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Stabilization and Solution of Two Diminution Nonlinear Hyperbolic Partial Differential Equations Using the Discretized Backstepping Method

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Abstract

Stabilizability and solvability of the two – dimensional nonlinear hyperbolic partial differential equation has experienced a growing popularity and of major interest of robust control theory. Therefore, in this paper, the backstepping transformation approach based on discretization of the space variable will be used to study the Stabilizability and solvability of nonlinear two dimensional hyperbolic partial differential equations by transforming the partial differential equation with unknown boundary control in to system of nonlinear ordinary differential equations and then using Lyapunov function method to stabilize and evaluate the control function, while the solution is obtained using Adem-bashforth method.

Keywords: Backstepping method, hyperbolic partial differential equation, Stabilization of boundary control problems, Lyapunov function.

1. Introduction

The study of nonlinear system is different than linear system or linearization of the nonlinear systems linearization approach may be used to study the behavior of nonlinear system but there are two basic limitations of linearization; first, since linearization is an approximation in a neighborhood of an operating point, it can only predict the "local" behavior of the nonlinear system in the vicinity of that point. It cannot predict the “nonlocal” behavior far from the operating point and certainly not the "global" behavior throughout the state space. Second, the dynamics of a nonlinear system are much richer than the dynamics of a linear system, [9]. There are "essentially nonlinear phenomena" that can take place only in the presence of nonlinearity; hence, they cannot be described or predicted by linear models. The following are examples of essentially nonlinear phenomena [8].

The backstepping is a particular approach for the stabilizing dynamical systems and is particularly successful approach that may be used in the area of nonlinear control theory.

Backstepping is unlike any of the methods previously developed in literatures for controlling ODEs and PDEs. It differs from optimal control methods in that it sacrifices optimality (though it can achieve a form of “inverse optimality”) for the sake of avoiding the operator of Riccati equations, which are very hard to solve for infinite or high dimensional systems, such as PDEs. Backstepping is also different from pole placement methods, because even though its objective is the stabilization of the system, which is also the same objective of the pole placement methods. In addition, backstepping does not pursue precise assignment of even a finite subset of the PDE’s eigenvalues [6].

As it is known, the Lyapunov stability may by achieved individually the eigenvalues, while using the backstepping method to achieve Lyapunov stability by collectively shifting all the eigenvalues in a favorable direction in the complex plane. This task can be achieved in a rather elegant way, where the control gains are easy to compute symbolically, numerically, and in some cases even explicitly [5].

The idea of backstepping method that will be introduced for designing nonlinear controllers and non-quadratic Lyapunov functions is intended in advance to the nonlinear control for PDEs where the state of nonlinear control for ODEs was given in, the early of 1990’s [6].

Most of the earlier studies using the backstepping method consensual with other type of nonlinear hyperbolic equations, such as M. Krstic and A. Smyshlyaev (see [4]) studied the first order nonlinear wave equation and M. Krstic and A. Smyshlyaev in 2008 (see [5]) studied the linear wave equation of different orders.

In this paper a new discretized backstepping control approach will be introduce to find for finding the boundary controller function which stabilizes the nonlinear hyperbolic PDE by transformation into an equivalent stable closed loop. This approach has its basis on transforming the PDE into an equivalent system of ordinary differential equations (ODEs) and using the backstepping control method to solve the resulting system which make our system stable. This approach is more easy and powerful than other approaches.

2. Fundamentals of Backstepping Method

The boundary control of nonlinear hyperbolic PDEs is still an open problem as far as general classes of systems are concerned, hyperbolic partial differential equations on a finite interval rather than on the whole real line. Most applications of partial differential equations involve domains with boundaries, and it is important to specify data correctly at these locations.

When attempting to develop general methods for nonlinear PDEs, it is better to take an idea about finite dimensional nonlinear systems. Clearly, one should be sure that the methods arise is successful there. This basically eliminates (direct) optimal control methods, because of the requirement to solve Hamilton-Jacobi-Bellman PDEs, and leaves feedback (linearization, backstepping, Lyapunov) approaches.

The stabilization problems for nonlinear systems are today the most commonly solved problems using the methods of feedback linearization and backstepping. These methods apply diffeomorphic coordinate transformations that transforms the system equations in the form where the stabilization problem becomes easy (the control input has access to all the nonlinearities).

