Control Theory and Informatics www.iiste.org
ISSN 2224-5774 (Paper) ISSN 2225-0492 (Online)
Vol.6, No.3, 2016

Optimal Control Equivalent Approach to Non-Linear Uncertain Descriptor Systems with Matching Condition

Radhi Ali Zaboon
Department of Mathematics, College of Science, Al Mustansiriyah University, Iraq

Sabeeh Lafta Jasim
Department of Mathematics, College of Science, Basrah University, Iraq

Abstract
In this paper, the solution of the robust control problem of some non-linear semi-explicit descriptor uncertain systems having matching condition and linear algebraic equation with rank deficient of the algebraic coefficient matrix is considered. An optimal control approach have been developed in the sense that, the solution of an equivalent optimal control problem to the uncertain nonlinear descriptor system, is the solution to the given descriptor one with matching condition. A relation between the robust control problem and its equivalent optimal control problem have been developed with theorems and illustration.

1 Introduction

A descriptor control system describes a natural representation for physical systems and can be represented by differential and algebraic equations which is a generalized representation of the state-space system. Descriptor systems can be found in electrical circuits, robots and many other practical systems which are modelled with additional algebraic constraints. This system is also referred to as singular system, implicit system, generalized state-space system, semi-state system, or differential-algebraic system (Debeljkovic & Buzurovic 2011).

The solvability of linear descriptor systems may be found in (Campbell 1980), (Dai 1989) and (Brenan et al. 1996), while, nonlinear descriptor systems is discussed by (Kunkel & Mehrmann 1994, 1995, 2001, 2004) under some suitable assumptions. Furthermore, Stability of linear and non-linear descriptor systems are studied by (Danielle et al. 2002), (Michael 2011), (Debeljkovic 2011), (Tadeusz 2012), (Shravan 2012) and (Xiaoming & Zhi 2013).

The descriptor control uncertain system have been interested and introduced to preserve various system properties under some perturbation in the model.

The insensitiveness of the system properties is called *robustness* and it is an important field of investigation. The fact is that in many practical situations the parameters of system components are not known exactly. Usually, there is only some information on the intervals to which they belong. Therefore, the robustness for any system property is an important theoretical and practical question.

Recently, much attention has been given to the design of controllers, so that system properties are preserved under various classes of uncertainties appearing in the system. Such controllers are called *robust controllers*, and the resulting system is said to be *robust control system*.

Descriptor systems, like other systems may contain many types of uncertainties. These uncertainties can be classified as with and without matching condition. In this paper, robust control with matching condition have been considered.

Due to the difficulty in solving general robust descriptor systems see (SUN & WANG 2012), in this paper, robust control problem is translated into a specific (equivalent) optimal control problem. The solution of optimal control problem is then a solution to the robust control problem based on the nominal system structure and the types of uncertainties.

The idea is the generalisation to the state-space approach of (FING LIN et al.1992) , (FING LIN 2000) and (Radhi *et al*. 2006, 2008).

The aim of this paper is to solve the semi-explicit descriptor robust control systems with some type of uncertainties. A new approach have been developed by finding an equivalent optimal control problem to robust one so that the optimal solution of the optimal control problem is solution to the given robust descriptor uncertain control system.

This approach up to our knowledge and survey, is a novel technique, which gives a procedure to study the uncertain descriptor system with matching condition via an equivalent optimal control problem. The construction of this approach is based on some theorems and lemmas which are developed in this paper with illustration.

2 Problem formulation

Consider the non-linear semi-explicit descriptor system with matching condition

$$E\dot{x} = Ax + Bu + Bf(x) \tag{1}$$

Where $E, A \in \mathbb{R}^{n\times n}$, $rank(E) = n_0$, $0 < n_0 \le n$ $x \in \mathbb{R}^n$, $u \in R^r$ and $B \in \mathbb{R}^{n\times r}$ are the system coefficients and $f \in C^1(\mathbb{R}^n; \mathbb{R}^r)$ represent the uncertainty of the system, satisfying some conditions that will be defined to ensure the solvability. System (1) is equivalent to

$$\dot{x}_1 = A_1x_1 + A_2x_2 + B_1u + B_1f(x_1, x_2) \quad (2a)$$

$$0 = A_3x_1 + A_4x_2 + B_2u + B_2f(x_1, x_2) \quad (2b)$$

Where $x_1 \in R^{n_0}$, $x_2 \in \mathbb{R}^{n-n_0}$ and $A_1 \in \mathbb{R}^{n_0\times n_0}, A_2 \in \mathbb{R}^{n_0\times(n-n_0)}$, $A_3 \in \mathbb{R}^{(n-n_0)\times n_0}$ and $A_4 \in \mathbb{R}^{(n-n_0)\times(n-n_0)}$, $B_1 \in \mathbb{R}^{n_0\times r}$, $B_2 \in \mathbb{R}^{(n-n_0)\times r}$ with $\begin{pmatrix}\Sigma A_1 & \Sigma A_2\\ A_3 & A_4\end{pmatrix} = \mathbb{Q}A\mathbb{P}$, $\begin{pmatrix}\Sigma B_1\\ B_2\end{pmatrix} = \mathbb{Q}B$ for some nonsingular matrices $\mathbb{Q}, \mathbb{P}$ and Σ and this can be obtained as follows:

Since $rank(E) = n_0$, then it follows that there always exist unitary matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{n\times n}$ such that

$$E = U\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^T \quad (3)$$

Where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_{n_0})$ and $\sigma_1 \ge \sigma_2 \ge \dots, \ge \sigma_{n_0} > 0$.
From (3), one can define

$$\mathbb{P} \triangleq V, \qquad \mathbb{Q} \triangleq U^{-1}$$

$$\mathbb{Q}E\mathbb{P} = U^{-1}\left[U\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^T\right]V = \begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix} \quad (4)$$

On using (1) and (4) as well as $rank(E) = n_0$, one gets
$\mathbb{Q}E\dot{x} = \mathbb{Q}Ax + \mathbb{Q}Bu + \mathbb{Q}Bf(x) \Rightarrow \mathbb{Q}E\mathbb{P}\mathbb{P}^{-1}\dot{x} = \mathbb{Q}A\mathbb{P}\mathbb{P}^{-1}x + \mathbb{Q}Bu + \mathbb{Q}Bf(x)$
From

$$\mathbb{P}^{-1}x = \begin{pmatrix}x_1\\ x_2\end{pmatrix} \Leftrightarrow x = \mathbb{P}\begin{pmatrix}x_1\\ x_2\end{pmatrix},\ x_1 \in \mathbb{R}^{n_0},\ x_2 \in \mathbb{R}^{n-n_0} \quad (5)$$

then

$$\mathbb{Q}E\mathbb{P}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \mathbb{Q}A\mathbb{P}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \mathbb{Q}Bu + \mathbb{Q}Bf\left(\mathbb{P}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\right) \quad (6)$$

where $\mathbb{Q}A\mathbb{P} = \begin{pmatrix}\bar{A}_1 & \bar{A}_2\\ A_3 & A_4\end{pmatrix}$ and $\mathbb{Q}B=\begin{pmatrix}\bar{B}_1\\ B_2\end{pmatrix}$, which gives

$$\dot{x}_1 = A_1x_1 + A_2x_2 + B_1u + B_1f(x_1, x_2)$$

$$0 = A_3x_1 + A_4x_2 + B_2u + B_2f(x_1, x_2)$$

Where $A_1 = \Sigma^{-1}\bar{A}_1$, $A_2 = \Sigma^{-1}\bar{A}_2$, $B_1 = \Sigma^{-1}\bar{B}_1$.

