Control Theory and Informatics www.iiste.org
ISSN 2224-5774 (Paper) ISSN 2225-0492 (Online)
Vol.6, No.3, 2016

Solution of Non-Linear Uncertain Descriptor Systems without Matching Conditions via an Optimal Control Approach

Radhi Ali Zaboon
Department of Mathematics, College of Science, Al Mustansiriyah University, Iraq.

Sabeeh Lafta Jasim
Department of Mathematics, College of Science, Basrah University, Iraq.

Abstract

In this paper, the solution of some (robust) control problem of non-linear semi-explicit descriptor uncertain systems without matching condition by defining an optimal control approach is considered. This approach has been developed in the sense that, the solution of an equivalent optimal control problem is the solution to the given descriptor one without matching condition. A relation between the robust control problem and its equivalent optimal control problem has been discussed with theoretical justifications and illustration.

1 Introduction

A descriptor control system can be represented by differential and algebraic equations which is a generalized representation of the state-space system. The application of these systems can be found in electrical circuits, robots,… (Kunkel & Mehrmann 2001). These systems are also referred to as singular systems, implicit systems, generalized state-space systems, semi-state systems, or differential-algebraic systems (Debeljkovic & Buzurovic 2011).

The solvability of linear descriptor systems may be found in (Brenan et al. 1996), (Campbell 1980) and (Dai 1989), and while, nonlinear descriptor systems is discussed by (Kunkel & Mehrmann 1994, 1995, 2001 and 2004). Furthermore, Stability of linear and non-linear descriptor systems are studied by (Danielle et al 2002), (Debeljkovic & Buzurovic 2011), (Michael 2011), (Shravan 2012), (Tadeusz 2012), and (Xiaoming & Zhi 2013).

The descriptor control uncertain system have been interested and introduced to preserve various system properties under some perturbation in the model.

The insensitiveness of the system properties is called *robustness* and it is an important field of investigation. The fact is that in many practical situations the parameters of system components are not known exactly. Usually, there is only some information on the intervals to which they belong. Therefore, the robustness for any system property is an important theoretical and practical question.

Recently, much attention has been given to the design of controllers, so that system properties are preserved under various classes of uncertainties appearing in the system. Such controllers are called *robust controllers*, and the resulting system is said to be *robust control system*.

If the uncertainties lying in the range of the input matrix (operator), they are called matched condition uncertainties. For state- space system and some class of control problem, matched condition have been discussed by (FING LIN 2000). If this condition is not satisfied, a decomposition approach may be used (FENG *et al* 1992) and (Radhi et al 2008).

Due to the difficulty in solving general robust descriptor systems see (SUN & WANG 2012), in this paper, robust control problem is translated into a specific (equivalent) optimal control problem. The solution of optimal control problem is then a solution to the robust control problem based on the nominal system structure and the types of uncertainties.

Descriptor systems, like other systems may contain many types of uncertainties. These uncertainties can be classified as with or without matching condition. In this paper, robust control without matching condition has been considered.

A novel method for design and analysis of some class of non-linear uncertain descriptor system with matching condition by using an optimal control approach have been developed and presented in (Radhi & Sabeeh 2016).

The following is needed later on.

2 Pseudo-inverse operator

Recalling that, not every matrix has inverse. If A is singular or not square, then there exists an inverse called *pseudo –inverse* for A. For more such inverses see (Dai 1989), (Nikuie at el 2010).

Pseudo–inverse of a matrix is one of those inverses who have called *generalized inverses*. All these inverses will reduce to the well-known inverse when the matrix is nonsingular.

Definition (Ayman 2012)

If $\mathrm{A} \in \mathbb{R}^{n\times m}$, then there exists a unique $\mathrm{A}^{\dagger} \in \mathbb{R}^{n\times m}$ that satisfies the three Penrose conditions:

1. $\mathrm{AA^{\dagger}A = A}$
2. $\mathrm{A^{\dagger}AA^{\dagger} = A^{\dagger}}$
3. $\mathrm{A^{\dagger}A = (A^{\dagger}A)^T}$

$\mathrm{A}^{\dagger}$ is called *pseudo –inverse* for A.

