Rational Model and Probability of Infection

Are S.O[1] Olaiju O.A[1] Ezikiel I.D[1] Afolayan O[2]
1.Department of Mathematics and Statistics, Federal Polytechnic Ilaro
2.Department of Science laboratory Technology, federal polytechnic Ilaro
oloruntoba22@yahoo.com, olaiju@yahoo.com, imekela@yahoo.com

Abstract
We generalized the solution to the problem of probability of infection from, (*mathforum.org/dr.math*). We derived a model for n hermits. The model was used to generate expectation for some number of hermits and the result tabulated and graphed. Due to computational limitation of the software used to evaluate factorial n where n is greater than 170, the model could be evaluated up to n=170 populations, we then used the curve fitting tool to fit some curves using the data generated by our model. Among all, the Rational function model(R4) performed best with SSE: 0.01654, R-square: 0.9999, Adjusted R-square: 0.9999 and RMSE: 0.0108.
Keywords: Probability, Mathematical model , Series , Infection, Population.

Introduction

We generalize the solution to the problem of probability of infection from math forum of Dr math. We reframe the Problem using n for the total population of the hermits where n can be any positive integer. Thus we have, n sociable hermits live on an otherwise deserted island. An infectious disease strikes the island. The disease has a 1-day infectious period, and after that, the person is immune.

Assume one of the hermits gets the disease. He randomly visits one of the other hermits during his infectious period.

If the visited hermit has not had the disease, he gets it and is infectious the following day. The visited hermit then randomly visits another hermit. The disease is transmitted until an infectious hermit visits an immune hermit, and the disease dies out.

There is one hermit visit per day. Assuming this pattern of behavior, how many hermits can be expected, on average, to get the disease? (*mathforum.org/dr.math/*)

We derived a model for n hermits. The model was used to generate expectation for some number of hermits and the result tabulated and graphed. Due to computational limitation of the software used the model could be evaluated up to n=170 Population, we then used the curve fitting tool to fit some curves using the data generated by our model.

A mathematical model is a description of a system using mathematical concepts and language. The process of developing a mathematical model is termed mathematical modelling. Mathematical models are used not only in the natural sciences (such as physics, biology, earth science, meteorology) and engineering disciplines (e.g. computer science, artificial intelligence), but also in the social sciences (such as economics, psychology, sociology and political science); physicists, engineers, statisticians, operations research analysts and economists use mathematical models most extensively. A model may help to explain a system and to study the effects of different components, and to make predictions about behaviour.

Rational function

A rational function is simply the ratio of two polynomial functions.

$$y = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_2x^2 + b_1x + b_0}$$

with n denoting a non-negative integer that defines the degree of the numerator and m is a non-negative integer that defines the degree of the denominator. For fitting rational function models, the constant term in the denominator is usually set to 1. Rational functions are typically identified by the degrees of the numerator and denominator. For example, a quadratic for the numerator and a cubic for the denominator is identified as a quadratic/cubic rational function. A rational function model is a generalization of the polynomial model: rational function models contain polynomial models as a subset (i.e., the case when the denominator is a constant

Related works

BE ´ La Suki et al(1998) , developed a mathematical model of the *P–V* curve of the first inflation by assuming that (1) Central airways are open leading to many subtrees of n generations that are initially closed; (2) an airway opens when inflation pressure reaches the opening threshold pressure of that segment; and (3) the opening threshold pressures do not depend on airway generation. In this model, airway opening occurs in cascades or avalanches. To test the model which contains only two parameters, n and a pressure, P_{low} , at which at least one subtree completely opens, They measured the first inflation *P–V* curves of 15 excised and degassed rabbit lungs.

By fitting these data, they found that $n = 17 \pm 5$, $P_{low} = 23 \pm 4$ cmH2O, and that there is a wide distribution of threshold pressures for airways with diameters ,2 mm. Analysis of the *P–V* curve in a lung which was lavaged with a liquid of constant surface tension and in which airways are presumably open demonstrated that the distribution of threshold pressures is narrow, and hence no avalanches occur during inflation. They conclude that in normal lungs the first inflation is dominated by avalanche behavior of airway opening providing information on the global distribution of threshold pressures and the average site of airway closure.