The main idea of backstepping method for nonlinear hyperbolic PDEs is to find the coordinate transformation is same that use in nonlinear parabolic PDEs as in [2]:

$$w = u - \alpha(u) \quad \ldots(1)$$

which transforms the unstable nonlinear hyperbolic PDEs:

$$u_{tt}(x,t) = u_{xx}(x,t) + f\big(u(x,t)\big) \quad \ldots(2)$$

with initial and boundary conditions:

$$u(x,0) = g_1(x),\ u_t(x,0) = g_2(x),\ x \in [0,1] \quad \ldots(3)$$

$$u(0,t) = 0,\ u(1,t) = U(t),\ t \geq 0 \quad \ldots(4)$$

into the exponentially stable target system:

$$w_{tt}(x,t) = w_{xx}(x,t) \quad \ldots(5)$$

with boundary conditions

$$w(0,t) = 0, w(1,t) = 0 \quad \ldots(6)$$

where $0 < x < 1$, $t \geq 0$ and f is a nonlinear functions of u and $U(t): C[0,1] \to C[0,1]$ is the nonlinear feedback control function.

3. Solution of Nonlinear Two Dimensions Hyperbolic Partial Differential Equations

The nonlinear hyperbolic PDEs (2)-(4), will be discretized into an equivalent system of nonlinear ODEs and upon using the coordinate transformation (1) to transform this system of ODEs in to an equivalent one related to the target system (5)-(6) which is exponentially stable.

This approach may be divided into four steps:

Step 1: Fix $n \in \mathbb{N}$ and $h = \frac{1}{n+1}$ as the step size of discretization over [0, 1]. Also let $u_i(t) = u(ih,t)$ for all $i = 0,1,\ldots,n+1$; where it is assumed that $u_0(t)$ is boundary condition at $x = 0$ and $u_{n+1}(t)$ is the control function at $x = 1$. Hence using the central difference discretization for $u_{xx}(x,t)$, we have:

$$u_0 = 0 \quad \ldots(7)$$

$$\frac{d^2u_i}{dt^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + f\big(u_i(t)\big) \quad \ldots(8)$$

$$u_{n+1} = U(t) \quad \ldots(9)$$

Similar discretization for the target system (5)-(6), will give:

$$w_0 = 0 \quad \ldots(10)$$

$$\frac{d^2w_i}{dt^2} = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} \quad \ldots(11)$$

$$w_{n+1} = 0 \quad \ldots(12)$$

Step 2: Reduce the second order systems of differential equations (8) and (11) into a system of the first order by letting for eq.(8):

$$\left.\begin{aligned} u_i &= u_{1,i} \\ \dot{u}_i &= \dot{u}_{1,i} = u_{2,i} \\ \dot{u}_{2,i} &= \frac{u_{1,i+1} - 2u_{1,i} + u_{1,i-1}}{h^2} + f\left(u_{1,i}(t)\right) \end{aligned}\right\} \quad \ldots(13)$$

and the boundary conditions (7) and (9) become

$$u_0 = u_{1,0} = 0,\ u_{n+1} = u_{1,n+1} = U(t) \quad \ldots(14)$$

Also, for eq.(11)let:

$$\left.\begin{aligned} w_i &= w_{1,i} \\ \dot{w}_i &= \dot{w}_{1,i} = w_{2,i} \\ \dot{w}_{2,i} &= \frac{w_{1,i+1}-2w_{1,i}+w_{1,i-1}}{h^2} \end{aligned}\right\} \quad \ldots(15)$$

and the boundary conditions (10) and (12) becomes:

$$w_0 = w_{1,0} = 0,\ w_{n+1} = w_{1,n+1} = 0 \quad \ldots(16)$$

Step 3: Using the discretized backstepping coordinate transformation:

$$\left.\begin{aligned} w_{1,i} &= u_{1,i} \\ w_{2,i} &= u_{2,i} - \alpha_i\left(u_{1,1}, u_{1,2}, u_{2,1} \ldots, u_{1,n}\right), i = 1,2,\ldots,n \end{aligned}\right\} \quad \ldots(17)$$

Then carrying out similar approach for the calculations followed in [2] to solve the obtained nonlinear system of ODEs.

At last from substitution equations (9) and (12) in equation (17) to get the controller boundary function $U(t)$, which is the nonlinear boundary condition that make equation (2) stable (see [3], [9]).