The singular values decomposition may be used to find pseudo –inverse. Recording that for any $n \times m$ matrix A has a rank n_0, then the singular values decomposition of A is $A = UDV^T$, where U and V are $n \times n$ and $m \times m$ orthogonal matrices and $D = \begin{pmatrix} D_{n_0} & 0 \\ 0 & 0 \end{pmatrix}$ is $n \times m$ matrix, here D_{n_0} is an $n_0 \times n_0$ diagonal matrix with

$D_{n_0} = diag(\sigma_1, \sigma_2, \dots, \sigma_{n_0})$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n_0} \geq 0$ and n_0 is the rank of A. The numbers σ_i is singular value of A, $\forall\, i = 1,2,\dots, n_0$; thus $A^{\dagger} = VD^{\dagger}U^T$, where $D^{\dagger} = D_{n_0}^{-1}$ and 0 otherewise. That is

$D_{n_0}^{-1} = diag(\sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_{n_0}^{-1})$.

3 Problem formulation

Consider the non-linear semi-explicit descriptor system without matching condition

$$E\dot{x} = Ax + Bu + Cf(x) \tag{1}$$

Where $E, A \in \mathbb{R}^{n\times n}$, $rank(E) = n_0$, $0 < n_0 \le n$ $x \in \mathbb{R}^n$, $u \in R^r$ and $B \in \mathbb{R}^{n\times r}$ are the system coefficients and $f \in C^1(\mathbb{R}^n; \mathbb{R}^r)$ represent the uncertainty of the system, satisfying some conditions that will be defined later on to ensure the solvability of the system (1). System (1) is equivalent to

$$\dot{x}_1 = A_1x_1 + A_2x_2 + B_1u + C_1f(x_1,x_2) \tag{2a}$$

$$0 = A_3x_1 + A_4x_2 + B_2u + C_2f(x_1,x_2) \tag{2b}$$

Where $x_1 \in R^{n_0}$, $x_2 \in R^{n-n_0}$ and $A_1 \in R^{n_0\times n_0}, A_2 \in R^{n_0\times(n-n_0)}$,
$A_3 \in R^{(n-n_0)\times n_0}$, $A_4 \in R^{(n-n_0)\times(n-n_0)}$ and $B_1 \in R^{n_0\times r}$, $B_2 \in R^{(n-n_0)\times r}$, $C_1 \in R^{n_0\times r}$, $C_2 \in R^{(n-n_0)\times r}$, which comes by the following

Since $rank(E) = n_0$, then it follows that there always exist unitary matrices $U \in R^{n\times n}$ and $V \in R^{n\times n}$ such that

$$E = U\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}V^T \tag{3}$$

Where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_{n_0})$ and $\sigma_1 \ge \sigma_2 \ge \dots, \ge \sigma_{n_0} > 0$.

From (3), one can define

$$\mathbb{P} \triangleq V, \qquad \mathbb{Q} \triangleq U^{-1}$$

$$\mathbb{Q}E\mathbb{P} = U^{-1}\left[U\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^T\right]V = \begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix} \tag{4}$$

On using (1) and (4) as well as $rank(E) = n_0$, one gets

$\mathbb{Q}E\dot{x} = \mathbb{Q}Ax + \mathbb{Q}Bu + \mathbb{Q}Bf(x) \Rightarrow \mathbb{Q}E\mathbb{P}\mathbb{P}^{-1}\dot{x} = \mathbb{Q}A\mathbb{P}\mathbb{P}^{-1}x + \mathbb{Q}Bu + \mathbb{Q}Cf(x)$