Catherine Heffernan and Jacob A. Dunningham in their paper 'Simplifying mathematical modelling to test intervention strategies for *Chlamydia' in* 2009 outlines a new approach to mathematical modelling that tests intervention efforts on *Chlamydia*. The aim was to produce a simple model that can be used when new data comes to hand without the need to re-run the simulation. They developed a simple model to study the effects of interventions in lowering rates of *Chlamydia* in a high-risk population of 16 to 24 year olds. Parameters were informed by the best available data. Their model was verified by running it backwards in time to see if it correctly 'retrodicts' rates of *Chlamydia* in the past. Their model predicted that *Chlamydia* would disappear long-term if there were 45% condom use, annual check-ups and 23.5% successful contact tracing among the high-risk 16 – 24 year old age group. Their model's expressions can be applied readily to different populations of interest and to address specific questions, indicating that the model is a quick and easy tool to apply in public health policy making.

Amritbir Singh1 & Ravi Kant Mishra (2010) proposed a simple mathematical model by using ordinary differential equation to know the spread rate of technological innovations in rural India. In support of model the data related to technological innovations in the state Punjab (India) has been collected and analysed graphically/mathematically. The outcomes verified the Sigmoid pattern growth (exponential, linear & asymptotic)

Short M. B. *et al* (2007), Motivated by empirical observations of spatio-temporal clusters of crime across a wide variety of urban settings, presented a model to study the emergence, dynamics, and steady-state properties of crime hotspots. They focus on a two-dimensional lattice model for residential burglary, where each site is characterized by a dynamic attractiveness variable, and where each criminal is represented as a random walker. The dynamics of criminals and of the attractiveness field are coupled to each other via specific biasing and feedback mechanisms. Depending on parameter choices, they observe and describe several regimes of aggregation, including hotspots of high criminal activity. On the basis of the discrete system, they also derive a continuum model; the two are in good quantitative agreement for large system sizes. By means of a linear stability analysis they are able to determine the parameter values that will lead to the creation of stable hotspots. they discuss their model and results in the context of established criminological and sociological findings of criminal behavior.

Shashi Kant (2011) discussed the problem of tumor growth with the function of phosphorous to both tumor cells and healthy cells. A detailed discussion about the growth of healthy cells and tumor cells was observed. A consideration about the solution part of tumor micro vessels was made. The governing equations were then solved using MATLAB (a differential equation solver) and graphs are drawn which show the results to the problem under consideration.

Harjeet Kumar et al (2002), studied the blood flow which was derived from Navier-Stokes equations. A system of non linear partial differential equations for blood flow and cross sectional area of the artery was obtained. The governing equations are solved numerically by using finite difference method

Derivation of the model

Suppose the hermits are I, II, III, IV, V, VI,……… and that I is the one hermit first infected. He visits another hermit who is not immune, say II, who is then infected.

I is now immune and II has probability $\frac{1}{n-1}$ of visiting an immune hermit, and $\frac{n-2}{n-1}$ of visiting a hermit who is not immune. This means there is probability $\frac{1}{n-1}$ that only 2 get the disease

This situation is best illustrated with a tree diagram. (*mathforum.org/dr.math/*)

The tree starts with I infecting II with probability 1, since no matter whom he visits that hermit is not immune. So assuming he visited II, we have the first branch at II having probability $\frac{1}{n-1}$ that II visits I and the process ends, or probability $\frac{n-2}{n-1}$ that he visits a vulnerable hermit.

