Performance of Estimates of Reliability Parameters for Compound Rayleigh Progressive Type II Censored Data

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Abstract
This paper develops Bayesian analysis in the context of progressively Type II censored data from the two-parameter compound Rayleigh distribution. The maximum likelihood and Bayes estimates along with the associated posterior risks are derived for unknown reliability parameters under the balanced logarithmic loss and balanced general entropy loss functions. A practical example and simulation study have been considered to illustrate the proposed estimation methods and compare the performance of derived estimates based on maximum likelihood and Bayesian frameworks. The study indicates that Bayesian approach is more preferable over the maximum likelihood approach for estimation of the reliability parameters, while in Bayesian approach, a balance general entropy loss function can effectively be employed.
Keywords: Maximum likelihood estimation, Bayes estimation, balanced logarithmic loss function, balanced general entropy loss function, posterior risk, Monte Carlo simulation.

1. Introduction

In life testing and reliability, the two-parameter compound Rayleigh distribution plays an important role and useful for modelling and analysis of lifetime data especially in medical and biological sciences. In the last couple of decades, significant inference procedures have been developed for this distribution (Mostert *et al.* 1999; Abushal 2011; Shojaee *et al.* 2012). In many industries, industrial experiments often terminate before failure of all the experimental units. Such experiments should be planned with an aim of significant reduction in total failures or time duration. Due to these reasons, an experimenter naturally prefers the most popular progressive Type II censoring scheme that allows the removal of experimental units at points other than the terminal point of an experiment. For a comprehensive review of this censoring scheme, one may refer Cohen (1963), Balakrishnan & Aggarwala (2000), Wu *et al.* (2006), and Barot & Patel (2013).

In decision-making theory, balanced loss function usually focuses on precision of estimation as well as goodness of fit. Zellner (1994) introduced this loss function in the context of the general linear model and used this for the estimation of a scalar mean, vector mean, and a regression coefficient vector. Various authors have done various inferential studies using this loss function under different set ups. For more information, one may refer Rodrigues & Zellner (1994), Chung *et al.* (1998), Dey *et al.* (1999), Sanjari & Asgharzadeh (2004), and Gruber (2004). Appreciating the popularity of balanced loss functions, we introduce and motivate the use of balanced logarithmic loss function (BLGLF) and balanced general entropy loss function (BGELF) in estimating reliability parameters of the compound Rayleigh model.

The present paper is an attempt to examine and compare the performance of Bayes estimates and maximum likelihood estimates when the data are progressively Type II censored from the compound Rayleigh distribution. In Section 2, the maximum likelihood and Bayes estimates of reliability parameters along with the corresponding posterior risks are obtained. In Section 3, an example with the real data is considered to illustrate the proposed methods of estimation. In Section 4, an extensive Monte-Carlo simulation study is carried out to compare the performance of the maximum likelihood and Bayes estimates. The paper concludes in Section 5.

2. Estimation of Reliability Parameters of Compound Rayleigh Model

The probability density, cumulative density, reliability, and failure rate functions (at mission time t) of a compound Rayleigh distribution with unknown shape parameter α and scale parameter β are given, respectively,

$$f(x \mid \alpha,\beta) = 2\alpha\beta^{\alpha} x\left(\beta + x^2\right)^{-(\alpha+1)} \; ; \; F(x \mid \alpha,\beta) = 1 - \left(1 + \frac{x^2}{\beta}\right) \; ; \; x,\alpha,\beta > 0 \,, \tag{1}$$

$$R(t) = \left(1 + \frac{t^2}{\beta}\right)^{-\alpha} \; ; \quad h(t) = \frac{2\alpha t}{\beta + t^2},\; t > 0. \tag{2}$$

Let n units are placed on a life-testing experiment and only m ($\le n$) units are completely observed until failure. At the time of each failure occurring prior to the termination point, one or more surviving units are removed from the test. Let $x_{(i)}$ be the lifetimes of completely observed units following (1), and r_i denotes the withdrawn units at i^{th} failure, $i = 1, 2, \dots, m$. Then the likelihood function based on progressively Type II censored sample $\underset{\sim}{x} = \left(x_{(1)}, x_{(2)}, \dots, x_{(m)}\right)$ is (Barot & Patel 2015)

$$L(\underset{\sim}{x} \mid \alpha, \beta) = A\, 2^m \alpha^m \beta^{\alpha m} \prod_{i=1}^{m} \frac{x_{(i)}}{\left(\beta + x_{(i)}^2\right)^{1+\alpha(1+r_i)}}, \tag{3}$$

where,

$$A = n(n-1-r_1)(n-2-r_1-r_2) \dots \left(n - m + 1 - \sum_{i=1}^{m-1} r_i\right).$$