From

$$\mathbb{P}^{-1}x = \begin{pmatrix}x_1\\ x_2\end{pmatrix} \Leftrightarrow x = \mathbb{P}\begin{pmatrix}x_1\\ x_2\end{pmatrix},\ x_1 \in \mathbb{R}^{n_0},\ x_2 \in \mathbb{R}^{n-n_0} \tag{5}$$

then

$$\mathbb{Q}E\mathbb{P}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \mathbb{Q}A\mathbb{P}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \mathbb{Q}Bu + \mathbb{Q}Cf\left(\mathbb{P}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\right) \tag{6}$$

where $\mathbb{Q}A\mathbb{P} = \begin{pmatrix}\bar{A}_1 & \bar{A}_2\\ A_3 & A_4\end{pmatrix}$, $\mathbb{Q}B=\begin{pmatrix}\bar{B}_1\\ B_2\end{pmatrix}$ and , $\mathbb{Q}C=\begin{pmatrix}\bar{C}_1\\ C_2\end{pmatrix}$ which gives

$$\dot{x}_1 = A_1x_1 + A_2x_2 + B_1u + C_1f(x_1,x_2)$$

$$0 = A_3x_1 + A_4x_2 + B_2u + C_2f(x_1,x_2)$$

Where $A_1 = \Sigma^{-1}\bar{A}_1$, $A_2 = \Sigma^{-1}\bar{A}_2$, $B_1 = \Sigma^{-1}\bar{B}_1$ and $C_1 = \Sigma^{-1}\bar{C}_1$.

In this paper, for simplicity, it is assumed that the matrix $\mathbb{Q}$ satisfies $\mathbb{Q}B = \begin{pmatrix}\bar{B}_1\\ 0\end{pmatrix}$, and $\mathbb{Q}C = \begin{pmatrix}\bar{C}_1\\ 0\end{pmatrix}$ then the system (2) is equivalent to

$$\dot{x}_1 = A_1x_1 + A_2x_2 + B_1u + C_1f(x_1,x_2) \tag{7a}$$

$$0 = A_3x_1 + A_4x_2 \tag{7b}$$

To study the solvability of the differential algebraic equations (7) which is equivalent to the descriptor system (1) by the invertible transformation $x = \mathbb{P}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$, the following assumption have been presented (FING LIN 2000).

Assumption A

3. Assume there exist an open set $\widetilde{\Omega}_x \subset D$ such that for all $\hat{x}_1 \in \widetilde{\Omega}_x$ it is possible to solve $A_3\hat{x}_1(t) + A_4\hat{x}_2(\mathrm{t}) = 0$ for $\hat{x}_2$. One can define the corresponding solution manifold as:

$\widetilde{\Omega} = \left\{ x_1 \in \widetilde{\Omega}_x, x_2 \in \mathbb{R}^{n-n_0} | \begin{pmatrix} x_1(t), \\ x_2(t) \end{pmatrix} \in \aleph([\mathrm{A}_3 \quad \mathrm{A}_4]), t \geq 0 \right\}$ where $\aleph(\cdot)$ denotes the kernel (null space) of the operator $(\cdot)$.

Let us denote the set of the consistent initial values of (7) by w_k,

$$w_k \triangleq \{x_0 = (x_{1,0}, x_{2,0}) | \, x_0 \in \aleph([\mathrm{A}_3 \quad \mathrm{A}_4])\} \tag{8}$$

4. $Rank[\mathrm{A}_3 \quad \mathrm{A}_4] = Rank\mathrm{A}_4 \Leftrightarrow w_k = \aleph[\mathrm{A}_3 \quad \mathrm{A}_4]$ (9)

Lemma 1

Consider the system

$$\dot{x}_1 = A_1 x_1 + A_2 x_2 + B_1 u + C_1 f(x_1, x_2) \tag{10a}$$

$$0 = A_3 x_1 + A_4 x_2 \tag{10b}$$

$$(x_1(0), x_2(0)) = (x_{1,0}, x_{2,0}) \in w_k \tag{10c}$$

Where $x_1 \in R^{n_0}$, $x_2 \in \mathbb{R}^{n-n_0}$ and $A_1 \in \mathbb{R}^{n_0 \times n_0}, A_2 \in \mathbb{R}^{n_0 \times (n-n_0)}$,
$A_3 \in \mathbb{R}^{(n-n_0) \times n_0}$, $A_4 \in \mathbb{R}^{(n-n_0)\times(n-n_0)}$ and $B_1, C_1 \in \mathbb{R}^{n_0 \times r}$, $f \in C^1(\mathbb{R}^n; \mathbb{R}^r)$,
$f(0,0) = 0$, such that Assumption A is satisfied.