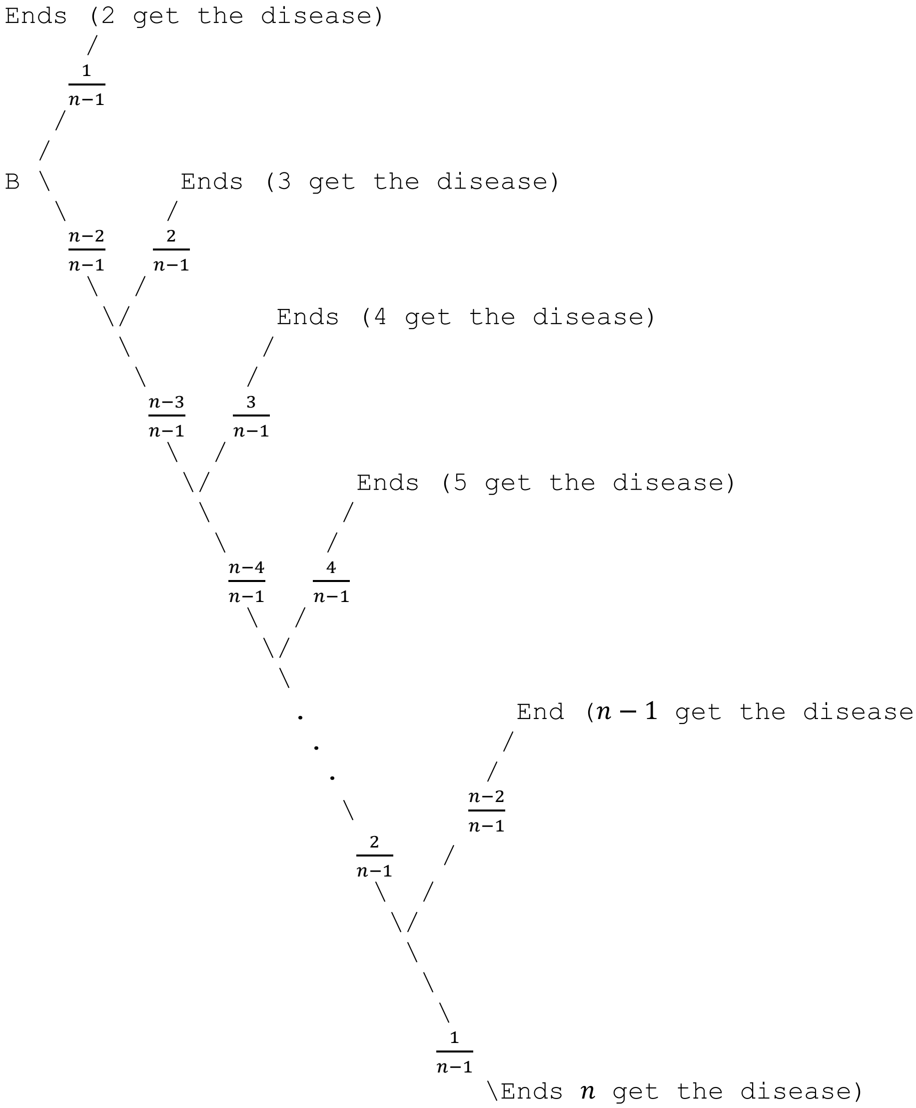

Figure 1: The tree diagram of the problem (*mathforum.org/dr.math/*)

The chance that 2 get the disease is $\frac{1}{n-1}$.

The chance that 3 get the disease is $\frac{n-2}{n-1} \times \frac{2}{n-1} = \frac{2(n-2)}{(n-1)^2}$

The chance that 4 get the disease is $\frac{n-2}{n-1} \times \frac{n-3}{n-1} \times \frac{3}{n-1} = \frac{3(n-2)(n-3)}{(n-1)^3}$

The chance that 5 get the disease is $\frac{n-2}{n-1} \times \frac{n-3}{n-1} \times \frac{n-4}{n-1} \times \frac{4}{n-1} = \frac{4(n-2)(n-3)(n-4)}{(n-1)^4}$

The chance that 6 get the disease is

$$\frac{n-2}{n-1} \times \frac{n-3}{n-1} \times \frac{n-4}{n-1} \times \frac{n-5}{n-1} \times \frac{5}{n-1} = \frac{5(n-2)(n-3)(n-4)(n-5)}{(n-1)^5}$$

And so on

If we add these probabilities, they give a total of 1, as they should.

