

$b_5 = -\, p_i\, P_{i-1}^3 \dfrac{11}{9} h^2$

and $x_{1_0} = \beta_{i-j}, x_{1_1} = x_i = x_{2_1} = y_{2_1} = y_{4_0}$

$x_{2_0} = x_{i-1}, x_{3_0} = \gamma_{i-1} - h, x_{3_1} = \gamma_{i-1} + h$

$y_{1_0} = \alpha_{i-1}, y_{1_1} = \beta_{i-1}, y_{2_0} = \alpha_i, y_{3_0} = \gamma_{i-1} - h$

$x_{3_1} = \gamma_{i-1} + h, y_{4_1} = \alpha_i, y_{5_0} = \gamma_i - h,$

$y_{5_1} = \gamma_i + h, x_{4_0} = \alpha_i, x_{4_1} = \beta_i, x_{5_0} = \alpha_i$

$x_{5_1} = x_{i+1}$

Now using equation (3.1) and (3.9) in (3.8)

$\|e(x_i)\| \le J(h)\, K(P,h)\, w(f', P)$

This is inequality (3.6) of Theorem (3.1). To obtain inequality (3.4) of theorem 3.1. Writing (2.6) in form of error functions as follows.

$$e(x) = e(x_i) a_4(t) + e(x_{i+1})\, a_5(t) + M_i(f) \qquad (3.12)$$

Where $M_1(f) = f(\alpha_i) q_1(t) + f(\beta_i) q_2(t)$

$+ P_{i-1}\, f^{\{1\}}(\gamma_i) q_3(t) + f_{i-1}\, q_4(t) + f_i\, q_5(f) - f(x)$

Again write $M_i(f)$ is of the form of divided difference as follows :

$$|M_i(f)| = \left| \sum_{i=1}^{3} a_i \left[x_{i_0}, x_{i_1}\right]_f - \sum_{j=1}^{2} b_j \left[y_{j_0}, y_{j_1}\right]_f \right| \qquad (3.13)$$

Where

$$a_1 = P_i \left[tG\left(\frac{3}{16}, \frac{-3}{4}\right) + G\left(\frac{-15}{16}, \frac{15}{4}\right) t^2 + G\left(\frac{2}{3}, -6\right) t^3 + G\left(\frac{-3}{4}, 3\right) t^4 \right]$$

$$a_2 = P_i \left[G\left(\frac{-1}{9}, \frac{2}{3}\right) t + G\left(\frac{23}{12}, \frac{-47}{18}\right) t^2 + G\left(\frac{-7}{18}, 4\right) t^3 + G\left(\frac{4}{3}, -2\right) t^4 \right]$$

$$a_3 = P_i \left[G\left(\frac{1}{9}, 0\right) t + G\left(\frac{2}{3}, 0\right) t^2 - G\left(\frac{11}{9}, 0\right) t^3 + G\left(\frac{2}{3}, 0\right) t^4 \right]$$

$$b_1 = P_i \left[G\left(\frac{-7}{144}, \frac{17}{36}\right) t + G\left(\frac{7}{144}, \frac{79}{36}\right) t^2 - G\left(\frac{1}{9}, 2\right) t^3 + G\left(\frac{1}{12}, 1\right) t^4 \right]$$

$$b_2 = P_i t\, G\left(\frac{5}{9}, \frac{1}{9}\right)$$

By using Lemma 3.1 of Lyche [9.10]

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{2} b_j = \left[-G\left(\frac{-5}{144}, \frac{1}{12}\right)t + t^2 G\left(\frac{7}{144}, \frac{41}{36}\right) - G\left(\frac{2}{9}, 2\right)t^3 + G\left(\frac{1}{4}, 1\right)t^4 \right]$$

and

$$x_{1_0} = \alpha_i, x_{1_1} = \beta_i = x_{2_1}$$

$$x_{2_0} = x_{i-1}, x_{3_0} = \gamma_i - h, x_{3_1} = \gamma_i + h$$

$$y_{1_0} = \beta_i, y_{1_1} = x_i, y_{2_0} = x_{i-1}, y_{2_1} = x$$

From equation (3.6), (3.12) and (3.13) gives inequality (3.5) of Theorem 3.1.