2.1. Maximum likelihood Estimation

The likelihood equations can be obtained by differentiating the natural logarithm of (3) partially with respect to β and α, and then equating the partial derivatives with zero. The resulting likelihood equations will be in the form

$$\frac{m \sum_{i=1}^{m} (1+r_i)\left(\frac{x_{(i)}^2}{\beta + x_{(i)}^2}\right)}{\beta \sum_{i=1}^{m} (1+r_i) \ln\left(\frac{\beta}{\beta + x_{(i)}^2}\right)} + \sum_{i=1}^{m} \left(\frac{1}{\beta + x_{(i)}^2}\right) = 0 \quad \text{and} \quad \alpha = -\frac{m}{\sum_{i=1}^{m} (1+r_i) \ln\left(\frac{\beta}{\beta + x_{(i)}^2}\right)}. \tag{4}$$

The Maximum likelihood (ML) estimates $\hat{\beta}_{ML}$ and $\hat{\alpha}_{ML}$ can be obtained by solving the equations in (4) numerically. The invariance property of ML estimation enables one to obtain the ML estimates $\hat{R}(t)_{ML}$ and $\hat{h}(t)_{ML}$ by using the ML estimates of α and β in (2).

2.2. Bayes Estimation

In Bayes estimation, it is required to assign prior distributions of the unknown scale parameter α and shape parameter β to consider subjective inputs from experienced experts or summary judgment of past research that yielded similar results. Following the idea of Soland (1969), β is restricted to finite positive real values β_j with probability $\frac{2j}{N(N+1)}$, $j = 1, 2, \dots, N$; and conditional upon $\beta = \beta_j$, α has a natural conjugate gamma (a_j, b_j) prior with a density function

$$\pi(\alpha \mid \beta = \beta_j) = \frac{b_j^{a_j} \alpha^{a_j - 1} e^{-b_j \alpha}}{\Gamma(a_j)}, \quad \alpha, a_j, b_j > 0, \tag{5}$$

where a_j and b_j are unknown hyper-parameters chosen to reflect prior beliefs on α given that $\beta = \beta_j$. Following the idea of Barot & Patel (2015), the marginal posterior probability distribution of β and α can be obtained, respectively, as

$$P_j = \Pr(\beta = \beta_j) = \left(\sum_{j=1}^{N} \frac{j\, b_j^{a_j}\, \Gamma(m + a_j) \prod_{i=1}^{m} \frac{1}{\beta_j + x_{(i)}^2}}{(b_j - T_j)^{m + a_j}\, \Gamma(a_j)} \right)^{-1} \left(\frac{j\, b_j^{a_j}\, \Gamma(m + a_j) \prod_{i=1}^{m} \frac{1}{\left(\beta_j + x_{(i)}^2\right)}}{(b_j - T_j)^{m + a_j}\, \Gamma(a_j)} \right) \tag{6}$$

and

$$\pi_1^*(\alpha) = \sum_{j=1}^{N} \frac{P_j (b_j - T_j)^{m + a_j} \alpha^{m + a_j - 1} e^{-(b_j - T_j)\alpha}}{\Gamma(m + a_j)}, \tag{7}$$

where,

$$T_j = \sum_{i=1}^{m} (1 + r_i) \ln\left(\frac{\beta_j}{\beta_j + x_{(i)}^2}\right). \tag{8}$$

For each β_j, the hyper-parameters (a_j, b_j) can be elicited from the expected value of reliability $R(t)$ conditional on $\beta = \beta_j$.