In this paper, for simplicity, it is assumed that the matrix $\mathbb{Q}$ satisfies $\mathbb{Q}B = \begin{pmatrix}\bar{B}_1\\ 0\end{pmatrix}$, then the system (2) is equivalent to

$$\dot{x}_1 = A_1x_1 + A_2x_2 + B_1u + B_1f(x_1, x_2) \quad (7a)$$

$$0 = A_3x_1 + A_4x_2 \quad (7b)$$

To study the solvability of the differential algebraic equations (7) which is equivalent to the descriptor system (1) by the invertible transformation $x = \mathbb{P}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$, the following assumption have been presented (Kunkel & Mehrmann 2001).

Assumptions A

1. Assume there exist an open set $\tilde{\Omega}_x \subset D$ such that for all $\hat{x}_1 \in \tilde{\Omega}_x$ it is possible to solve $A_3\hat{x}_1(t) + A_4\hat{x}_2(\mathrm{t}) = 0$ for $\hat{x}_2$. One can define the corresponding solution manifold as:
$\tilde{\Omega} = \left\{x_1 \in \tilde{\Omega}_x, x_2 \in \mathbb{R}^{n-n_0} \middle| \begin{pmatrix}x_1(t),\\ x_2(t)\end{pmatrix} \in \aleph([\mathrm{A}_3 \quad \mathrm{A}_4]), t \ge 0\right\}$ where $\aleph(\cdot)$ denotes the kernel (null space) of the operator $(\cdot)$.
Let us denote the set of the consistent initial values of (7) by w_k,
$$w_k \triangleq \{x_0 = (x_{1,0}, x_{2,0}) |\, x_0 \in \aleph([\mathrm{A}_3 \quad \mathrm{A}_4])\} \quad (8)$$
2. $Rank[\mathrm{A}_3 \quad \mathrm{A}_4] = Rank\mathrm{A}_4 \Leftrightarrow w_k = \aleph[\mathrm{A}_3 \quad \mathrm{A}_4]$ (9)

Lemma 1 (solvability)

Consider the system

$$\dot{x}_1 = A_1x_1 + A_2x_2 + B_1u + B_1f(x_1, x_2) \quad (10a)$$

$$0 = A_3x_1 + A_4x_2 \quad (10b)$$

$$\big(x_1(0), x_2(0)\big) = (x_{1,0}, x_{2,0}) \in w_k \quad (10c)$$

where $x_1 \in R^{n_0}$, $x_2 \in \mathbb{R}^{n-n_0}$ and $A_1 \in \mathbb{R}^{n_0\times n_0}, A_2 \in \mathbb{R}^{n_0\times(n-n_0)}$,
$A_3 \in \mathbb{R}^{(n-n_0)\times n_0}$, $A_4 \in \mathbb{R}^{(n-n_0)\times(n-n_0)}$ and $B_1 \in \mathbb{R}^{n_0\times r}$, $f \in C^1(\mathbb{R}^n; \mathbb{R}^r)$, $f(0,0) = 0$ and the system satisfies Assumption A.

If A_4 is of rank deficient matrix, i.e., $ank\ \mathrm{A}_4 < n - n_0$, then there exists a matrix L of dimension $(n -$

$n_0) \times n_0$ then the system (10) will be in the reduced form, for $x_1 \in \widetilde{\Omega}_x$ which is open subset of R^{n_0}

$$\dot{x}_1 = (A_1 + A_2 L)x_1 + B_1 u + B_1 f\big(x_1, \phi(x_1)\big) \tag{11}$$

Where $\phi(x_1) \triangleq L x_1$.

Which is solvable for a given $\big(x_1(0), x_2(0)\big) \in w_k$ and $u \in \mathfrak{U}[0,t]$, where

$$w_k = \big\{\big(x_1(0), x_2(0)\big)\big| x_1(0) \in \widetilde{\Omega}_x, x_2(0) = \phi(x_1(0))\big\}$$

and

$$\mathfrak{U}[0,t] = \{u(\cdot)|u(t) \textit{ is differntiable on } [0,t]\}$$

And the solution to system (11) is $x_1(t) \in \widetilde{\Omega}_x$ and $x_2(t) = \phi(x_1(t))$.

Proof

If A_4 is of rank deficient matrix in (10), then we may consider the existence of a matrix L of dimension $(n - n_0) \times n_0$ such that

$$x_2(t) = L x_1(t) \tag{12}$$

$$L x_1(t) - x_2(t) = 0 \Rightarrow \big(L - I_{n-n_0}\big)\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0$$

So $(x_1(t), x_2(t)) \in \aleph(L \quad I_{n-n_0})$ and L satisfies $A_3 x_1 + A_4 L x_1 = 0$ or $(A_3 + A_4 L)x_1 = 0$ for all $x_1 \neq 0$, $x_1 \in \widetilde{\Omega}_x$. This means that

$$A_3 + A_4 L = 0 \tag{13}$$

Such a matrix L is always exists when condition (8) is satisfied (see [8]). So the solution of (10b) have to belong to the set $\aleph(L \quad I_{n-n_0})$, so the solution manifold is

$$\widetilde{\Omega} \triangleq \{x \in \mathbb{R}^n; x(t) \in \aleph(L \quad I_{n-n_0})\} \tag{14}$$

Therefore, the solution of (10b) will be found locally in $\widetilde{\Omega}$ and the system is then given in the reduced form for $x_1 \in \widetilde{\Omega}_x$

$$\dot{x}_1 = (A_1 + A_2 L)x_1 + B_1 u + B_1 f(x_1, L x_1) \tag{15}$$

From state space analysis and $f \in C^1(\mathbb{R}^n; \mathbb{R}^r)$, the problem has a solution for a given $u \in \mathfrak{U}[0,t]$ and $x_1 \in \widetilde{\Omega}_x$, hence the original system solution is $x_1(t) \in \widetilde{\Omega}_x$ and $x_2(t) = \phi(x_1(t))$ with $x_2(0) = \phi(x_1(0))$. ■

3 Robust Descriptor Control Problem

Consider the nonlinear semi-explicit descriptor robust control system with matching condition defined by (1) which is (by using lemma 1)) equivalent to the system (15)

$\dot{x}_1 = (A_1 + A_2 L)x_1 + B_1 u + B_1 f(x_1, L x_1)$, $x_1 \in \widetilde{\Omega}_x$

$x_2 = \phi(x_1)$, $\big(x_1(0), x_2(0)\big) \in w_k$

The equilibrium states of the robust control system (15) can be calculated when the control function u is identically 0 or is a constant vector u_0. Since $f(0,0) = 0$, then the unique equilibrium state of the system is the origin $(x_1, x_2) = (0,0)$.