If A_4 is of rank deficient matrix, i.e., $\mathrm{A}_4 < n - n_0$, then there exists a matrix L of dimension $(n - n_0) \times n_0$ such that the system (10) will be in the reduced form, for $x_1 \in \widetilde{\Omega}_x$ which is open subset of R^{n_0},

$$\dot{x}_1 = (A_1 + A_2 L) x_1 + B_1 u + C_1 f(x_1, \phi(x_1)) \tag{11}$$

Where $\phi(x_1) \triangleq Lx_1$.

Which is solvable for a given $(x_1(0), x_2(0)) \in w_k$ and $u \in \mathfrak{U}[0, t]$, where

$$w_k = \{(x_1(0), x_2(0)) | x_1(0) \in \widetilde{\Omega}_x, x_2(0) = \phi(x_1(0))\}$$

and

$$\mathfrak{U}[0, t] = \{u(\cdot) | u(t) \; is \; differntiable \; on \; [0, t]\}$$

And the solution to system (10) is $(x_1(t), x_2(t))$.

Proof

If A_4 is of rank deficient matrix in (10), then we may consider the existence of a matrix L of dimension $(n - n_0) \times n_0$ such that

$$x_2(t) = Lx_1(t) \tag{12}$$

$$Lx_1(t) - x_2(t) = 0 \Rightarrow (L - I_{n-n_0}) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0$$

So $(x_1(t), x_2(t)) \in \aleph(L \quad I_{n-n_0})$ and L satisfies $A_3 x_1 + A_4 L x_1 = 0$ or $(A_3 + A_4 L) x_1 = 0$ for all $x_1 \neq 0$, $x_1 \in \widetilde{\Omega}_x$. This means that

$$A_3 + A_4 L = 0 \tag{13}$$

Such a matrix L is always exists when condition (8) is satisfied (Kunkel & Mehrmann 2001). So the solution of (10b) have to belong to the set $\aleph(L \quad I_{n-n_0})$, so the solution manifold is

$$\widetilde{\Omega} \triangleq \{x \in \mathbb{R}^n; x(t) \in \aleph(L \quad I_{n-n_0})\} \tag{14}$$

Therefore, the solution of (10b) will be found locally in $\widetilde{\Omega}$ and the system is then given in the reduced form for $x_1 \in \widetilde{\Omega}_x$

$$\dot{x}_1 = (A_1 + A_2 L) x_1 + B_1 u + B_1 f(x_1, Lx_1) \tag{15}$$

From state space analysis and $f \in C^1(\mathbb{R}^n;\mathbb{R}^r)$, the problem has a solution for a given $u \in \mathfrak{U}[0,t]$ and $x_1 \in \widetilde{\Omega}_x$, hence the original system solution is $x_1(t) \in \widetilde{\Omega}_x$ and

$x_2(t) = \phi(x_1(t))$ with $x_2(0) = \phi(x_1(0))$. ■

4 Robust descriptor control problem

Consider the nonlinear semi-explicit descriptor robust control system without matching condition defined by (1) which is (by using lemma 1) equivalent to the system

$$\dot{x}_1 = (A_1 + A_2L)x_1 + B_1u + C_1f(x_1, Lx_1),$$
$$x_2 = \phi(x_1) = Lx_1, \big(x_1(0), x_2(0)\big) \in w_k$$

The equilibrium states of the robust control system (15) can be calculated when the control function u is identically 0 or is a constant vector u_0. Since $f(0,0) = 0$, then the equilibrium state of the system is the origin $(x_1, x_2) = (0,0)$.

Suppose that the feedback control is defined by

$$u(t) = -kx_1(t) \tag{16}$$

Now, the aim of the following work under a suitable assumptions is to find a suitable matrix k such that the closed loop nonlinear dynamical system

$$\dot{x}_1 = (A_1 + A_2L - B_1k)x_1 + C_1f(x_1, Lx_1) \tag{17a}$$
$$x_2 = \phi(x_1) = Lx_1, \big(x_1(0), x_2(0)\big) \in w_k, x_1 \in \widetilde{\Omega}_x \tag{17b}$$

is asymptotically stable.

To find the conditions which make the nonlinear descriptor robust control system without matching condition (1) is asymptotically stable, the following theorem has been developed.