In order to create a balance between Bayesian and classical approach and provide an estimate that is a linear combination of Bayes and ML estimates, the BLGLF and the BGELF with shape parameter d $(d \neq 0)$ have been proposed, respectively, in the forms

$$L_1(\hat{\phi},\phi)=\omega(\ln\hat{\phi}-\ln\hat{\phi}_{ML})^2+(1-\omega)(\ln\hat{\phi}-\ln\phi)^2 \tag{9}$$

and

$$L_2(\hat{\phi},\phi)=\omega\left[\left(\frac{\hat{\phi}}{\hat{\phi}_{ML}}\right)^d-d\ln\left(\frac{\hat{\phi}}{\hat{\phi}_{ML}}\right)-1\right]+(1-\omega)\left[\left(\frac{\hat{\phi}}{\phi}\right)^d-d\ln\left(\frac{\hat{\phi}}{\phi}\right)-1\right], \tag{10}$$

where $\hat{\phi}$ is a Bayes estimates of ϕ, and $\omega\in[0,1]$ is a weight.

The Bayes estimates $\hat{\phi}_{BLG}$ and $\hat{\phi}_{BGE}$ relative to BLGLF and BGELF are, respectively, the values of $\hat{\phi}$ that minimizes the corresponding posterior expectations $E^{\pi^*}(L_1(\hat{\phi},\phi))$ and $E^{\pi^*}(L_2(\hat{\phi},\phi))$; and their posterior risks are, respectively, the posterior expectations $E^{\pi^*}\left[L_1(\hat{\phi}_{BLG},\phi)\right]$ and $E^{\pi^*}\left[L_2(\hat{\phi}_{BGE},\phi)\right]$. Moreover, the posterior risks of ML estimate $\hat{\phi}_{ML}$ under BLGLF and BGELF are, respectively, the posterior expectations $E^{\pi^*}\left[L_1(\hat{\phi}_{ML},\phi)\right]$ and $E^{\pi^*}\left[L_2(\hat{\phi}_{ML},\phi)\right]$.

2.2.1. Bayes estimates and posterior risks under BLGLF

Based on the progressively Type II censored data and the posterior densities (6) and (7), the Bayes estimates $\hat{\alpha}_{BLG}$, $\hat{\beta}_{BLG}$, $\hat{R}(t)_{BLG}$ and $\hat{h}(t)_{BLG}$ are obtained, respectively, as

$$\hat{\alpha}_{BLG}=\exp\left[\omega\ln\hat{\alpha}_{ML}+(1-\omega)S_1\right], \tag{11}$$

$$\hat{\beta}_{BLG}=\exp\left[\omega\ln\hat{\beta}_{ML}+(1-\omega)S_2\right], \tag{12}$$

$$\hat{R}(t)_{BLG}=\exp\left\{-\left[\omega\hat{\alpha}_{ML}t_m+(1-\omega)S_3\right]\right\}, \tag{13}$$

$$\hat{h}(t)_{BLG}=\exp\left[\omega\delta+(1-\omega)S_4\right], \tag{14}$$

and the corresponding posterior risks are obtained, respectively, as

$$PR_{L_1}(\hat{\alpha}_{BLG})=(1-\omega)\left[\omega(\ln\hat{\alpha}_{ML})^2-2\omega(\ln\hat{\alpha}_{ML})S_1-(1-\omega)S_1^2+S_5\right], \tag{15}$$

$$PR_{L_1}(\hat{\beta}_{BLG})=(1-\omega)\left[\omega(\ln\hat{\beta}_{ML})^2-2\omega(\ln\hat{\beta}_{ML})S_2-(1-\omega)S_2^2+S_6\right], \tag{16}$$

$$PR_{L_1}(\hat{R}(t)_{BLG})=(1-\omega)\left[\omega\hat{\alpha}_{ML}^2t_m^2-2\omega\hat{\alpha}_{ML}t_mS_3-(1-\omega)S_3^2+S_7\right], \tag{17}$$

$$PR_{L_1}(\hat{h}(t)_{BLG})=(1-\omega)\left[\omega\delta^2+S_4+2S_8+S_9-2\omega\delta S_4-(1-\omega)S_4^2\right], \tag{18}$$