Step 4: Substitute $U(t)$ back into equation (8) for $i = n$, a system of nonlinear n first order ODEs, is obtained.

$$\dot{u} = F(u,t) \quad \ldots(18)$$

where the function F is the vector of $u_{j,i}$, $i = 1,2,\ldots,n, j = 1,2$.

The solution of the obtained system of ODEs may be achieved by linearization method or any other numerical method for solving systems of nonlinear ODEs (see [1]) and (see [7]). Numerical method is used to find the solution by using computer programs based on Adem-Bashforth method.

4. Illustrative Example

Consider the nonlinear wave equation:

$$u_{tt}(x,t) = u_{xx}(x,t) + \left(u(x,t)\right)^2,\ t \geq 0,\ 0 \leq x \leq 1$$

with boundary conditions:

$$u(0,t) = 0,\ u(1,t) = U(t)$$

and initial conditions:

$$u(0,x) = 1,\ u_t(0,x) = 0$$

Hence using the same steps presented above, we proceed for this example as follows:

Step 1: Using the finite difference discretization:

$$\frac{d^2u_i}{dt^2} = \frac{u_{i+1}-2u_i+u_{i-1}}{h^2} + f(u_i)$$

where $h = \frac{1}{N+1}$, where, $u_i = u(ih,t), i = 1,2,\ldots,N$; and u_0, u_{N+1} are the boundary conditions.

Step 2: Let us reduce the order of the resulting system of ODEs by letting:

$$u_{1,i} = u_i,$$

$$\dot{u}_{1,i} = \dot{u}_i = u_{2,i},$$

$$\dot{u}_{2,i} = \ddot{u}_i = \frac{u_{1,i+1}-2u_{1,i}+u_{1,i-1}}{h^2} + \left(u_{1,i}\right)^2$$

For simplicity, let $N = 3$ and therefore $h = \frac{1}{4}$, $u_{1,0} = 0$, $u_{1,4} = U(t)$, the first order of ODEs take the forms:

$$\dot{u}_{1,1} = u_{2,1},$$

$$\dot{u}_{2,1} = \frac{u_{1,2}-2u_{1,1}+u_{1,0}}{(0.25)^2} + \left(u_{1,1}\right)^2 = 16u_{1,2} - 32u_{1,1} + \left(u_{1,1}\right)^2$$

$$\dot{u}_{1,2} = u_{2,2},$$

$$\dot{u}_{2,2} = \frac{u_{1,3}-2u_{1,2}+u_{1,1}}{(0.25)^2} + \left(u_{1,2}\right)^2 = 16u_{1,3} - 32u_{1,2} + 16u_{1,1} + \left(u_{1,2}\right)^2$$

$$\dot{u}_{1,3} = u_{2,3},$$

$$\dot{u}_{2,3} = \frac{u_{1,4}-2u_{1,3}+u_{1,2}}{(0.25)^2} + \left(u_{1,3}\right)^2 = 16U(t) - 32u_{1,3} + 16u_{1,2} + \left(u_{1,3}\right)^2$$

Step 3: Applying the backstepping method:

For $j = 1$, $i = 1$.

Let $w_{1,1} = u_{1,1}$ and therefore $\dot{w}_{1,1} = \dot{u}_{1,1} = u_{2,1}$

Let $u_{2,1} = \alpha_1(u_{1,1})$, with error $w_{2,1} = u_{2,1} - \alpha_1(u_{1,1})$.

Since $u_{2,1} = w_{2,1} + \alpha_1$, hence $\dot{w}_{1,1} = w_{2,1} + \alpha_1$

Now, consider the control Lyapunov function $V_1 = \frac{1}{2}w_{1,1}^2$. Then:

$$\dot{V}_1 = \frac{dV_1}{dt} = \frac{dV_1}{dw_{1,1}}\frac{dw_{1,1}}{dt}$$
$$= w_{1,1}\dot{w}_{1,1} = w_{1,1}(w_{2,1} + \alpha_1)$$
$$= (\alpha_1 + k_1 w_{1,1})w_{1,1} - k_1 w_{1,1}^2 + w_{1,1}w_{2,1}$$

Now, select:

$$\alpha_1 = -k_1 w_{1,1}$$
$$\dot{\alpha}_1 = -k_1\dot{w}_{1,1} = -k_1 u_{2,1}$$

Then:

$$\dot{V}_1 = -k_1 w_{1,1}^2 + w_{1,1}w_{2,1}, \text{ where } k_1 > 0.$$

Clearly if $w_{2,1} = 0$, then $\dot{V}_1 = -k_1 w_{1,1}^2$ and $w_{1,1}$ is guaranteed to converge to zero asymptotically.