Suppose that the control is feedback control defined by

$$u(t) = -k x_1(t) \tag{16}$$

Now, the aim of the following work is to find a suitable matrix k such that the feedback nonlinear dynamical system

$$\dot{x}_1 = (A_1 + A_2 L - B_1 k)x_1 + B_1 f(x_1, L x_1) \tag{17a}$$

$$x_2 = \phi(x_1) = L x_1, \big(x_1(0), x_2(0)\big) \in w_k \tag{17b}$$

Where $x_1 \in \widetilde{\Omega}_x$, is asymptotically stable.

To find the conditions which make the nonlinear descriptor robust control system with matching condition (1) is asymptotically stable, the following theorem has been developed.

Theorem 1

Consider the nonlinear descriptor robust control system with matching condition (1) which is locally equivalent to the system (15), that satisfy

1. The system satisfies Assumption A.
2. The eigenvalues of $A_1 + A_2 L$ satisfies $(A_1 + A_2 L) = \{\lambda_i | \lambda_i + \lambda_j \neq 0 , \ \forall\, i \neq j\}$.
3. $(A_1 + A_2 L , B_1)$ is state space controllable, where $x_2 = L x_1$.
4. $f(0,0) = 0$.
5. $\|f(x)\| \leq f_{max}(x) = \eta \|x\|_Q$, $x = (x_1, x_2)$.
   Where $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$ and Q_1 *and* Q_2 are symmetric positive semi-definite matrices and $Q_1 = L^T Q_2 L$.
6. The control is defined by $u(t) = -k x_1 = -R^{-1} B_1^T P x_1$,
   Where R is symmetric positive definite matrix and P is the symmetric positive definite matrix that solves Riccati equation

$$(A_1+A_2L)^TP+P(A_1+A_2L)-2PB_1R^{-1}B_1^TP+2\alpha P+2Q_1=0 \quad (18)$$

7. $\alpha=\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\lambda_{max}(Q_1)(1+\|L\|)\|B_1\|$, where $\|.\|$ is suitable norm.

Then the equilibrium point $(x_1,x_2)=(0,0)$ of (15) is asymptotically stable.

proof

For $x_1\in\tilde{\Omega}_x$, and $(x_1,x_2)\in\tilde{\Omega}=\{x_1\in\tilde{\Omega}_x,x_2\in\mathbb{R}^{n-n_0}|\ x_2=Lx_1\}$ define Lyapunov function for (15) as $V(x_1(t))=x_1^TPx_1$, $P^T=P>0$

Since $\lambda_{min}(P)\|x_1\|^2\le x_1^TPx_1\le\lambda_{max}(P)\|x_1\|^2$ for $P^T=P>0$ (19)

Where $\lambda_{min}(P)$ and $\lambda_{max}(P)$ are the minimum and maximum eigenvalues of P respectively, from (15) and condition (7) as well as $u(t)=-R^{-1}B_1^TPx_1$ with some computations, one gets

$\dot{V}(x_1)=x_1^T[(A_1+A_2L)^TP+P(A_1+A_2L)-2PB_1R^{-1}B_1^TP]x_1+x_1^TPB_1f+f^TB_1^TPx_1$

By solving (18) for P , we have that

$\dot{V}(x_1)=-2x_1^TQ_1x_1-2\alpha x_1^TPx_1+x_1^TPB_1f+f^TB_1^TPx_1$, deletion of the positive term $x_1^TQ_1x_1$ gives that

$\dot{V}(x_1)\le-2\alpha x_1^TPx_1+x_1^TPB_1f+f^TB_1^TPx_1$

Since

$$\begin{aligned}x_1^TPB_1f+f^TB_1^TPx_1&\le 2\eta\lambda_{max}(P)\|B_1\|\|x_1\|\|x_1\|_{Q_1}(1+\|L\|)\\&\le 2\eta\lambda_{max}(P)\lambda_{max}(Q_1)\|B_1\|\|x_1\|^2(1+\|L\|)\end{aligned}$$

And

$$x_1^TPx_1\ge\lambda_{min}(P)\|x_1\|^2\Rightarrow -x_1^TPx_1\le-\lambda_{min}(P)\|x_1\|^2$$

Therefore, from the two inequalities above, we have that

$$\dot{V}(x_1)\le-2\alpha\,\lambda_{min}(P)\|x_1\|^2+2\eta\lambda_{max}(P)\lambda_{max}(Q_1)\|B_1\|\|x_1\|^2(1+\|L\|)$$

From assumption 6, we get that

$\dot{V}(x_1)\le 2(\eta-1)\lambda_{max}(P)\lambda_{max}(Q_1)\|B_1\|(1+\|L\|)\|x_1\|^2$

Putting the condition $\eta-1<0$ on η i.e.,

$$0<\eta<1$$

Gives that

$$\dot{V}(x_1)\le-2\lambda_{max}(P)\lambda_{max}(Q_1)\|B_1\|(1+\|L\|)\|x_1\|^2 \quad\Rightarrow\quad \dot{V}(x_1)<0$$

This proves that $x_1=0$ is asymptotically stable. i.e.,

$$\lim_{t\to\infty}\|x_1(t)\|=0,\qquad x_1\in\tilde{\Omega}_x$$

And from the continuity of the norm $\|\cdot\|$, then we have that

$$\begin{aligned}\lim_{t\to\infty}\|x_2(t)\|&=\left\|\lim_{t\to\infty}x_2(t)\right\|\\&=\left\|\lim_{t\to\infty}Lx_1(t)\right\|,\ x_1\in\tilde{\Omega}_x\\&=\left\|L\lim_{t\to\infty}x_1(t)\right\|,\ x_1\in\tilde{\Omega}_x\\&=0\end{aligned}$$

Therefore the equilibrium point $(x_1,x_2)=(0,0)$ is asymptotically stable. ■

Theorem1 above gives as a class for the uncertainties $f(x)$ which can be defined as

$$f_\alpha=\left\{f\middle|\|f(x)\|\le\eta\|x\|_Q,0<\eta<1,\alpha=\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\lambda_{max}(Q_1)(1+\|L\|)\|B_1\|\right\}$$

i.e., the nonlinear descriptor robust control system with matching condition (1) is stable for all $f\in f_\alpha$ and its solution is defined by $u(t)=-R^{-1}B_1^TPx_1$.