The expected number getting the disease on average is then

$$E(n) = 2\left(\frac{1}{n-1}\right) + 3\left(\frac{2(n-2)}{(n-1)^2}\right) + 4\left(\frac{3(n-2)(n-3)}{(n-1)^3}\right) + 5\left(\frac{4(n-2)(n-3)(n-4)}{(n-1)^4}\right) + 6\left(\frac{5(n-2)(n-3)(n-4)(n-5)}{(n-1)^5}\right) + \cdots$$

$$E(n) = 2\left(\frac{1}{n-1}\right) + 6\left(\frac{(n-2)}{(n-1)^2}\right) + 12\left(\frac{(n-2)(n-3)}{(n-1)^3}\right) + 20\left(\frac{(n-2)(n-3)(n-4)}{(n-1)^4}\right) + 30\left(\frac{(n-2)(n-3)(n-4)(n-5)}{(n-1)^5}\right) + \cdots$$

$$E(n) = 2\left[\left(\frac{1}{n-1}\right) + 3\left(\frac{(n-2)}{(n-1)^2}\right) + 6\left(\frac{(n-2)(n-3)}{(n-1)^3}\right) + 10\left(\frac{(n-2)(n-3)(n-4)}{(n-1)^4}\right) + 15\left(\frac{(n-2)(n-3)(n-4)(n-5)}{(n-1)^5}\right) + \cdots\right]$$

$$E(n) = 2\left[\left(\frac{(n-1)}{(n-1)^2}\right) + 3\left(\frac{(n-1)(n-2)}{(n-1)^3}\right) + 6\left(\frac{(n-1)(n-2)(n-3)}{(n-1)^4}\right) + 10\left(\frac{(n-1)(n-2)(n-3)(n-4)}{(n-1)^5}\right) + 15\left(\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{(n-1)^6}\right) + \cdots\right]$$

$$E(n) = 2\left[\left(\frac{1.(n-1)}{(n-1)^2}\right) + (1+2)\left(\frac{(n-1)(n-2)}{(n-1)^3}\right) + (1+2+3)\left(\frac{(n-1)(n-2)(n-3)}{(n-1)^4}\right) + (1+2+3+4)\left(\frac{(n-1)(n-2)(n-3)(n-4)}{(n-1)^5}\right) + (1+2+3+4+5)\left(\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{(n-1)^6}\right) + \cdots\right]$$

But $1+2+3+4+5+\cdots+\mathrm{n} = \sum_{r=1}^{n}\frac{r(r+1)}{2}$
and
$\sum_{r=1}^{n}(n-1)(n-2)(n-3)(n-4)(n-5)\ldots.(n-r-1)! = \frac{(n-1)!}{(n-r-1)!}$

Therefore we have

$$E(n) = 2\sum_{r=1}^{n-1}\frac{r(r+1)}{2}\frac{(n-1)!}{(n-1)^{r+1}(n-r-1)!}$$

$$E(n) = \sum_{r=1}^{n-1}\frac{r(r+1)(n-1)!}{(n-1)^{r+1}(n-r-1)!}$$

$$E(n) = \sum_{r=1}^{n-1}\frac{r(r+1)(n-2)!}{(n-1)^{r}(n-r-1)!}$$

As n getting large, using starling formulae $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ we have,

$$E(n) = \sum_{r=1}^{n-2}\frac{r(r+1)\sqrt{2\pi(n-2)}\left(\frac{n-2}{e}\right)^{n-2}}{(n-1)^r\sqrt{2\pi(n-r-1)}\left(\frac{n-r-1}{e}\right)^{n-r-1}}$$

$$E(n) = \sum_{r=1}^{n-2}\frac{r(r+1)\sqrt{2\pi(n-2)}(n-2)^{n-2}e^{1-r}}{(n-1)^r\sqrt{2\pi(n-r-1)}(n-r-1)^{n-r-1}}$$

$$E(n) = \sum_{r=1}^{n-2} \frac{r(r+1)\sqrt{(n-2)}(n-2)^{n-2}e^{1-r}}{(n-1)^r\sqrt{(n-r-1)}(n-r-1)^{n-r-1}}$$