For $j = 2, i = 1$

From the equation (17) and the results when $i = j = 1$

$$w_{2,1} = u_{2,1} - \alpha_1(u_{1,1})$$
$$\dot{w}_{2,1} = \dot{u}_{2,1} - \dot{\alpha}_1(u_{1,1})$$
$$= 16u_{1,2} - 32u_{1,1} + (u_{1,1})^2 + k_{1,1}u_{2,1} \quad \ldots(19)$$

in which $u_{1,2}$ is considered as a virtual control input.

At $i = 2$ the first equation of system (17) is:

$$w_{1,2} = u_{1,2} \quad \ldots(20)$$

then equation (19) takes the form:

$$\dot{w}_{2,1} = 16w_{1,2} - 32u_{1,1} + (u_{1,1})^2 + k_1 u_{2,1}$$

the objective here is to ensure $w_{2,1} \to 0$, and one may consider the Lyapunov function:

$$V_2 = V_1 + \frac{1}{2}w_{2,1}^2$$

Therefore:

$$\dot{V}_2 = \dot{V}_1 + w_{2,1}\dot{w}_{2,1}$$
$$= -k_1 w_{1,1}^2 - k_2 w_{2,1}^2 + 16w_{2,1}w_{1,2} + w_{2,1}\left(w_{1,1} - 32u_{1,1} + (u_{1,1})^2 + k_1 u_{2,1} + k_2 w_{2,1}\right)$$

while $w_{2,1}$ cannot be removed, let:

$$32u_{1,1} - w_{1,1} - (u_{1,1})^2 - k_1 u_{2,1} - k_2 w_{2,1} = 0$$
$$32u_{1,1} - u_{1,1} - (u_{1,1})^2 - k_1 u_{2,1} - k_2 u_{2,1} - k_1 k_2 u_{1,1} = 0$$

hence:

$$\dot{V}_2 = -k_1 w_{1,1}^2 - k_2 w_{2,1}^2 + 16w_{2,1}w_{1,2}$$

if $w_{1,2} = 0$, then $\dot{V}_2 = -\sum_{i=1}^{2} k_i w_{i,1}^2$, and $w_{1,1}, w_{2,1}$ are converge to zero asymptotically.

For $j = 1, i = 2$

As in the cases of $i = j = 1$ and from equation (20) one can get:

$$w_{1,2} = u_{1,2}$$
$$\dot{w}_{1,2} = \dot{u}_{1,2} = u_{2,2}$$

Now, define a virtual control low α_2 (error) of $u_{2,2}$ by:

$$u_{2,2} = w_{2,2} + \alpha_2 \quad \ldots(21)$$

and the new Lyapunov function will reads as follows:

$$V_3 = V_3 + \frac{1}{2}w_{1,2}^2$$

with total derivative:

$$\dot{V}_3 = \dot{V}_2 + w_{1,2}\dot{w}_{1,2}$$
$$= -\sum_{i=1}^{2} k_i w_{i,1}^2 + 16w_{2,1}w_{1,2} + w_{1,2}(u_{2,2})$$
$$= -\sum_{i=1}^{2} k_i w_{i,1}^2 + 16w_{2,1}w_{1,2} + w_{1,2}(w_{2,2} + \alpha_2)$$
$$= -\sum_{i=1}^{2} k_i w_{i,1}^2 - k_3 w_{1,2}^2 + w_{1,2}w_{2,2} + w_{1,2}(16w_{2,1} + k_3 w_{1,2} + \alpha_2)$$

Since $w_{1,2} \neq 0$, then:

$$\alpha_2 = -16w_{2,1} - k_3 w_{1,2} = -16(u_{2,1} + k_1 u_{1,1}) - k_3 u_{1,2}$$

Hence:

$$\dot{V}_3 = -\sum_{i=1}^{2} k_i w_{i,1}^2 - k_3 w_{1,2}^2 + w_{1,2} w_{2,2}$$

if $w_{2,2} = 0$, then $\dot{V}_3 = -\sum_{i=1}^{2} k_i w_{i,1}^2 - k_3 w_{1,2}^2$, and $w_{1,1}, w_{2,1}, w_{1,2}$ are converge to zero asymptotically.