Due to the difficulty in solving the equivalent robust control descriptor problem (15) in the presence of system uncertainties, leads to develop a novel approach by finding an equal control problem (in reduced system form) which is equivalent to the robust one in the sense that the solution of the equivalent optimal control problem is the solution to the robust one. The following theorems present this fact.

4 Optimal Control Equivalent Problem

Based on the optimal control theorem for state space systems [6], the following optimal control problem that equivalent to the robust control problem (15) is presented.

For all $x_1\in\tilde{\Omega}_x$, the nominal system of (15) is defined by

$$\dot{x}_1=(A_1+A_2L)x_1+B_1u,\ \left(x_1(t_0),x_2(t_0)\right)=\left(x_{1,0},x_{2,0}\right) \quad (20a)$$
$$x_2=Lx_1,\ \left(x_{1,0},x_{2,0}\right)\in w_k$$

Where

$x_1\in R^{n_0}$, $x_2\in\mathbb{R}^{n-n_0}$, $A_1\in\mathbb{R}^{n_0\times n_0}$, $A_2\in\mathbb{R}^{n_0\times(n-n_0)}$, $B_1\in\mathbb{R}^{n_0\times r}$ and $L\in\mathbb{R}^{(n-n_0)\times n_0}$.

Which depends on the known part of the system (15) and the cost functional or performance criterion

$$J(u(\cdot))\triangleq\int_0^\infty(f_{max}^2(x)+x_1^T(t)Q_1x_1(t)+u^T(t)Ru(t))dt \quad (20b)$$

Where $x_1(t) \triangleq x_1(t; x_{1,0}, u(\cdot))$ and $f_{max}^2(x)$ is a given upper bound of $f(x)$, R is symmetric positive definite matrix and Q_1 is symmetric positive semi-definite matrix. Now the *optimal control problem* is to steer $x_{1,0}$ to the target state 0, using a control $u(\cdot)$ from the appropriate class for the problem, in such a way that J is a minimum. Let the class of successful controls is denoted by Δ, i.e.,

$$\Delta = \{u(\cdot) \in \mathfrak{U}[0,t] | \exists t > 0 \text{ such that } x_1(t; x_{1,0}, u(\cdot)) = 0\}$$

Then a control $u_*(\cdot) \in \mathfrak{U}[0,t]$ is optimal if it is successful, *i.e.*, $u_*(\cdot) \in \Delta$, and satisfies that

$$J(u_*(\cdot)) \leq J(u(\cdot)) \text{ for all } u(\cdot) \in \Delta.$$

To prove the necessary condition for optimality for the equivalent optimal control problem (20), the following lemma will be introduced.

Lemma 2 (necessary condition for optimality)

Consider the equivalent optimal control system (20) of the robust descriptor control system (1), and there is a positive definite continuously differentiable function $V(x_1)$ such that

$$V(x_1) \triangleq \min_{u \in \Delta} \int_{t_0}^{t_f} (f_{max}^2(x) + x_1^T Q_1 x_1 + u^T R u) dt.$$

Then the necessary condition for existence of optimal control is that $V(x_1)$ must satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \min_{u \in \Delta} \left[f_{max}^2(x) + x_1^T Q_1 x_1 + u^T R u + V_{x_1}^T((A_1 + A_2 L) x_1 + B_1 u) \right]$$

Where $V_{x_1} = \frac{dV}{dx_1}$.

Proof the same as derivation in the state space proof, See Sage [8].

Main Theorem 2 (Equivalency theorem)

Consider the robust control problem

$$E\dot{x} = Ax + Bu + Bf(x), \quad x(0) = x_0 \tag{21}$$

Where $E, A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$ and $B \in \mathbb{R}^{n \times r}$ are the system coefficients and $f \in C^1(\mathbb{R}^n; \mathbb{R}^r)$ represent the uncertainty of the system and

1. $rank(E) = n_0 < n$
2. $\|f(x)\| \leq f_{max}(x)$, $f_{max}(0) = 0$,
3. $f(0) = 0, u \in \Delta$.

And the optimal control problem

$$\min_{u \in \Delta} J = \int_{t_0}^{\infty} (f_{max}^2(x) + x_1^T Q_1 x_1 + u^T R u) dt \tag{22a}$$

Subject to the controllable system

$$\dot{x}_1 = (A_1 + A_2 L) x_1 + B_1 u, \left(x_1(t_0), x_2(t_0)\right) = \left(x_{1,0}, x_{2,0}\right) \tag{22b}$$

$$x_2 = L x_1, \left(x_{1,0}, x_{2,0}\right) \in w_k, x_1 \in \widetilde{\Omega}_x$$

Where Q_1 is positive semi-definite matrix and R is positive definite matrix $x = \mathbb{P}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $x_1 \in \mathbb{R}^{n_0}$, $x_2 \in \mathbb{R}^{n-n_0}$, L is the solution of $A_3 + A_4 L = 0$ and $\mathbb{Q}E\mathbb{P} = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbb{Q}A\mathbb{P} = \begin{pmatrix} \Sigma A_1 & \Sigma A_2 \\ A_3 & A_4 \end{pmatrix}$, $QB = \begin{pmatrix} \Sigma B_1 \\ 0 \end{pmatrix}$ for some suitable nonsingular matrices $\mathbb{Q}$ and $\mathbb{P}$.

Then, the solution of the optimal control problem (22) is the solution of the robust control problem (21).

proof

From lemma 1 and the conditions above, we get that the robust control problem (21) is equivalent to the robust control problem

$$\dot{x}_1 = A_1 x_1 + A_2 x_2 + B_1 u + B_1 f(x_1, x_2) \tag{23a}$$

$$0 = A_3 x_1 + A_4 x_2 \tag{23b}$$

And this problem is reduced locally to the problem

$$\dot{x}_1 = (A_1 + A_2 L) x_1 + B_1 u + B_1 f(x_1, L x_1) \tag{24}$$

For all $x_1 \in \widetilde{\Omega}_x$, $x_2 = L x_1$, $\left(x_1(t_0), x_2(t_0)\right) \in w_k$

Now to prove the theorem, it is enough to prove that the solution of (22) is the solution of (24). To do so, one can define

$$V(x_1) \triangleq \int_{t_0}^{\infty} (f_{max}^2(x) + x_1^T Q_1 x_1 + u^T R u) dt$$

to be the minimum cost of bringing the system (22b) from $x_1(t_0) = x_{1,0}$ to $x_1 = 0$.