Theorem 1

Consider the nonlinear descriptor robust control system without matching condition (1) which is locally equivalent to the system (15), that satisfy

8. The system satisfies Assumption A.
9. The eigenvalues of $A_1 + A_2L$ satisfies $\delta(A_1 + A_2L) = \{\lambda_i | \lambda_i + \lambda_j \neq 0\,,\ \forall\, i \neq j\}$
10. $(A_1 + A_2L\,, B_1)$ is state space controllable, where $x_2 = Lx_1$.
11. $f(0,0) = 0$.
12. $\|f(x)\| \le f_{max}(x) = \eta\|x\|_Q\,,\ x = (x_1, x_2)$.

    Where $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$ and $Q_1 and\ Q_2$ are symmetric positive semi-definite matrices.
13. The control is defined by $u(t) = -kx_1 = -R^{-1}B_1^TPx_1$,

    Where R is symmetric positive definite matrix and P is the symmetric positive definite matrix that solves the following Riccati equation

$$(A_1 + A_2L)^TP + P(A_1 + A_2L) - 2PB_1R^{-1}B_1^TP + 2\alpha P + 2Q_1 = 0 \tag{18}$$

$\alpha = \frac{\lambda_{max}(P)}{\lambda_{min}(P)}\lambda_{max}(Q_1)(1 + \|L\|)\|C_1\|$, where $\|.\|$ is suitable norm.

Then the equilibrium point $(x_1, x_2) = (0,0)$ of (15) is asymptotically stable.

proof

For $x_1 \in \tilde{\Omega}_x$, and $(x_1, x_2) \in \tilde{\Omega} = \{x_1 \in \tilde{\Omega}_x,\ x_2 \in R^{n-n_0} |\ x_2 = Lx_1\}$, define Lyapunov function for system (15) as $V(x_1(t)) = x_1^T P x_1$, $P^T = P > 0$

since $\lambda_{min}(P)\|x_1\|^2 \le x_1^T P x_1 \le \lambda_{max}(P)\|x_1\|^2$ for $P^T = P > 0$ (19)

Where $\lambda_{min}(P)$ and $\lambda_{max}(P)$ are the minimum and maximum eigenvalues of P respectively, from (15) and condition (7) as well as $u(t) = -R^{-1}B_1^T P x_1$ with some computations, one gets

$$\dot{V}(x_1) = x_1^T[(A_1 + A_2L)^T P + P(A_1 + A_2L) - 2PB_1R^{-1}B_1^T P]x_1 + x_1^T PC_1 f + f^T C_1^T P x_1$$

By solving (18) for P , we have that

$$\dot{V}(x_1) = -2x_1^T Q_1 x_1 - 2\alpha x_1^T P x_1 + x_1^T P C_1 f + f^T C_1^T P x_1 ,$$

Deletion of the negative term $-2x_1^T Q_1 x_1$ gives that

$$\dot{V}(x_1) \le -2\alpha x_1^T P x_1 + x_1^T P C_1 f + f^T C_1^T P x_1$$

Since

$$x_1^T P B_1 f + f^T B_1^T P x_1 \le 2\eta\lambda_{max}(P)\|C_1\|\|x_1\|\|x_1\|_{Q_1}(1 + \|L\|)$$

$$\le 2\eta\lambda_{max}(P)\lambda_{max}(Q_1)\|C_1\|\|x_1\|^2(1 + \|L\|)$$

And

$$x_1^T P x_1 \ge \lambda_{min}(P)\|x_1\|^2 \Rightarrow -x_1^T P x_1 \le -\lambda_{min}(P)\|x_1\|^2$$

Therefore, from the two inequalities above, we have that

$$\dot{V}(x_1) \le -2\alpha\, \lambda_{min}(P)\|x_1\|^2 + 2\eta\lambda_{max}(P)\lambda_{max}(Q_1)\|C_1\|\|x_1\|^2(1 + \|L\|)$$

From assumption 6, we get that

$$\dot{V}(x_1) \le 2(\eta - 1)\lambda_{max}(P)\lambda_{max}(Q_1)\|C_1\|(1 + \|L\|)\|x_1\|^2$$