For $j = 2,\ i = 2$

As in the cases of $i = 1$, $j = 2$ and from equation (21) one can get:

$$w_{2,2} = u_{2,2} - \alpha_2(u_{1,1}, u_{2,1}, u_{1,2})$$

$$\dot{w}_{2,2} = \dot{u}_{2,2} - \dot{\alpha}_2 = \dot{u}_{2,2} - \frac{\partial \alpha_2}{du_{1,1}}\dot{u}_{1,1} - \frac{d\alpha_2}{du_{2,1}}\dot{u}_{2,1} - \frac{d\alpha_2}{du_{1,2}}\dot{u}_{1,2}$$

$$w_{1,3} = u_{1,3} \Rightarrow u_{1,3} = w_{1,3} \quad \ldots(22)$$

and the new Lyapunov function:

$$V_4 = V_3 + \frac{1}{2} w_{2,2}^2$$

with total derivative:

$$\dot{V}_4 = \dot{V}_3 + w_{2,2}\dot{w}_{2,2}$$

$$= -\sum_{i=1}^{2} k_i w_{i,1}^2 - k_3 w_{1,2}^2 + w_{1,2} w_{2,2} + w_{2,2}\left(\dot{u}_{2,2} - \frac{d\alpha_2}{du_{1,1}}\dot{u}_{1,1} - \frac{d\alpha_2}{du_{2,1}}\dot{u}_{2,1} - \frac{d\alpha_2}{du_{1,2}}\dot{u}_{1,2}\right)$$

$$= -\sum_{i=1}^{2} k_i w_{i,1}^2 - k_3 w_{1,2}^2 - k_4 w_{2,2}^2 + 16 w_{1,3} w_{2,2} + w_{2,2}\Big(w_{1,2} + k_4 w_{2,2} - 32u_{1,2} + 16u_{1,1} + (u_{1,2})^2 - \frac{d\alpha_2}{du_{1,1}} u_{2,1} - \frac{d\alpha_2}{du_{2,1}}\left(16u_{1,2} - 32u_{1,1} + (u_{1,1})^2\right) - \frac{d\alpha_2}{du_{1,2}} u_{2,2}\Big)$$

Since $w_{2,2} \neq 0$, then:

$$\Big(w_{1,2} + k_4 w_{2,2} - 32u_{1,2} + 16u_{1,1} + (u_{1,2})^2 - \frac{d\alpha_2}{du_{1,1}} u_{2,1} - \frac{d\alpha_2}{du_{2,1}}\left(16u_{1,2} - 32u_{1,1} + (u_{1,1})^2\right) - \frac{d\alpha_2}{du_{1,2}} u_{2,2}\Big) = 0$$

$$\Big(u_{1,2} + k_4 u_{2,2} + 16k_4 u_{2,1} + 16k_1 k_4 u_{1,1} + k_3 k_4 u_{1,2} - 32u_{1,2} + 16u_{1,1} + (u_{1,2})^2 + 16k_1 u_{2,1} + 16\left(16u_{1,2} - 32u_{1,1} + (u_{1,1})^2\right) + k_3 u_{2,2}\Big) = 0$$

Hence:

$$\dot{V}_4 = -k_1 w_{1,1}^2 - k_2 w_{2,1}^2 - k_3 w_{1,2}^2 - k_4 w_{2,2}^2 + w_{2,2} w_{1,3}$$

if $w_{1,3} = 0$, then $\dot{V}_4 = -k_1 w_{1,1}^2 - k_2 w_{1,2}^2 - k_3 w_{2,1}^2 - k_4 w_{2,2}^2$, and $w_{1,1}, w_{2,1}, w_{1,2}, w_{2,2}$ are converge to zero asymptotically.

For $j = 3, i = 1$

From equation (22) and repeat the same procedure followed above, one can get the following:

$$w_{1,3} = u_{1,3}$$

$$\dot{w}_{1,3} = \dot{u}_{1,3} = u_{2,3}$$

Now, define a virtual control low α_3 (error) of $u_{2,3}$ by:

$$w_{2,3} = u_{2,3} - \alpha_3 \Rightarrow u_{2,3} = w_{2,3} + \alpha_3 \quad \ldots(23)$$

and the new Lyapunov function:

$$V_5 = V_4 + \frac{1}{2} w_{1,3}^2$$

with total derivative:

$$\dot{V}_5 = \dot{V}_4 + w_{1,3}\dot{w}_{1,3}$$

$$= -k_1 w_{1,1}^2 - k_2 w_{2,1}^2 - k_3 w_{1,2}^2 - k_4 w_{2,2}^2 + w_{2,2} w_{1,3} + w_{1,3}\dot{u}_{1,3}$$

$$= -k_1 w_{1,1}^2 - k_2 w_{2,1}^2 - k_3 w_{1,2}^2 - k_4 w_{2,2}^2 + w_{2,2} w_{1,3} + w_{1,3}(w_{2,3} + \alpha_3)$$

$$= -k_1 w_{1,1}^2 - k_2 w_{2,1}^2 - k_3 w_{1,2}^2 - k_4 w_{2,2}^2 - k_5 w_{1,3}^2 + w_{1,3} w_{2,3} + w_{1,3}(\alpha_3 + k_5 w_{1,3} + w_{2,2})$$

Since $w_{1,3} \neq 0$, then:

$$\alpha_3 = -k_5 w_{1,3} - w_{2,2}$$
$$= -k_5 u_{1,3} - u_{2,2} + \alpha_2$$
$$= -16k_1 u_{1,1} - 16u_{2,1} - k_3 u_{1,2} - u_{2,2} - k_5 u_{1,3}$$

Hence:

$$\dot{V}_5 = -k_1 w_{1,1}^2 - k_2 w_{2,1}^2 - k_3 w_{1,2}^2 - k_4 w_{2,2}^2 - k_5 w_{1,3}^2 + w_{2,3} w_{1,3}$$

if $w_{2,3} = 0$, then $\dot{V}_5 = -k_1 w_{1,1}^2 - k_2 w_{2,1}^2 - k_3 w_{1,2}^2 - k_4 w_{2,2}^2 - k_5 w_{1,3}^2$, and $w_{1,1}, w_{2,1}, w_{1,2}$, $w_{2,2}$, $w_{1,3}$ are converge to zero asymptotically.

For $j = 3, i = 2$

From equation (23) and also repeating the same procedure, the following is obtained:

$$w_{2,3} = u_{2,3} - \alpha_3(u_{1,1}, u_{2,1}, u_{1,2}, u_{2,2}, u_{1,3})$$

$$\dot{w}_{2,3} = \dot{u}_{2,3} - \dot{\alpha}_3$$
$$= \dot{u}_{2,3} - \frac{d\alpha_3}{du_{1,1}}\dot{u}_{1,1} - \frac{d\alpha_3}{du_{2,1}}\dot{u}_{2,1} - \frac{d\alpha_3}{du_{1,2}}\dot{u}_{1,2} - \frac{d\alpha_3}{du_{2,2}}\dot{u}_{2,2} - \frac{d\alpha_3}{du_{1,3}}\dot{u}_{1,3}$$

and the new Lyapunov function:

$$V_6 = V_5 + \frac{1}{2} w_{2,3}^2$$

with total derivative:

$$\dot{V}_6 = \dot{V}_5 + w_{2,3}\dot{w}_{2,3}$$
$$= -k_1 w_{1,1}^2 - k_2 w_{2,1}^2 - k_3 w_{1,2}^2 - k_4 w_{2,2}^2 - k_5 w_{1,3}^2 + w_{2,3} w_{1,3} + w_{2,3}\dot{w}_{2,3}$$
$$= -k_1 w_{1,1}^2 - k_2 w_{2,1}^2 - k_3 w_{1,2}^2 - k_4 w_{2,2}^2 - k_5 w_{1,3}^2 + w_{2,3} w_{1,3} + w_{2,3}\left(\dot{u}_{2,3} - \frac{d\alpha_3}{du_{1,1}}\dot{u}_{1,1} - \frac{d\alpha_3}{du_{2,1}}\dot{u}_{2,1} - \frac{d\alpha_3}{du_{1,2}}\dot{u}_{1,2} - \frac{d\alpha_3}{du_{2,2}}\dot{u}_{2,2} - \frac{d\alpha_3}{du_{1,3}}\dot{u}_{1,3}\right)$$