From lemma 2 , $V(x_1)$ must satisfies H-J-B equation

$$0 = \min_{u \in \Delta} \left[f_{max}^2(x) + x_1^T Q_1 x_1 + u^T R u + V_{x_1}^T((A_1 + A_2 L)x_1 + B_1 u) \right] \quad (25)$$

Now if $x_1 \in \widetilde{\Omega}_x$ and $u(0) = 0, \exists\, u \in \Delta$ such that (25) is satisfied then there exists a solution to the optimal control problem (25), $u = \mu(x_1)$ satisfies

$$f_{max}^2(x) + x_1^T Q_1 x_1 + \mu^T R \mu + V_{x_1}^T((A_1 + A_2 L)x_1 + B_1 \mu) = 0 \quad (26)$$

$$2\mu^T R + V_{x_1}^T B_1 = 0 \quad (27)$$

$$x_2 = L x_1 \quad (28)$$

Now, one can show that $u = \mu(x_1)$ of (22) is the solution of the robust control problem (24), *i.e.*, $x_1 = 0$ of (24) is globally asymptotically stable for all admissible uncertainty $f(x)$.

To do so, we show that $V(x_1)$ is a Lyapunov function of the system (24).

1. Since $u(0) = 0, f_{max}(0) = 0$, then $V(0) = 0$.
2. And since $f_{max}^2(x) > 0$, $x_1^T Q_1 x_1 > 0$, $u^T R u > 0$,
   $\forall\, x = (x_1, L x_1) \neq (0,0)$, $x_1 \in \widetilde{\Omega}_x$ then $V(x_1) > 0$.

$$\begin{aligned} \dot{V}(x_1) &= V_{x_1}^T \dot{x}_1 \\ &= V_{x_1}^T[(A_1 + A_2 L)x_1 + B_1 u + B_1 f] \\ &= V_{x_1}^T[(A_1 + A_2 L)x_1 + B_1 u] + V_{x_1}^T B_1 f \end{aligned}$$

Substitution (26) and (27), yields

$$\dot{V}(x_1) = -f_{max}^2(x) - x_1^T Q_1 x_1 - \mu^T R \mu - 2\mu^T R$$

By adding and subtracting the term $f^T R f$, we have the following:

$$\begin{aligned} \dot{V}(x_1) &= -(f_{max}^2(x) - f^T R f) - x_1^T Q_1 x_1 - \mu^T R \mu - 2\mu^T R - f^T R f \\ &= -(f_{max}^2(x) - \|f(x)\|_R^2) - x_1^T Q_1 x_1 - (f + \mu)^T R (f + \mu) \\ &= -(f_{max}^2(x) - \|f(x)\|_R^2) - \|x_1\|_{Q_1}^2 - \|f(x) + \mu(x)\|_R^2 \end{aligned}$$

From the condition $\|f(x)\|_R^2 \leq f_{max}^2(x)$ and $\|f(x) + \mu(x)\|_R^2 \geq 0$,

Therefore, $\dot{V}(x_1) \leq -\|x_1\|_{Q_1}^2 < 0$

Thus, the condition of asymptotically stable is satisfied.

Consequently, there exists a neighborhood $N_C = \{x_1 \in \widetilde{\Omega}_x ; \|x_1\|_{Q_1} < C\}$ for some $C > 0$. Such that if $x_1(t)$ enters N_C then $\lim_{t\to\infty} \|x_1(t)\|_{Q_1} = 0$

But $x_1(t)$ cannot remains forever outside N_C, otherwise $\|x_1(t)\|_{Q_1} > C$ for all $t > 0$, therefore

$$\begin{aligned} V(x_1(t)) - V(x_1(0)) &= \int_0^t \dot{V}(x_1(s))ds \\ &\leq -\int_0^t \|x_1(s)\|_{Q_1}^2 ds \\ &\leq -\int_0^t C^2 ds \\ &= -C^2 t \end{aligned}$$

$$V(x_1(t)) \leq V(x_1(0)) - C^2 t$$

Letting $\to \infty$, we have $V(x_1(t)) \to -\infty$ which contradicts the fact that $V(x_1(t)) > 0$ for all $x_1 \in \widetilde{\Omega}_x$. Therefore $\lim_{t\to\infty} \|x_1(t)\|_{Q_1} = 0$. But $x_2(t) = L x_1(t)$ such that $x(t) = (x_1(t), x_2(t)) \in \widetilde{\Omega}$.

Then

$$\begin{aligned} \lim_{t\to\infty} \|x_2(t)\| &= \left\| \lim_{t\to\infty} x_2(t) \right\| \\ &= \left\| \lim_{t\to\infty} L x_1(t) \right\|, \ x_1 \in \widetilde{\Omega}_x \\ &= \left\| L \lim_{t\to\infty} x_1(t) \right\|, \ x_1 \in \widetilde{\Omega}_x \\ &= 0 \end{aligned}$$

So $\lim_{t\to\infty} \|x(t)\| = \left\| \lim_{t\to\infty} (x_1(t), x_2(t)) \right\| = 0.$

For all $x(t) \in \widetilde{\Omega}$ ■

Based on the previous results, the following illustration have been developed.

5 Illustration

Consider the robust descriptor system

$$E\dot{x} = Ax + Bu + Bf(x)$$

Where $x^T = (x_1, x_2, x_3, x_4)$ and

$$E = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} -12 & 3 & -3 & 3 \\ -6 & 2 & -2 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

Let the uncertain $f(x) \in f_\alpha$ be assumed as:
$f(x) = q\, x_1 sin(x_2), \quad q \in [-1,1]$ unknown parameter.

Using the transformation matrices $\mathbb{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\mathbb{Q} = I_4$ and putting

$\mathbb{P}^{-1}x = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}$, the system above can be transformed to

$$\dot{Y}_1 = A_1Y_1 + A_2Y_2 + B_1(u + qy_2\, sin(y_1))$$
$$0 = A_3Y_1 + A_4Y_2$$

Where $A_1 = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Or

$$\dot{y}_1 = y_1 - 4y_2 - y_3 + y_4 + u + qy_2\, sin(y_1)$$
$$\dot{y}_2 = y_1 - 3y_2 - y_3 + u + qy_2\, sin(y_1)$$
$$0 = 2y_1 + y_2 + y_3 + y_4$$
$$0 = 2y_1 + y_2 + y_3 + y_4$$

Clearly $\|f(y)\| \le |q|\, |y_2|\, |sin(y_1)| \le |y_2|$

$$\|f(y)\|^2 \le y_2^2 = f_{max}^2(y).$$

Here, the consistency space is

$\widetilde{\Omega} = \{Y_1 \in \widetilde{\Omega}_y = \mathbb{R}^2, Y_2 \in \mathbb{R}^2 |\, A_3Y_1 + A_4Y_2 = 0\}$

Since A_4 is rank deficient and $Rank[A_3\ A_4] = Rank\, A_4 = 1$, then there exists a non singular matrix L such that $Y_2 = LY_1$ with

$$(A_3 + A_4L)Y_1 = 0, \text{ or } A_3 + A_4L = 0 \Rightarrow L = \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}$$

$\widetilde{\Omega} = \{Y_1 \in \widetilde{\Omega}_y = \mathbb{R}^2,\ Y_2 = LY_1\}$.