Putting the condition $\eta - 1 \le 0$ on η *i.e.,*

$$0 < \eta \le 1$$

Gives that

$$\dot{V}(x_1) \le -2\lambda_{max}(P)\lambda_{max}(Q_1)\|C_1\|(1 + \|L\|)\|x_1\|^2 \quad \Rightarrow \quad \dot{V}(x_1) < 0$$

This proves that $x_1 = 0$ is asymptotically stable. *i.e.,*

$$\lim_{t\to\infty}\|x_1(t)\| = 0, \qquad x_1 \in \tilde{\Omega}_x$$

And from the continuity of the norm $\|\cdot\|$, then we have that

$$\lim_{t\to\infty}\|x_2(t)\| = \left\|\lim_{t\to\infty} x_2(t)\right\|$$
$$= \|\lim_{t\to\infty} Lx_1(t)\|,\ x_1 \in \tilde{\Omega}_x$$
$$= \|L\lim_{t\to\infty} x_1(t)\|,\ x_1 \in \tilde{\Omega}_x$$
$$= 0$$

Therefore the equilibrium point $(x_1, x_2) = (0,0)$ is asymptotically stable. ■

Theorem1 above gives us a class for the uncertainties $f(x)$ which can be defined as

$$f_\alpha = \left\{ f \middle| \|f(x)\| \le \eta\|x\|_Q, 0 < \eta \le 1, \alpha = \frac{\lambda_{max}(P)}{\lambda_{min}(P)}\lambda_{max}(Q_1)(1 + \|L\|)\|C_1\| \right\}$$

i.e., the nonlinear descriptor robust control system with matching condition (1) is stable for all $f \in f_\alpha$ and its solution is defined by $u(t) = -R^{-1}B_1^T P x_1$.

Due to the difficulty in solving the equivalent robust control descriptor problem (15) in the presence of system uncertainties, leads to develop a novel approach by finding an equivalent control problem (in reduced system form) which is equivalent to the robust one in the sense that the solution of the equivalent optimal control problem is the solution to the robust one. The following theorems present this fact.

In robust control problem (17), one can decompose the uncertainty $C_1 f(x)$ into the sum of a matched component and unmatched component by projecting $C_1 f(x)$ onto the range of B_1, thus

$$C_1 f(x) = B_1 B_1^\dagger C_1 f(x) + \left(I - B_1 B_1^\dagger\right) C_1 f(x)$$

So the system (17) will be

$$\dot{x}_1 = (A_1 + A_2 L)x_1 + B_1 u + B_1 B_1^\dagger C_1 f(x) + \left(I - B_1 B_1^\dagger\right) C_1 f(x) \quad (20)$$

Where $B_1^\dagger$ is the pseudo inverse of B_1.

Consider a condition on (17) as:

$$\left\| B_1^\dagger C_1 f(x) \right\| \le f_{max}(x) \quad (21)$$

Where $f_{max}(x)$ is a nonnegative function.

Now, the robust control problem (20) with the condition (21) will be translated into the following problem:

5 Optimal control equivalent problem

The robust control problem (17) can be put in the equivalent quadratic optimal control problem:

For all $x_1 \in \widetilde{\Omega}_x$, the nominal system will be

$$\dot{x}_1 = (A_1 + A_2 L)x_1 + B_1 u + \left(I - B_1 B_1^\dagger\right) C_1 v \quad (22a)$$

$$x_2 = L x_1, \ \left(x_{1,0}, x_{2,0}\right) \in w_k$$

Where $x_1 \in R^{n_0}$, $x_2 \in \mathbb{R}^{n-n_0}$, $v \in R^r$, $A_1 \in \mathbb{R}^{n_0 \times n_0}$, $A_2 \in \mathbb{R}^{n_0 \times (n-n_0)}$, $B_1 \in \mathbb{R}^{n_0 \times r}$ and $L \in \mathbb{R}^{(n-n_0) \times n_0}$.