We now proceed to obtain an upper bound of $e_i^{\{1\}}$ for we use first difference operator in (2.6) equation we get

$$AP_i e_i^{\{1\}}(x) = e_{i-1}\, q_4^{\{1\}}(t) = e_i\, q_5^{\{1\}}(t) + U_i(f) \tag{3.14}$$

Where $U_i(f) = f(\alpha_i)\, q_1^{\{1\}}(t) + f(\beta_i) q_2^{\{1\}}(t) + P_i f^{\{1\}}(\gamma_i)$

$$q_3^{\{1\}}(t) + f_{i-1}\, q_4^{\{1\}}(t) + f_i q_5^{\{1\}}(t) - P_i\, f^{\{1\}}(x) \text{A and } A = G\left(\frac{-1}{36}, \frac{1}{9}\right)$$

Now writing $U_i(f)$ is of the form of divided difference. We get

$$|U_i(f)| = \left| \sum_{i=1}^{3} a_i \left[x_{1_0}, x_{i_1}\right]_f - \sum_{j=1}^{2} b_j \left[y_{j_0}, y_{j_1}\right]_f \right|$$

Where

$$a_1 = P_i \left[G\left(\frac{-1}{9}, \frac{2}{3}\right) + 2G\left(\frac{23}{36}, \frac{-47}{9}\right)t + \left(3t^2 + h^2\right) \right.$$

$$\left. G\left(\frac{-59}{6}, 4\right) + 4t\left(t^2 + h^2\right)G\left(\frac{1}{6}, -2\right) \right]$$

$$a_2 = P_i \left[G\left(\frac{1}{144}, \frac{7}{36}\right) + 2t\, G\left(\frac{-7}{72}, \frac{-59}{18}\right) + \left(3t^2 + h^2\right) \right.$$

$$\left. G\left(\frac{1}{9}, 2\right) + 4G\left(\frac{-1}{12}, -1\right)t\left(t^2 + h^2\right) \right]$$

$$a_3 = P_i\left[\frac{-1}{9} + \frac{11}{3}t - \frac{11}{9}\left(3t^2 + h^2\right) + \frac{8}{3}t\left(t^2 + h^2\right)\right]$$

$$b_1 = P_i\left[G\left(\frac{-3}{16}, \frac{-3}{4}\right) + 2t\left(\frac{15}{8}, \frac{-15}{2}\right) + G\left(\frac{-3}{2}, 6\right)\left(3t^2 + h^2\right)\right.$$

$$\left. + G\left(\frac{3}{4}, -3\right)4t\left(t^2 + h^2\right)\right],$$

$$b_2 = P_i\, G\left(\frac{-1}{36}, \frac{1}{9}\right)$$

and $y_{1_0} = \alpha_i, y_{1_1} = \beta_i, y_{2_0} = x + h$

$$y_{2_1} = x - h, x_{1_0} = \beta_i, x_{1_1} = x_i, x_{3_0} = \gamma_i - h$$

$$x_{2_0} = \beta_i, x_{2_1} = x_{i+1}, x_{3_1} = \gamma_i + h$$

Now, using Lemma 3.1 of Lyche [9,10] we get

$$\left|U_i(f)\right| \le \left(\sum_{i=1}^{3} a_1 = \sum_{j=1}^{2} b_j\right) w\left(f^{\{1\}}, 1 - P\right) \tag{3.15}$$

Where

$$\sum_{i=1}^{3} a_1 = \sum_{k=1}^{2} b_j = P_i\left[G\left(\frac{-105}{8}, \frac{31}{36}\right) + 2 + G\left(\frac{15}{8}, \frac{-15}{2}\right) + \left(3t^2 + h^2\right)\right.$$

$$\left. G\left(\frac{-3}{2}, 6\right) + 4t\left(t^2 + h^2\right)G\left(\frac{3}{4}, -3\right)\right]$$

Now using equation (3.6), (3.14) and (3.15) we get inequality (3.7) of theorem 3.1. This complete proof of theorem 3.1.

Future scope: we have find out existence and uniqueness, error and convergence of deficient discrete spline interpolation in interval [0, 1] by this spline method. The deficient discrete quartic spline will match the function at two interior points and first difference at middle point of the function with boundary condition.

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