$\widetilde{\Omega} = \left\{(y_1, y_2)^T \in \widetilde{\Omega}_y = \mathbb{R}^2, (y_3, y_4)^T = \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\}$

$\widetilde{\Omega} = \{(y_1, y_2)^T \in \widetilde{\Omega}_y = \mathbb{R}^2, y_3 = y_1 + y_2, y_4 = -3y_1 - 2y_2\}$

Therefore, the initial condition $(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0})$ is consistent iff

$y_{3,0} = y_{1,0} + y_{2,0}$ and $y_{4,0} = -3y_{1,0} - 2y_{2,0}$ for a given $(y_{1,0}, y_{2,0})$,

$\therefore\ w_k = \{(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0}) |\, y_{3,0} = y_{1,0} + y_{2,0}, y_{4,0} = -3y_{1,0} - 2y_{2,0}\}$

Therefore the robust control problem of finding a feedback control law that stabilizes the system for all possible q can be transformed into the following optimal control problem:

For nominal system

$$\dot{Y}_1 = AY_1 + Bu$$

where $A = \begin{pmatrix} -3 & -7 \\ 0 & -4 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Since $Rank(B_1 : (A_1 + A_2L)B_1) = Rank(B : AB) = 2$, then the system is controllable.

Find a feedback control law $u = -kY_1$ that minimize

$$\int_0^\infty (f_{max}^2(y) + Y_1^TY_1 + u^Tu)dt = \int_0^\infty (y_1^2 + 2y_2^2 + u^2)dt$$
$$= \int_0^\infty \left(Y_1^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} Y_1 + u^Tu\right) dt$$
$$= \int_0^\infty (Y_1^TQY_1 + u^Tu)dt$$

This linear quadratic optimal control problem can be solved by solving the algebraic Riccati equation

$$A^TP + PA - PBR^{-1}B^TP + Q = 0$$

Suppose $P = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ then one can get that

$$\begin{pmatrix} -3 & 0 \\ -7 & -4 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} -3 & -7 \\ 0 & -4 \end{pmatrix} - \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Solving the system above gives

$a = 0.1667$, $b = -0.1667$, $c = 0.5256$ and $P = \begin{pmatrix} 0.1667 & -0.1667 \\ -0.1667 & 0.5256 \end{pmatrix}$. The eigenvalues of P are $\{0.1012, 0.5910\}$. Then P is positive definite symmetric matrix. And the feedback control is

$$u = -R^{-1}B^T PY_1 = -0.3589y_2$$

By theorem 2, this is a solution to the original robust control problem.
Solutions of the optimal control problem and the robust control of the equivalent system are shown in the following figures.

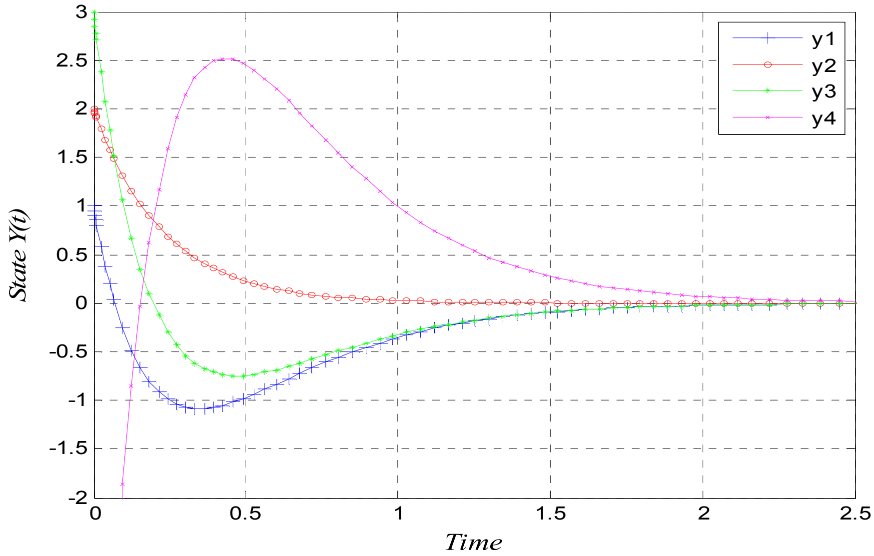

Figure (1) Optimal solution represents (y_1, y_2, y_3, y_4) *with*

$$(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0}) = (1,2,3,-7) \in w_k$$

It's very important to notice that in contracts to the state-space (o.d.e) systems, when the eigenvalues of the nominal system have negative real part then the system is stable for all initial condition. While in descriptor systems the initial condition divided into two parts, the first one concerns the dynamic (o.d.e) and the second part is the algebraic equation which is called the consistent initial conditions. And these initial conditions effect the system stability even when the spectrum of the dynamic system lie in the left half of $\mathbb{C}$ as one can see the figure (2), the dynamic state space and the non dynamic state space vector are far from the equilibrium point $(0,0)$ when choosing the initial condition out of the consistency region.

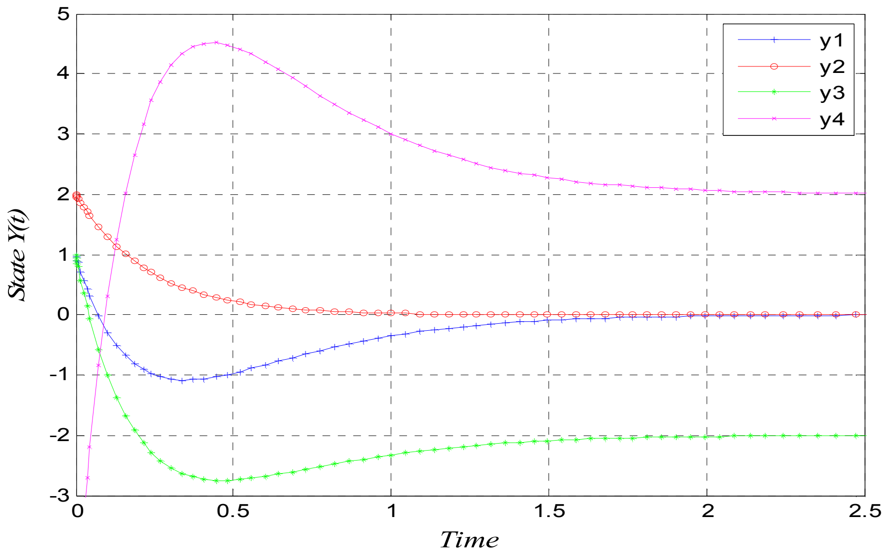

Figure (2) Optimal solution represents (y_1, y_2, y_3, y_4) *with*

$$(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0}) = (1,2,1,-5) \notin w_k$$