Since $w_{23} \neq 0$, then:

$$16U(t) = 32u_{1,3} - 16u_{1,2} - u_{1,3} - k_6(u_{2,3} - \alpha_3) - (u_{1,3})^2 + \frac{d\alpha_3}{du_{1,1}} u_{2,1} + \frac{d\alpha_3}{du_{2,1}}\left(16u_{1,2} - 32u_{1,1} + (u_{1,1})^2\right) + \frac{d\alpha_3}{du_{1,2}} u_{2,2} + \frac{d\alpha_3}{du_{2,2}}\left(16u_{1,3} - 32u_{1,2} + 16u_{1,1} + (u_{1,2})^2\right) + \frac{d\alpha_3}{du_{1,3}} u_{2,3}$$

which make the system stable, i.e.,

$$\dot{V}_6 = -k_1 w_{1,1}^2 - k_2 w_{2,1}^2 - k_3 w_{1,2}^2 - k_4 w_{2,2}^2 - k_5 w_{1,3}^2 - k_6 w_{2,3}^2 \leq 0,$$

Finally, the controller function $U(t) = u(1,t)$ is given by:

$$16U(t) = 32u_{1,3} - 16u_{1,2} - (u_{1,3})^2 - u_{1,3} - 16k_1 k_6 u_{1,1} - k_6 u_{2,2} - k_6 u_{2,3} - 16k_6 u_{2,1} - k_3 k_6 u_{1,2} - k_5 k_6 u_{1,3} - 16k_1 u_{2,1} - 16\left(16u_{1,2} - 32u_{1,1} + (u_{1,1})^2\right) - \left(-32u_{1,2} + 16u_{1,1} + (u_{1,2})^2\right) - 16u_{1,3} - k_3 u_{2,2} - k_5 u_{2,3}$$

Step 4: Since $k_i > 0, i = 1,2,\dots,6$ and for computation and comparison purpose let $k_i = 1, i = 1,2,\dots,6$, then:

$$16U(t) = 32u_{1,3} - 16u_{1,2} - (u_{1,3})^2 - 18u_{1,3} + u_{1,2} - 2u_{2,3}$$

Therefore the resulting nonlinear system of ODEs is given by:

$$\left.\begin{aligned}
\dot{u}_{1,1} &= u_{2,1} \\
\dot{u}_{2,1} &= 16u_{1,2} - 32u_{1,1} + (u_{1,1})^2 \\
\dot{u}_{1,2} &= u_{2,2} \\
\dot{u}_{2,2} &= 16u_{1,3} - 32u_{1,2} + 16u_{1,1} + (u_{1,2})^2 \\
\dot{u}_{1,3} &= u_{2,3} \\
\dot{u}_{2,3} &= -18u_{1,3} + u_{1,2} - 2u_{2,3}
\end{aligned}\right\} \quad \dots(24)$$

Figure(1) illustrate the numerical solution of system (24) for different values of $t \in [0, T], T > 0$ with initial condition $u(x,0) = 1$ and $u_t(x,0) = 0$ which is equivalent to the solution of the original PDE.

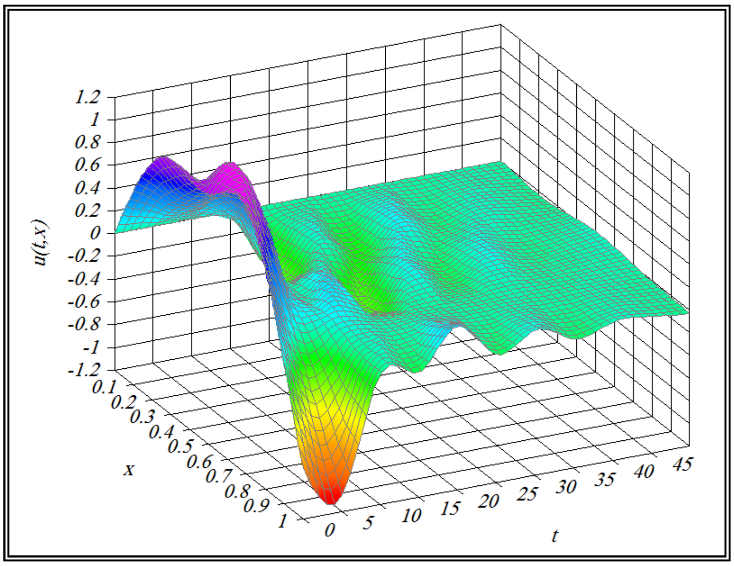

Figure 1. Closed-loop response with controller for the wave equation.

The table (1) illustrates the numerical solution of system (24) for different values of $t \in [0,50]$ with initial condition $u(x,0) = 1$.