Moreover, the posterior risks of ML estimates $\hat{\alpha}_{ML}$, $\hat{\beta}_{ML}$, $\hat{R}(t)_{ML}$, and $\hat{h}(t)_{ML}$ under BLGLF are obtained, respectively, as

$$PR_{L_1}(\hat{\alpha}_{ML})=(1-\omega)\left[(\ln\hat{\alpha}_{ML})^2-2(\ln\hat{\alpha}_{ML})S_1+S_5\right], \tag{19}$$

$$PR_{L_1}(\hat{\beta}_{ML})=(1-\omega)\left[(\ln\hat{\beta}_{ML})^2-2(\ln\hat{\beta}_{ML})S_2+S_6\right], \tag{20}$$

$$PR_{L_1}(\hat{R}(t)_{ML})=(1-\omega)(\hat{\alpha}_{ML}^2t_m^2-2\hat{\alpha}_{ML}t_mS_3+S_7), \tag{21}$$

$$PR_{L_1}(\hat{h}(t)_{ML})=(1-\omega)(\delta^2-2\delta S_4+S_5+2S_8+S_9), \tag{22}$$

where

$$t_m=\ln\left(1+\frac{t^2}{\hat{\beta}_{ML}}\right);\ S_1=\sum_{j=1}^{N}P_j\left[\ln\left(\frac{m+a_j}{b_j-T_j}\right)-\frac{1}{2(m+a_j)}\right];\ S_2=\sum_{j=1}^{N}P_j\ln\beta_j;$$

$$S_3=\sum_{j=1}^{N}P_j\frac{(m+a_j)\ \ln\left(1+\frac{t^2}{\beta_j}\right)}{b_j-T_j};\ \ S_4=\sum_{j=1}^{N}P_j\left\{\ln\left[\frac{2t\,(m+a_j)}{(\beta_j+t^2)(b_j-T_j)}\right]-\frac{1}{2(m+a_j)}\right\};$$

$$S_5=\sum_{j=1}^{N}P_j\left\{\left[\ln\left(\frac{m+a_j}{b_j-T_j}\right)-\frac{1}{2(m+a_j)}\right]^2+\sum_{n_1=0}^{\infty}\frac{1}{(m+a_j+n_1)^2}\right\};\ S_6=\sum_{j=1}^{N}P_j\,(\ln\beta_j)^2;$$

$$S_7=\sum_{j=1}^{N}P_j\frac{(m+a_j+1)(m+a_j)t_j^2}{(b_j-T_j)^2};\qquad S_8=\sum_{j=1}^{N}P_j\ln\left(\frac{2t}{\beta_j+t^2}\right)\left[\ln\left(\frac{m+a_j}{b_j-T_j}\right)-\frac{1}{2(m+a_j)}\right];$$

$$S_9=\sum_{j=1}^{N}P_j\left[\ln\left(\frac{2t}{\beta_j+t^2}\right)\right]^2;\ \delta=\ln\left(\frac{2\hat{\alpha}_{ML}t}{\hat{\beta}_{ML}+t^2}\right).$$

2.2.2. Bayes estimates and posterior risks under BGELF

Based on the progressively Type II censored data and posterior densities (6) and (7), the Bayes estimates $\hat{\alpha}_{BGE}$, $\hat{\beta}_{BGE}$, $\hat{R}(t)_{BGE}$ and $\hat{h}(t)_{BGE}$ are obtained, respectively, as

$$\hat{\alpha}_{BGE}=\left[\omega\hat{\alpha}_{ML}^{-d}+(1-\omega)S_{10}\right]^{-\frac{1}{d}},\tag{23}$$

$$\hat{\beta}_{BGE}=\left[\omega\hat{\beta}_{ML}^{-d}+(1-\omega)S_{11}\right]^{-\frac{1}{d}},\tag{24}$$

$$\hat{R}(t)_{BGE}=\left[\omega\exp(dt_m\hat{\alpha}_{ML})+(1-\omega)S_{12}\right]^{-\frac{1}{d}},\tag{25}$$

$$\hat{h}(t)_{BGE}=2t\left[\omega\left(\frac{\hat{\alpha}_{ML}}{\hat{\beta}_{ML}+t^2}\right)^{-d}+(1-\omega)S_{13}\right]^{-\frac{1}{d}}.\tag{26}$$

with the corresponding posterior risks are obtained, respectively, as

$$PR_{L_2}(\hat{\alpha}_{BGE})=d\left[\omega\ln\hat{\alpha}_{ML}\ +\ (1-\omega)S_1\right]+\ln\left[\omega\hat{\alpha}_{ML}^{-d}\ +\ (1-\omega)S_{14}\right],\tag{27}$$