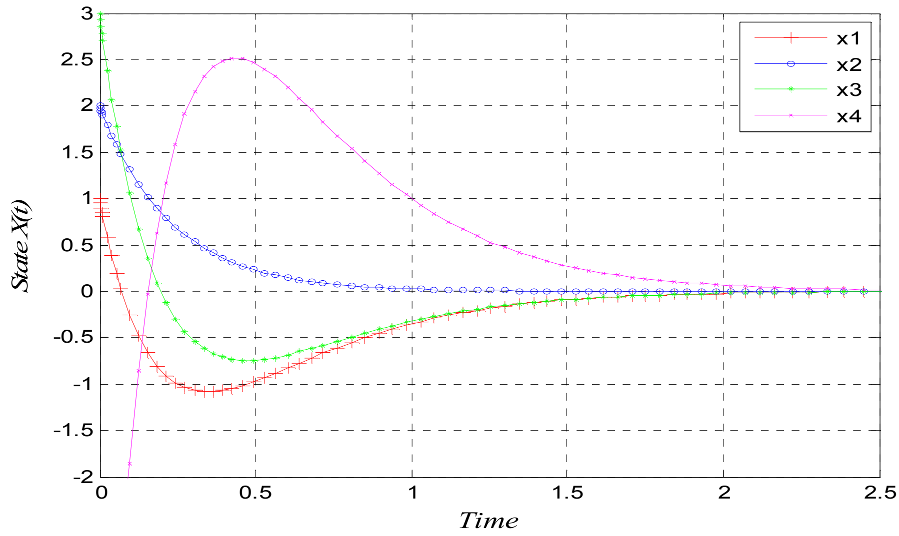

Figure (3) Optimal solution represents (x_1, x_2, x_3, x_4) *with*
$(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) = (2,1,3,-7)$

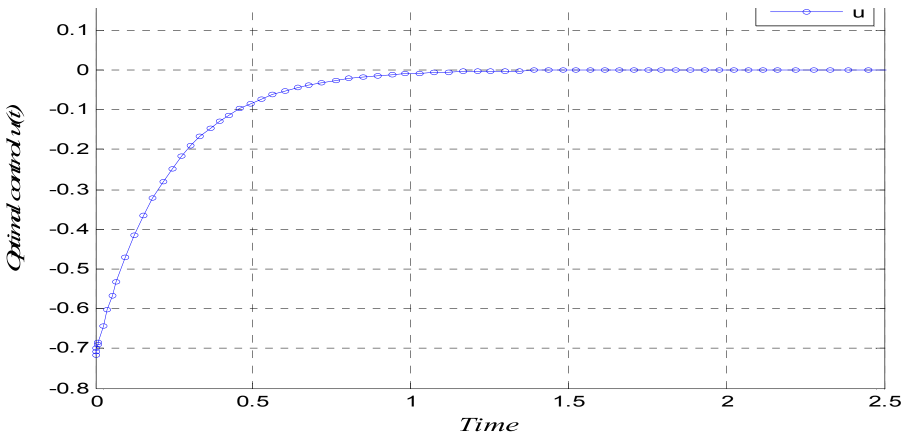

Figure (4) The optimal control $u(t)$

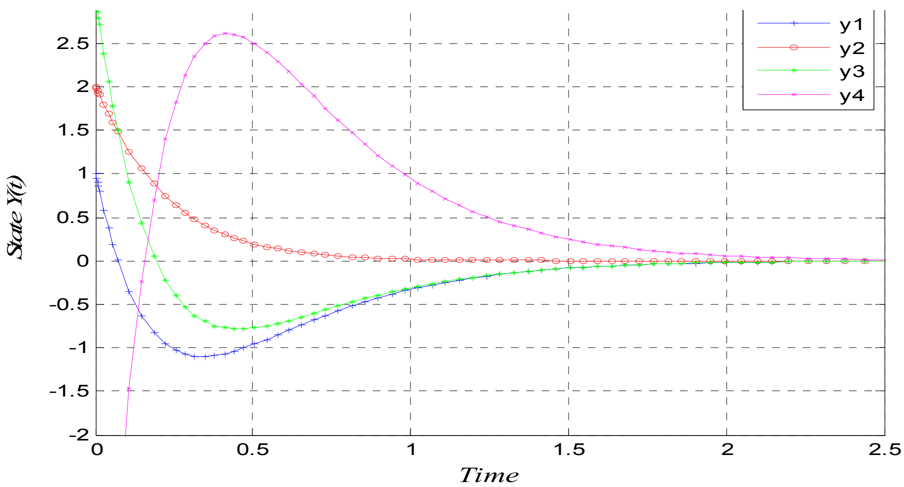

Figure (5) Robust solution represents (y_1, y_2, y_3, y_4) *with*
$(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0}) = (1,2,3,-7) \in w_k$, $q = 0.5$

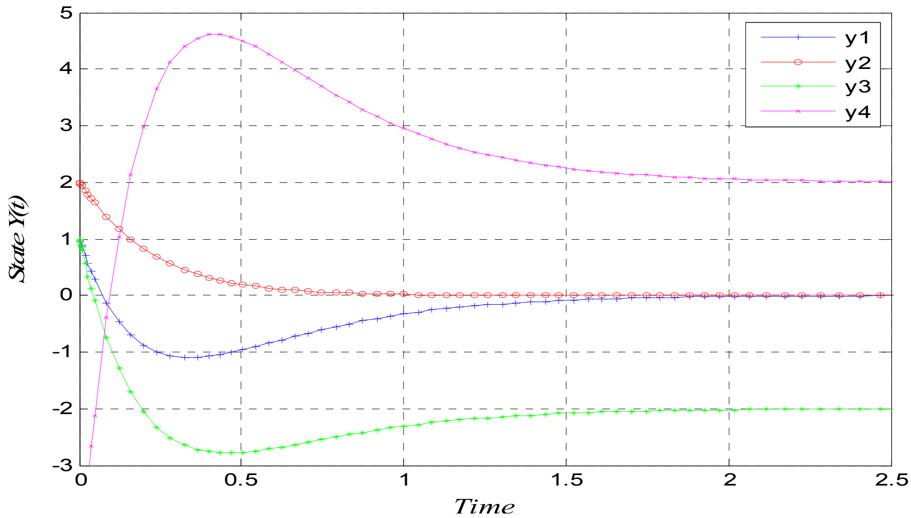

Figure (6) Robust solution represents (y_1, y_2, y_3, y_4) *with*

$(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0}) = (1,2,1,-5) \notin w_k$, $q = 0.5$