Which depends on the known part of the system (17) and the aim is to find a feedback control $\left(\mu(x_1), v(x_1)\right)$ that minimize the cost function

$$J = \int_0^\infty (f_{max}^2(x) + \rho^2 g_{max}^2(x) + \beta^2 x_1^T Q_1 x_1 + \mu^T R \mu + \rho^2 v^T W v) dt \quad (22b)$$

Where ρ and β are some positive constants that serves as design parameters, $f_{max}(x)$ is the upper bound of $f(x)$, $g_{max}(x)$ is the upper bound of $B_1^\dagger C_1 f(x)$, Q_1 is positive semi definite matrix and R, W are positive definite matrices.

Lemma 2 (necessary condition for optimality)

Consider the equivalent optimal control system (22) of the robust descriptor control system (1), and there is a positive definite continuously differentiable function $V(x_1)$ such that

$$V(x_1) = \min_{u,v \in \mathfrak{U}[0,t]} \int_0^\infty (f_{max}^2(x) + \rho^2 g_{max}^2(x) + \beta^2 x_1^T Q_1 x_1 + \mu^T R \mu + \rho^2 v^T W v) dt$$

Then the necessary condition for existence of optimal control is that $V(x_1)$ must satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \min_{u,v\in\mathfrak{U}[0,t]} \left[f_{max}^2(x) + \rho^2 g_{max}^2(x) + \beta^2 x_1^T Q_1 x_1 + \mu^T R \mu + \rho^2 v^T W v \right.$$
$$\left. + V_{x_1}^T \left((A_1 + A_2 L) x_1 + B_1 u + \left(I - B_1 B_1^{\dagger} \right) C_1 v \right) \right]$$

Where $V_{x_1} = \frac{dV}{dx_1}$.

Proof the same as derivation in the state space proof.

Main Theorem 2 (Equivalency theorem)

Consider the robust control problem

$$E\dot{x} = Ax + Bu + Cf(x), \quad x(0) = x_0 \tag{23}$$

Where $E, A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$ and $B, C \in \mathbb{R}^{n\times r}$ are the system coefficients and $f \in C^1(\mathbb{R}^n; \mathbb{R}^r)$ represent the uncertainty of the system and

1. $rank(E) = n_0 < n$
2. $f(0) = 0,$
3. $\|f(x)\| \le g_{max}(x)$, $g_{max}(0) = 0,$
4. $\left\|B_1^{\dagger} C_1 f(x)\right\| \le f_{max}(x)$, $f_{max}(0) = 0$

And the optimal control problem

$$\int_0^{\infty} (f_{max}^2(x) + \rho^2 g_{max}^2(x) + \beta^2 x_1^T Q_1 x_1 + \mu^T R \mu + \rho^2 v^T W v) dt \tag{24a}$$

Subject to

$$\dot{x}_1 = (A_1 + A_2 L) x_1 + B_1 u + \left(I - B_1 B_1^{\dagger} \right) C_1 v \tag{24b}$$

$x_2 = Lx_1$, $\left(x_{1,0}, x_{2,0}\right) \in w_k$, $x_1 \in \widetilde{\Omega}_x$

Where $Q_1$is positive semi-definite matrix and R and W are positive definite symmetric matrices, $x = P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $x_1 \in R^{n_0}$, $x_2 \in R^{n-n_0}$, L is the solution of $A_3 + A_4 L = 0$ and $\mathbb{Q}E\mathbb{P} = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbb{Q}A\mathbb{P} = \begin{pmatrix} \Sigma A_1 & \Sigma A_2 \\ A_3 & A_4 \end{pmatrix}$, $\mathbb{Q}B = \begin{pmatrix} \Sigma B_1 \\ 0 \end{pmatrix}$, $\mathbb{Q}C = \begin{pmatrix} \Sigma C_1 \\ 0 \end{pmatrix}$ for some suitable nonsingular matrices $\mathbb{Q}$ and $\mathbb{P}$.