$$PR_{L_2}(\hat{\beta}_{BGE})=d\left[\omega\ln\hat{\beta}_{ML}\ +\ (1-\omega)S_2\right]+\ln\left[\omega\hat{\beta}_{ML}^{-d}\ +\ (1-\omega)S_{15}\right],\tag{28}$$

$$PR_{L_2}(\hat{R}(t)_{BGE})=-d\left[\omega\hat{\alpha}_{ML}t_m\ +\ (1-\omega)S_3\right]+\ln\left[\omega\exp(dt_m\hat{\alpha}_{ML})\ +\ (1-\omega)S_{16}\right],\tag{29}$$

$$PR_{L_2}(\hat{h}(t)_{BGE})=d\,[\omega\delta+(1-\omega)S_4-\ln 2t]+\ln\left[\omega\left(\frac{\hat{\alpha}_{ML}}{\hat{\beta}_{ML}+t^2}\right)^{-d}+(1-\omega)S_{17}\right].\tag{30}$$

Moreover, the posterior risks of the ML estimates $\hat{\alpha}_{ML}$, $\hat{\beta}_{ML}$, $\hat{R}(t)_{ML}$, and $\hat{h}(t)_{ML}$ under BGELF are, obtained, respectively, as

$$PR_{L_2}(\hat{\alpha}_{ML})=(1-\omega)\left[\hat{\alpha}_{ML}^{d}\,S_{14}-d\,(\ln\hat{\alpha}_{ML}-S_1)-1\right],\tag{31}$$

$$PR_{L_2}(\hat{\beta}_{ML})=(1-\omega)\left[\hat{\beta}_{ML}^{d}\,S_{15}-d\,(\ln\hat{\beta}_{ML}-S_2)-1\right],\tag{32}$$

$$PR_{L_2}(\hat{R}(t)_{ML})=(1-\omega)\left[\exp(-d\hat{\alpha}_{ML}t_m)\ \ S_{16}-d(S_3-\hat{\alpha}_{ML}t_m)-1\right],\tag{33}$$

$$PR_{L_2}(\hat{h}(t)_{ML})==(1-\omega)\left[\left(\frac{\hat{\alpha}_{ML}}{\hat{\beta}_{ML}+t^2}\right)^{d}S_{17}-d\,(\delta-S_4)-1\right],\tag{34}$$

where

$$S_{14}=\sum_{j=1}^{N}P_j\frac{\Gamma(m+a_j-d)(b_j-T_j)^d}{\Gamma(m+a_j)};\ S_{15}=\sum_{j=1}^{N}P_j\,\beta_j^{-d};$$

$$S_{16}=\sum_{j=1}^{N}P_j\left(1-\frac{dt_j}{b_j-T_j}\right)^{-(m+a_j)};\ S_{17}=\sum_{j=1}^{N}P_j\frac{\Gamma(m+a_j-d)\left[(\beta_j+t^2)(b_j-T_j)\right]^d}{\Gamma(m+a_j)}.$$

3. Numerical Example (Real data)

In this section, the real data consisting of survival times (in years) of 46 patients given chemotherapy treatment alone reported in Bekker *et al.* (2000) is presented to illustrate the estimation methods developed in the preceding sections. They showed that the compound Rayleigh model is acceptable for these data. As a numerical illustration, we have generated the artificial progressive Type II censored sample of size m = 12 from the given data set. Let the vector of observed failure times be $\underline{x}$ =(0.115, 0.132, 0.164, 0.203, 0.296, 0.458, 0.501, 0.534, 0.641, 0.841, 1.219, 1.447) with the censoring scheme $\underline{r}$ =(4,4,4,4,1,0,0,1,4,4,4,4).

Firstly, we have obtained the ML estimate of parameter β by solving the equation (4) numerically. Concerning the estimated value of $\beta\,(=0.1237)$, we have assumed that β_j takes 25 values $(0.070\,(0.005)\,0.190)$ with the probability $\frac{2j}{N(N+1)}$, $j=1,2,\ldots,25$. Based on $\underline{x}$, we have obtained two values of reliability function $R(t_1)=0.86735$ and $R(t_2)=0.37755$ using a nonparametric approach (Martz & Waller 1982) by setting $t_1=0.132$ and $t_2=0.534$. For each assumed β_j, we have elicited the hyper-parameters (a_j,b_j) with the help of the reliability values. We have also obtained the prior probabilities P_j for each β_j.