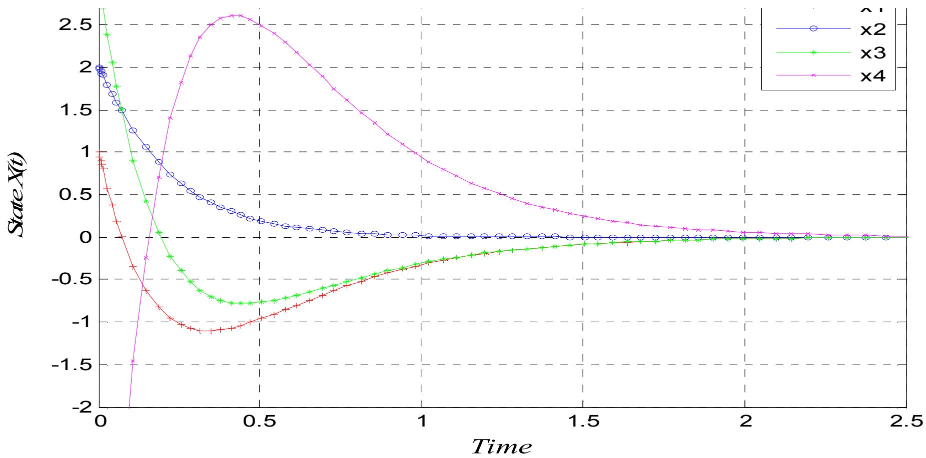

Figure (7) Robust solution represents (x_1, x_2, x_3, x_4) *with*

$(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) = (2,1,6,-5)$, $q = 0.5$

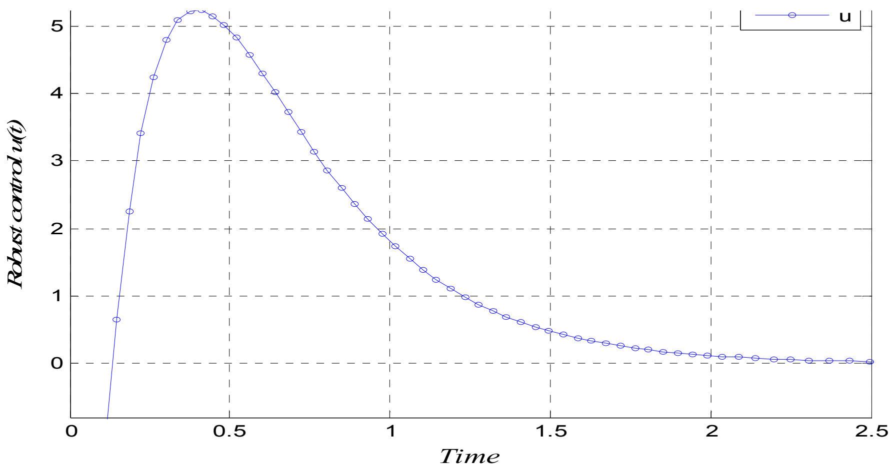

Figure (8) Robust control with

$(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0}) = (1,2,6,-5) \in w_k$, $q = 0.5$

4 Conclusions

From this work, we can conclude the following points:

1. The solvability and Stabilizability of the robust control problem of some non-linear semi-explicit descriptor uncertain systems having matching condition is discussed via an optimal control approach in the sense that, the solution of an equivalent optimal control problem to the uncertain nonlinear descriptor system, is the solution to the given descriptor one with matching condition.
2. This novel approach is very applicable for a large class of systems and make the original problems tractable and easy for point of applications.

5 future work

The following work have been considered for publication:

1. The solution of the robust control problem of some non-linear semi-explicit descriptor uncertain systems without matching condition and linear algebraic equation with rank deficient of the algebraic coefficient.
2. The solution of the robust control problem of some non-linear semi-explicit descriptor uncertain systems with matching condition and non-linear algebraic equation.

References

Brenan, K. E., Campbell, S., and Petzold, L. R. (1996), "Numerical Solution of Initial-Value Problems Differential-Algebraic Equations". *SIAM, New York.*

Campbell S.L. (1980), "Singular systems of differential equations", volume 1. *Pitman London.*

Dai, L. (1989), "S*ingular Control Systems", Lecture Notes in Control and Information Sciences.* Springer-Verlag, Berlin.

Danielle C. Tarraf and H. Harry A. (2002), "On the Nature and Stability of Differential-Algebraic Systems". *Proceedings of the American control conference*, Anchorage, A K May 8-10

Debeljkovic D. and Buzurovic I. (2011)., "Lyapunov stability of linear continuous singular systems: an overview", *International journal of information and systems sciences computing and Information,* Volume 7, Number 2-3, Pages 247–268.

FENG, LIN. , (2000), "An optimal control approach to robust control design". *INT.J. Control*, VOL.73, No.3, 177-186.

FENG, LIN. ROBERT, D. BRANDT, JING SUN, (1992), "Robust control of nonlinear systems: compensation for uncertainty*". INT.J. Control*, VOL.56, No.6, 1453-1459.

Kunkel, P. and Mehrmann, V. (2001), "Analysis of over- and underdetermined nonlinear differential-algebraic systems with application to nonlinear control problems", *Mathematics of Control, Signals, and Systems*, 14:233–256.

Kunkel, P. and Mehrmann, V. (2004), "Index reduction for differential-algebraic equations by minimal extension"*, Zeitschrift für Angewandte Mathematik und Mechanik*, VOL.84, No.9:579–597.

Kunkel, P. and Mehrmann, V. (1994), "Canonical forms for linear differential-algebraic equations with variable coefficients". *Journal of Computational and Applied Mathematics* 56 , 22.5-251.

Kunkel, P. and Mehrmann, V. (1995), "Regular solution of nonlinear differential-algebraic equations and their determination". *Deutsche Foreschungsgemein schaft, Research grant Me* 790/5-2 Differentiell-algebraische Gleichungen.

Michael M. (2011),"Stability criteria for nonlinear fully implicit differential-algebraic systems". *PhD thesis*, Universität zu Köln.

Radhi A. Zaboon, Hadi Jaber Mustafa, Ghazwan M. J. Al-Hashimi, , (2006), "Stability of optimal solution to robust control problem without matching condition", *Journal of college of education*, No. 1, pp. 125-162, Iraq.

Radhi A. Zaboon, Hadi Jaber Mustafa, Ghaswan M. J. AL-Hashimi, (2008), "Robust control of nonlinear dynamical control system with matching condition"*, Journal of college of education* No. 3,Vol. 28, pp.291-308, Iraq.

Sharvan S.S. (2012), "Stability results for constrained dynamical systems". *PhD thesis*, Hamilton Institute National University of Ireland Maynooth Ollscoil na hÉireann, Má Nuad.

SUN Liying and WANG Yuzhen, (2012), "Robust Stabilization of a Class of Nonlinear Descriptor Systems", *proceedings of the 31st Chinese control conference* July 25-27, Hefei, China.

Tadeusz K., (2012), "Positivity of descriptor linear systems with regular pencils". Archives Of Electrical Engineering, VOL. 61(1), pp. 101-113.

Xiaoming Su and Yali Zhi, (2013), "Sufficient Conditions of Asymptotic Stability of the Time-Varying Descriptor Systems", *Hindawi Publishing Corporation International Journal of Engineering Mathematics,* Volume 2013, Article ID 920851, 4 pages.