Table 1. The Numerical Solution for The Nonlinear System (24).

t	u_1	u_2	u_3	$U(t)$
0	1	1	1	-0.75
5	0.616516528	1.083484689	0.023024301	-1.063882447
10	0.453535379	0.071778445	0.060008135	-0.058098866
15	-0.248279082	0.118846075	0.022172248	-0.102375455
20	0.045570968	-0.249267252	0.002366636	0.247086064
25	-0.152742363	0.076224993	-0.007278467	-0.076607085
30	0.090009357	-0.080098205	-0.003362685	0.078679400
35	-0.039333364	0.065387947	0.000565961	-0.067038145
40	0.028566397	-0.013832884	-0.000997936	0.016992397
45	0.008667344	-0.007289252	0.002987476	3.494526632e-3
50	-0.034904065	0.035787381	-0.002944730	-0.030574397

The obtained controlled function $U(t)$ is presented in Figure(2), Figure(3) illustrates the numerical solution of $u_1(t)$, $u_2(t)$ and $u_3(t)$ for different values of $t \in [0,600]$ with initial condition $u(x,0) = 1$, which are clear that they are asymptotically stable since that tends to zero as t increases.

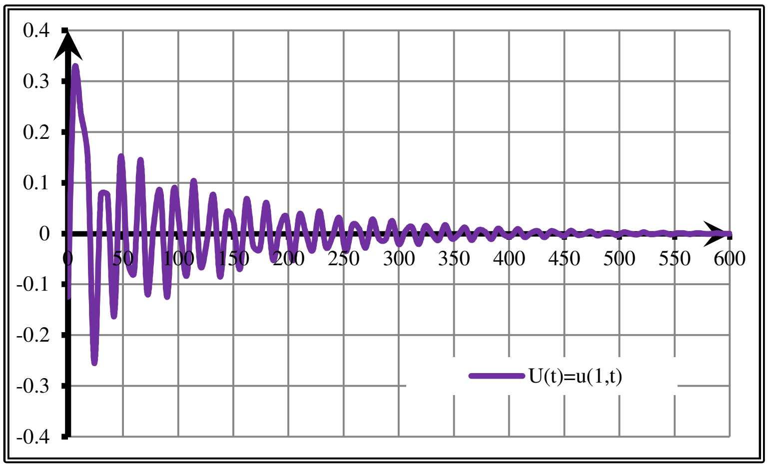

Figure 2. The control function $U(t) = u(1,t)$.

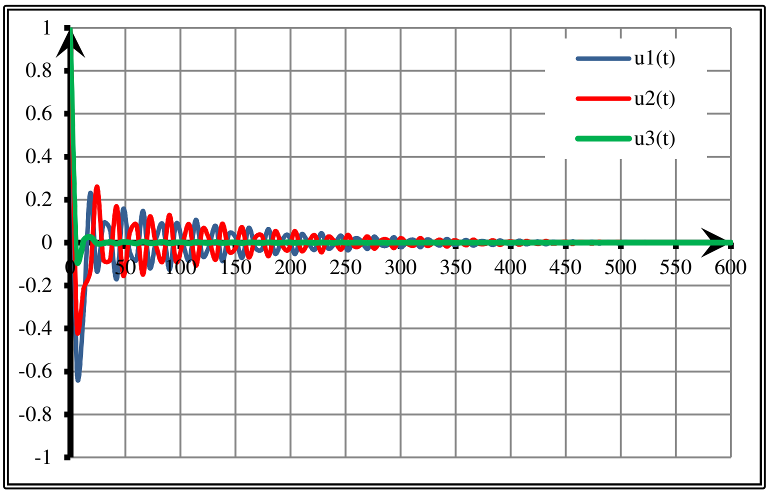

Figure 3. The numerical solution of $u_1(t)$, $u_2(t)$ and $u_3(t)$.

5. Conclusions

From the present study of this paper, the following conclusions may be drawn:

1. A nonlinear controller based on Lyapunov function method and backstepping design achieves global asymptotic stabilization of unstable nonlinear wave equation has been derived. The result holds for any finite discretization in space of the original PDE model.
2. The followed approach in this work indicates that a control law designed using only three steps of backstepping can be successfully used to stabilize the nonlinear wave equation.
3. The followed approach of derivation is easy to apply for stabilizing and solving PDEs which depends on mixing the straightforward approach in the theory of discretization of PDEs, theory of system of ODEs and theory of stability using Lyapunov functions.
4. The obtained results of the undertaken illustrative example are very accurate in comparison with results obtained by other researchers.

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