Table 1 summarized the elicited values of the hyper-parameters (a_j,b_j) and the posterior probabilities for each β_j. The posterior risks of ML estimates, Bayes estimates under BLGLF and BGELF were computed using the results outlined in Section 2, and reported in Table 2.

4. Simulation study

Since the performance of the different methods cannot be compared theoretically, we have performed an extensive Monte Carlo simulation study to compare the performance of Bayes and ML estimates in terms of posterior risks, for different sample sizes (n), effective sample sizes (m) and censoring schemes $(\underline{r})$ according to the following steps:

1. For a particular n, m and censoring scheme $\underline{r}=(r_1,r_2,\ldots,r_m)$, we have generated a progressive Type II censored samples $\underline{x}$ from the compound Rayleigh distribution with the parameters $(\alpha,\beta)=(1.2,4.5)$ according to the algorithm given in Balakrishnan and Sandhu (1995).
2. Concerning the assumed value of the parameter β, we have assumed that β_j takes 20 values $(3.6\ (0.1)\ 5.5)$ with the probability $\frac{2j}{N(N+1)}, j=1,2,\ldots,20\cdot$
3. According to the generated sample $\underline{x}$, we have estimated two values of the reliability using a nonparametric approach (Martz & Waller 1982). Using these reliability values, we have elicited the hyper-parameters (a_j,b_j) for each β_j via numerical method.
4. The ML estimates, Bayes estimates under BLGLF and BGELF, and the corresponding posterior risks were computed according to the results outlined in Section 2 by utilizing computer software Microsoft Visual Studio 2008.

The different censoring schemes applied in the simulation study are summarized in Table 3. The averaged values of posterior risks of ML and Bayes estimates based on 1000 simulated data sets were computed, and reported, respectively, in Tables 4 – 7. From the simulation results, the following points can be drawn:

1. It is clear that each of the Bayes estimates has smaller posterior risk than the ML estimates, i.e., the Bayes estimates perform better than the ML estimates.
2. In case of Bayes estimation under the mentioned balanced loss functions, the Bayes estimates under BGELF have the smallest posterior risk as compared with those under BLGLF. This indicates that the Bayes estimates relative to BGELF perform better than the Bayes estimates relative to BLGLF.
3. In case of ML estimation, the posterior risks of ML estimates of reliability parameters are smaller under BGELF with negative shape parameter d.
4. For fixed sample size n, as the effective sample size m increases, the posterior risks of both ML and Bayes estimates decrease, i.e., the performance becomes better with increasing effective sample size m.
5. For fixed effective sample size m, as the sample size n decreases, the posterior risks of both ML and Bayes estimates decrease, i.e., the posterior risk of the estimates gets smaller with decreasing sample sizes.
6. The Bayes estimates relative to BGELF are sensitive to the values of the corresponding shape parameter d.

7. Different values of the parameters α and β, weight ω, shape parameter d have been examined, and the same conclusions stated above were observed. It may be mentioned here that because of space restriction, results for all the variations in the reliability parameters are not shown.

5. Conclusion

Based on the progressive Type II censored data, the present paper proposes classical and Bayesian approaches to estimate the two unknown parameters as well as the reliability and failure rate functions for the compound Rayleigh model. The Bayes estimates are obtained under both the BLGLF and BGELF. The use of a discrete distribution for the shape parameter resulted in a closed form expression for the posterior pdf. The posterior risks of ML estimates and Bayes estimates relative to BGELF are obtained under BLGLF; and the estimates are compared in terms of posterior risks by considering real life data and simulated data. The findings from the analysis of real life data and simulated data are in accordance with those of the simulation study, suggesting that Bayesian approach is superior to ML approach. The motivation is also to explore the most appropriate loss function among the proposed loss functions. The effect of BLGLF and BGELF is therefore examined, and it is observed that importance should not solely rest upon the choice of prior distribution, but also the choice of loss function for optimum decision-making. As the Bayes estimates under BGELF perform better than BLGLF, we recommend employing BGELF for optimal decision-making. It is also noticed that the posterior risk of estimates gets smaller with increasing ratio *m/n*.