For n=6 we have

$$E(6) = \sum_{r=1}^{6-1} \frac{r(r+1)(6-2)!}{(6-1)^r(6-r-1)!}$$

$$E(6) = \sum_{r=1}^{5} \frac{r(r+1)(4)!}{(5)^r(6-r-1)!} = 3.5104$$

So, on average, 3.5 of the hermits can be expected to be infected.
Using the model and the mathlab command,

```
r=[1:n-1]
sum(((r.*(r+1))*factorial(n-2))./(((n-1).^r).*factorial(n-1-r))),
```

we generated the Expectation for n=2 to 100. The results were tabulated in table 1 and the graph plotted in figure 1

Number of Hermits	**Expectation of infection Population**
2.	2.000
3.	2.500
4.	2.888
5.	3.218
6.	3.510
.	.
.	.
97	12.9569
98	13.0207
99	13.0841
100	13.0961

Table 1: Expected population and Total population of hermits

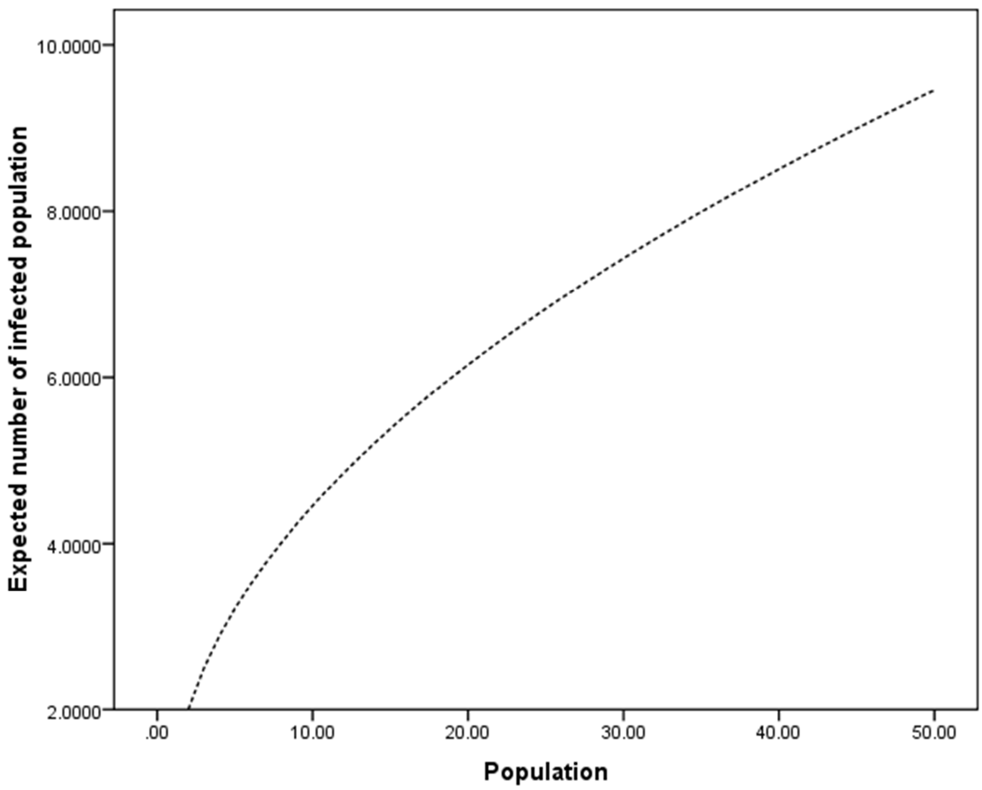

Figure 1: Graph of expected infectious population vs population

We used Mathlab curve fitting tools to fit curves to our result in figure 1, then Power and Rational models were compered. Rational model performed best on our data set with the following criteria,