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Table 1. Prior Information, Hyper-parameter Values, and Posterior Probabilities

j	β_j	e_j	a_j	b_j	P_j	j	β_j	e_j	a_j	b_j	P_j
1	0.070	0.00308	5.530	8.457	0.00113	14	0.135	0.04308	1.370	1.100	0.04545
2	0.075	0.00615	4.084	5.843	0.00343	15	0.140	0.04615	1.324	1.025	0.04847
3	0.080	0.00923	3.293	4.425	0.00647	16	0.145	0.04923	1.279	0.956	0.05147
4	0.085	0.01231	2.797	3.543	0.00992	17	0.150	0.05231	1.244	0.899	0.05426
5	0.090	0.01538	2.456	2.942	0.01358	18	0.155	0.05538	1.208	0.845	0.05706
6	0.095	0.01846	2.205	2.505	0.01734	19	0.160	0.05846	1.179	0.799	0.05966
7	0.100	0.02154	2.016	2.178	0.02109	20	0.165	0.06154	1.149	0.755	0.06225
8	0.105	0.02462	1.864	1.919	0.02484	21	0.170	0.06462	1.124	0.717	0.06470
9	0.110	0.02769	1.743	1.714	0.02851	22	0.175	0.06769	1.101	0.682	0.06703
10	0.115	0.03077	1.645	1.549	0.03209	23	0.180	0.07077	1.079	0.650	0.06933
11	0.120	0.03385	1.559	1.407	0.03559	24	0.185	0.07385	1.061	0.622	0.07147
12	0.125	0.03692	1.487	1.289	0.03899	25	0.190	0.07692	1.041	0.594	0.07360
13	0.130	0.04000	1.424	1.187	0.04227						

Table 2. Posterior risks of ML and Bayes estimates with $(\omega = 0.4, t = 2)$

Estimate	$PR_{L_1}(\,.\,)_{ML}$	$PR_{L_2}(\,.\,)_{ML}$		$PR_{L_1}(\,.\,)_{BLG}$	$PR_{L_2}(\,.\,)_{BGE}$	
		$d = -0.7$	$d = 1.5$		$d = -0.7$	$d = 1.5$
$\hat{\beta}$	0.04479	0.01172	0.04472	0.03386	0.00831	0.03792
$\hat{\alpha}$	0.03983	0.01582	0.06195	0.03370	0.01343	0.06035
$\hat{R}(t)$	0.04159	0.00938	0.05883	0.03810	0.00886	0.04892
$\hat{h}(t)$	0.03823	0.01533	0.06079	0.03268	0.01316	0.05945

Table 3. Progressive Type II censoring schemes (C.S.) applied in the simulation study

n	***m***	**C.S. No.**	$\underline{r} = (r_1, r_2, \ldots, r_m)$
20	8	[1]	2, 2, 2, 0, 0, 2, 2, 2
	10	[2]	2, 2, 1, 0, 0, 0, 0, 1, 2, 2
	12	[3]	1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1
50	8	[4]	8, 8, 5, 0, 0, 5, 8, 8
	10	[5]	5, 5, 5, 5, 0, 0, 5, 5, 5, 5
	12	[6]	4, 4, 4, 4, 3, 0, 0, 3, 4, 4, 4, 4
100	8	[7]	18, 18, 10, 0, 0, 10, 18, 18
	10	[8]	15, 15, 15, 0, 0, 0, 0, 15, 15, 15
	12	[9]	10, 10, 10, 10, 4, 0, 0, 4, 10, 10, 10, 10

Table 4. Averaged posterior risks of ML and Bayes estimates of parameter α with $\omega = 0.4$

C.S.	$PR_{L_1}(\hat{\alpha}_{ML})$	$PR_{L_2}(\hat{\alpha}_{ML})$		$PR_{L_1}(\hat{\alpha}_{BLG})$	$PR_{L_2}(\hat{\alpha}_{BGE})$	
		$d = -0.7$	$d = 1.5$		$d = -0.7$	$d = 1.5$
[1]	0.52082	0.18283	0.36874	0.23478	0.06278	0.25809
[2]	0.26953	0.09042	0.22041	0.12885	0.03746	0.15377
[3]	0.16206	0.05361	0.15153	0.08198	0.02573	0.10646
[4]	1.60272	0.65892	0.62312	0.66763	0.15522	0.45604
[5]	0.74752	0.27339	0.48421	0.32061	0.08067	0.34432
[6]	0.69798	0.24806	0.46673	0.29691	0.07519	0.32301
[7]	2.69403	1.27232	1.27756	1.10319	0.23844	1.01901
[8]	2.04948	0.89656	1.04645	0.84127	0.18732	0.81194
[9]	1.79258	0.75238	0.95503	0.73507	0.16588	0.72746

Table 5. Averaged posterior risks of ML and Bayes estimates of parameter β with $\omega = 0.4$

C.S.	$PR_{L_1}(\hat{\beta}_{ML})$	$PR_{L_2}(\hat{\beta}_{ML})$		$PR_{L_1}(\hat{\beta}_{BLG})$	$PR_{L_2}(\hat{\beta}_{BGE})$	
		$d = -0.7$	$d = 1.5$		$d = -0.7$	$d = 1.5$
[1]	1.38686	0.57409	0.74668	0.55868	0.11854	0.55553
[2]	0.88994	0.32945	0.53736	0.36003	0.07931	0.37725
[3]	0.70585	0.27621	0.49586	0.28647	0.07434	0.34758
[4]	2.77120	1.36296	1.25925	1.11222	0.22248	1.01197
[5]	1.51599	0.60558	0.80348	0.67027	0.12994	0.66002
[6]	1.47034	0.60366	0.79634	0.59208	0.12606	0.59253
[7]	3.56024	1.88663	1.91492	1.42776	0.27831	1.24713
[8]	3.09487	1.58501	1.66420	1.24173	0.24524	1.10759
[9]	3.00988	1.51181	1.54496	1.20782	0.24006	1.08821

Table 6. Averaged posterior risks of ML and Bayes estimates of reliability $R(t)$ with $(\omega = 0.4, t = 2)$

C.S.	$PR_{L_1}\left(\hat{R}(t)_{ML}\right)$	$PR_{L_2}\left(\hat{R}(t)_{ML}\right)$		$PR_{L_1}\left(\hat{R}(t)_{BLG}\right)$	$PR_{L_2}\left(\hat{R}(t)_{BGE}\right)$	
		$d = -0.7$	$d = 1.5$		$d = -0.7$	$d = 1.5$
[1]	0.03992	0.00874	0.06599	0.03425	0.00783	0.04608
[2]	0.01809	0.00437	0.02193	0.01652	0.00396	0.01982
[3]	0.01369	0.00354	0.01479	0.01096	0.00268	0.01244
[4]	0.12269	0.02470	0.26254	0.07446	0.01668	0.10029
[5]	0.03980	0.00877	0.05929	0.03077	0.00710	0.03979
[6]	0.02831	0.00633	0.04050	0.02275	0.00530	0.02874
[7]	0.25341	0.04864	0.67931	0.13253	0.02982	0.16938
[8]	0.12115	0.02528	0.21169	0.06803	0.01564	0.08659
[9]	0.07914	0.01701	0.12569	0.04598	0.01069	0.05744

Table 7. Averaged posterior risks of ML and Bayes estimates of failure rate $h(t)$ with $(\omega = 0.4, t = 2)$

C.S.	$PR_{L_1}\left(\hat{h}(t)_{ML}\right)$	$PR_{L_2}\left(\hat{h}(t)_{ML}\right)$		$PR_{L_1}\left(\hat{h}(t)_{BLG}\right)$	$PR_{L_2}\left(\hat{h}(t)_{BGE}\right)$	
		$d = -0.7$	$d = 1.5$		$d = -0.7$	$d = 1.5$
[1]	0.13806	0.04712	0.13530	0.08035	0.02601	0.10801
[2]	0.05704	0.02112	0.07928	0.04275	0.01595	0.07247
[3]	0.03161	0.01296	0.06154	0.02896	0.01199	0.05886
[4]	0.63944	0.22633	0.28830	0.28018	0.07302	0.20272
[5]	0.24147	0.08081	0.20185	0.11611	0.03424	0.14160
[6]	0.19823	0.06572	0.17249	0.09502	0.02878	0.11905
[7]	1.29814	0.51039	0.75239	0.54236	0.12889	0.55321
[8]	0.89958	0.33200	0.56688	0.37881	0.09330	0.40293
[9]	0.73427	0.26212	0.48625	0.30926	0.07753	0.33689