Rational models

Rational Regression Models	Constants	SSE	R-SQUARE	DFE	ADJ R-SQUARE	RMSE
$R(1)=\frac{p_1x+p_2}{x+q_1}$	$p_1=26.93$ $p_2=338.7$ $q_1=128.5$	4.441852	0.99765	146	0.99762	0.1744
$R(2)=\frac{p_1x^2+p_2x+p_3}{x^2+q_1x+q_2}$	$p_1=39.29$ $p_2=2321$ $p_3=3689$ $q_1=344$ $q_2=3383$	0.043330	0.99997	144	0.99997	0.0173
$R(3)=\frac{p_1x^3+p_2x^2+p_3x+p_4}{x^3+q_1x^2+q_2x+q_3}$	$p_1=43.89$ $p_2=4728$ $p_3=3.703e^{+004}$ $p_4=-2.313e^{+004}$ $q_1=483.7$ $q_2=1.364e^{+004}$ $q_3=8457$	0.05546	0.99997	142	0.99996	0.0197
$***R(4)=\frac{p_1x^4+p_2x^3+p_3x^2+p_4x+p_5}{x^3+q_1x^2+q_2x+q_3}$	$p_1=0.03482$ $p_2=17.69$ $p_3=319.2$ $p_4=-1935$ $p_5=-1378$ $q_1=84.94$ $q_2=-87.61$ $q_3=-2071$	0.01654	0.99999	141	0.99999	0.0108
$R(5)=\frac{p_1x^4+p_2x^3+p_3x^2+p_4x+p_5}{x^4+q_1x^3+q_2x^2+q_3x+q_4}$	$p_1=30.1$ $p_2=2104$ $p_3=1037$ $p_4=-1.756e^{+004}$ $p_5=2.453e^{+004}$ $q_1=239$ $q_2=4948$ $q_3=-1.841e^{+004}$ $q_4=2.162e^{+004}$	4.202956	0.99778	140	0.99765	0.1732
$R(7)=\frac{p_1x^5+p_2x^4+p_3x^3+p_4x^2+p_5x+p_6}{x^5+q_1x^4+q_2x^3+q_3x^2+q_4x+q_5}$	$p_1=39.43$ $p_2=2351$ $p_3=2320$ $p_4=187.8$ $p_5=-1066$ $p_6=-898.1$ $q_1=347.2$ $q_2=3316$ $q_3=-2143$ $q_4=1183$ $q_5=1562$	0.0295479	0.9999844	138	0.99998	0.0146

Table1: Analysis of the Rational regression models

Out of all the rational regression models , R4 perform best from table 1 , with SSE= 0.01654, rsquare =0.99999, adj R-square =0.99999 , RMSE= 0.01083 and dfe=141

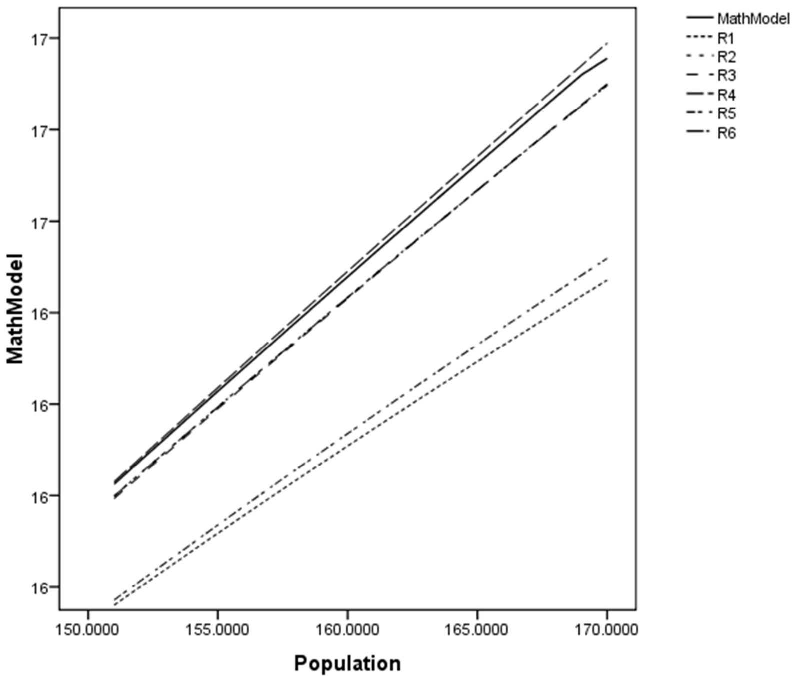

Figure 2: curves of the Rational Regression models

Power models

Power Models	Constants	SSE	R-SQUARE	DFE	ADJ R-SQUARE	RMSE
$PW(1) = ax^b + c$	a = 1.418 b = 0.4814 c = 0.1393	0.044425	0.999976	146	0.999976	0.01744
$PW(2) = ax^b$	a = 1.481 b = 0.4742	0.065186	0.999965	147	0.999965	0.02105

Table2: Power models analysis.

From Table2, The power model PW1 performed better than PW2 with SSE= 0.044425, rsquare =0.999976, adj R-square =0. 0.999976 and RMSE= 0.01744 and dfe=146

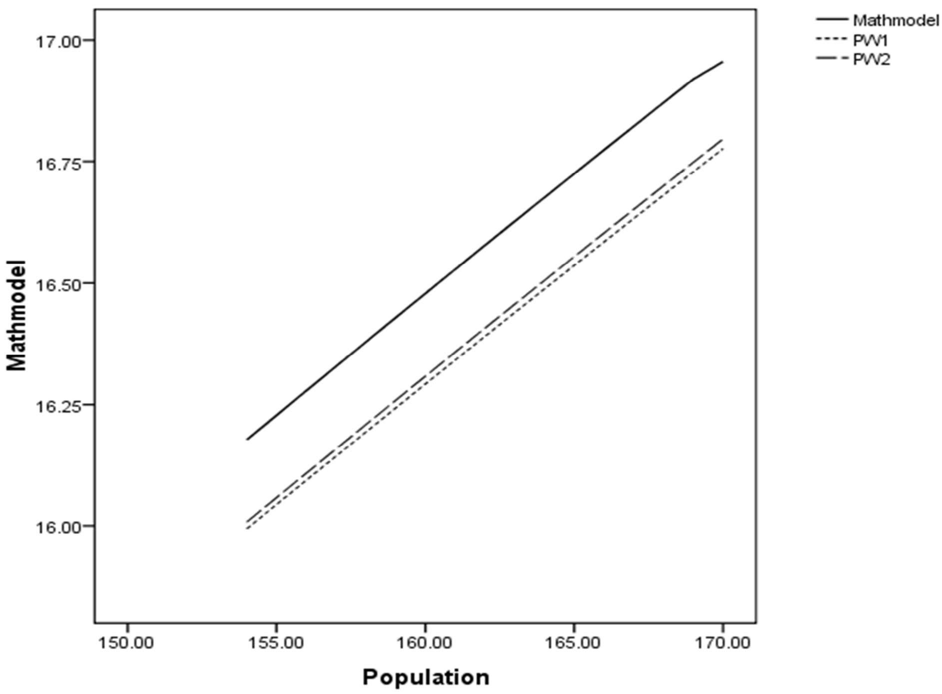

Figure 3: curves of Power models.

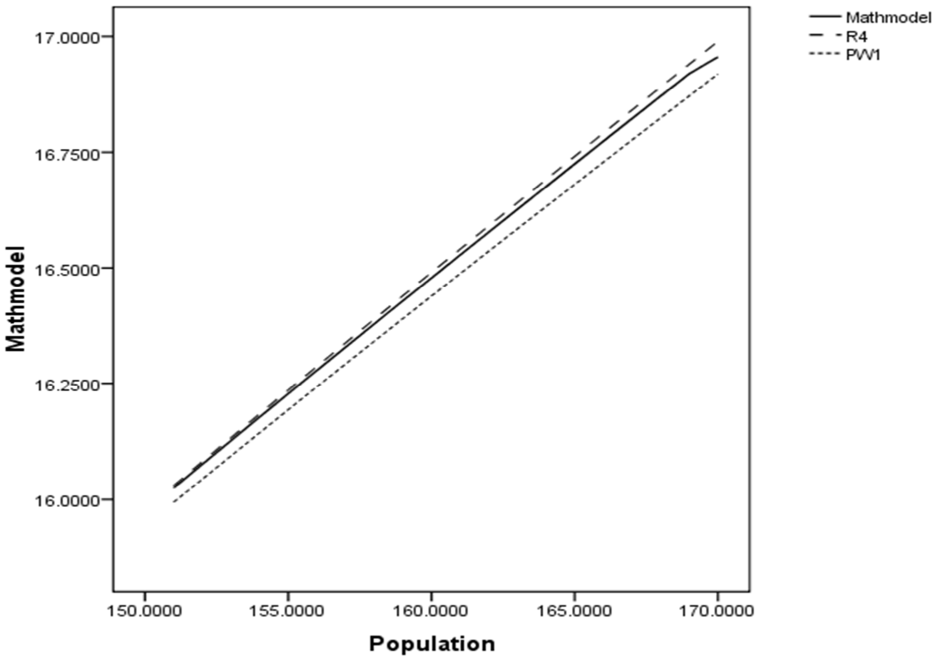

Figure4: curves of PW1 , R4 and Math model.

From both figure 4 and table 1 and 2 , it is clear that Rational model outperformed Power model . Thus we represent confidently our Mathematical model by the Rational model (R4).

$$E(n)=\sum_{r=1}^{n-1}\frac{r(r+1)(n-2)!}{(n-1)^r(n-r-1)!}\approx\frac{p_1x^4+p_2x^3+p_3x^2+p_4x+p_5}{x^3+q_1x^2+q_2x+q_3}$$

We used the Rational model (R4) to predict from 101 to 150 and then compare the result to the one obtained from the original model in table 1 and the curve plotted in figure 3. Both the table and the curve showed that Rational model is a better approximation to our Model.

Conclusion

We generalized the solution to the problem of probability of infection and derived a significant model for n hermits. The model was in the form of series, we used the model to generate some expectations for N=2-100, the results were tabulated and a graph of expected infectious population vs number of hermits was plotted. The shape of the graph indicated a rising trend. The trend was fitted by Rational and power models and the Rational model outperformed the power model this shows that our model is closely related to rational model, thus the model can be approximated by Rational model. This is possible because Rational function models have a moderately simple form, they are not dependent on the underlying metric, they can take on an extremely wide range of shapes, accommodating a much wider range of shapes. Rational function models have better interpolatory properties than polynomial models, they are typically smoother and less oscillatory than some other models.

Rational functions have excellent extrapolatory powers. They can typically be tailored to model the function not only within the domain of the data, but also so as to be in agreement with theoretical/asymptotic behavior outside the domain of interest. Rational function models have excellent asymptotic properties. So they can be used to model complicated structure with a fairly low degree in both the numerator and denominator. This means that fewer coefficients will be required.

REFERENCES

http://mathforum.org/ (*mathforum.org/dr.math/*)

Be´ la suki, jose´ s. Andrade, jr. Mark f. Coughlin, Dimitrije Stamenovic´ h, Eugene Stanley, Mamatha Sujeer, and Stefano Zapperi (1998) Mathematical Modeling of the First Inflation of Degassed Lungs *Annals of Biomedical Engineering,* Vol. 26,

Harjeet Kumar, R.S. Chandel, Sanjeev Kumar & Sanjeet Kumar(2002) A Mathematical Model for Blood flow and Cross Sectional Area of an Artery

Catherine Heffernan and Jacob A. Dunningham (2009) Simplifying mathematical modelling to test intervention strategies for *Chlamydia* Journal of Public Health and Epidemiology Vol. 1(1), pp. 022-030 Available online at

http://www.academicjournals.org/jphe
Amritbir Singh & Ravi Kant (2010) A mathematical modeling approach to study growth rate of grassroots technological innovations IJRRAS 3 (2) ● May 2010
Short M. B, D'orsogna M. R, Pasour V. B, Tita, G. E, Brantingham P. J, (2007) A statistical model of criminal behaviour. Mathematical Models and Methods in Applied Sciences Vol. 18, 1249–1267 World Scientific Publishing Company
Shashi Kant, Deepak Kumar and Sanjeev Kumar (2011) Mathematical model of homogenious tumour growth with delay in time. African Journal of Mathematics and Computer Science Research Vol. 4(5), pp. 201-207, Available online at http://www.academicjournals.org/AJMCSR ISSN 2006-9731 ©2011 Academic Journals