If one can choose α and β such that the solution to the optimal control problem (24) , denoted by $(\mu(x_1), v(x_1))$, exists and the following condition is satisfy

$$2\rho^2 \|v\|_W^2 \le \acute{\beta}^2 \|x_1\|_{Q_1}^2 \quad \text{for all } x_1 \in \widetilde{\Omega}_x$$

for some $\acute{\beta}$ such that $\left|\acute{\beta}\right| \le |\beta|$, then $\mu(x_1)$ the u-component of the solution to the optimal control problem (24), is the solution of the robust control problem (23).

proof

From lemma1 and the conditions above, we get that the robust control problem (23) is equivalent to the robust control problem

$$\dot{x}_1 = A_1 x_1 + A_2 x_2 + B_1 u + C_1 f(x) \tag{25a}$$
$$0 = A_3 x_1 + A_4 x_2 \tag{25b}$$

And this problem is reduced locally to the problem

$$\dot{x}_1 = (A_1 + A_2 L) x_1 + B_1 u + C_1 f(x) \tag{26a}$$
$$x_2 = Lx_1 \tag{26b}$$

Or

$$\dot{x}_1 = (A_1 + A_2 L)x_1 + B_1 u + B_1 B_1^{\dagger} C_1 f(x) + \left(I - B_1 B_1^{\dagger}\right) C_1 f(x) \quad (27a)$$

$$x_2 = L x_1 \quad (27b)$$

For all $x_1 \in \widetilde{\Omega}_x$.

Now to prove the theorem, it is enough to prove that the solution of (24) is the solution of (26). To do so, define

$V(x_1) = \int_0^{\infty} (f_{max}^2(x) + \rho^2 g_{max}^2(x) + \beta^2 x_1^T Q_1 x_1 + \mu^T R \mu + \rho^2 v^T W v) dt$

For all $x_1 \in \widetilde{\Omega}_x$ and $u(0) = 0, u \in \mathfrak{U}[0,t]$, to be the minimum cost of bringing the system (24b) from $x_1(t_0) = x_{1,0}$ to $x_1 = 0$.

From lemma 2 , $V(x_1)$ must satisfies H-J-B

$$0 = \min_{u,v \in \mathfrak{U}[0,t]} \Big[f_{max}^2(x) + \rho^2 g_{max}^2(x) + \beta^2 x_1^T Q_1 x_1 + \mu^T R \mu + \rho^2 v^T W v + V_{x_1}^T \left((A_1 + A_2 L)x_1 + B_1 u + \left(I - B_1 B_1^{\dagger}\right) C_1 v \right) \Big]$$

Now if $(u, v) = (\mu(x_1), v(x_1))$ is the solution of the optimal control problem (24), then

$$f_{max}^2(x) + \rho^2 g_{max}^2(x) + \beta^2 x_1^T Q_1 x_1 + \mu^T R \mu + \rho^2 v^T W v + V_{x_1}^T \left((A_1 + A_2 L)x_1 + B_1 u + \left(I - B_1 B_1^{\dagger}\right) C_1 v \right) = 0 \quad (28a)$$

$$2\mu^T R + V_{x_1}^T B_1 = 0 \quad (28b)$$

$$2\rho^2 v^T W + V_{x_1}^T \left(I - B_1 B_1^{\dagger}\right) C_1 = 0 \quad (28c)$$

$$x_2 = L x_1 \quad (28d)$$

Now, we will show that the u-component of the solution to the optimal control problem (25) is the solution to the robust control problem (23), *i.e.*, $x_1 = 0$ of (26) is globally asymptotically stable for all admissible uncertainty $f(x)$.

To do so, we show that $V(x_1)$ is a Lyapunov function of the system (26).

1. Since $u(0) = 0, f_{max}(0) = 0, g_{max}(0) = 0$, then $V(0) = 0$.
2. And $f_{max}^2(x) > 0$, $g_{max}^2(x) > 0$, $x_1^T Q_1 x_1 > 0$, $u^T R u > 0$, $v^T W v > 0$ $\forall\, x = (x_1, L x_1) \neq (0,0)$, then $V(x_1) > 0$, $\forall\, x_1 \neq 0$.

$$\begin{aligned}
\dot{V}(x_1) &= V_{x_1}^T \dot{x}_1 \\
&= V_{x_1}^T \left[(A_1 + A_2 L)x_1 + B_1 u + B_1 B_1^{\dagger} C_1 f + \left(I - B_1 B_1^{\dagger}\right) C_1 f \right] \\
&= V_{x_1}^T \left[(A_1 + A_2 L)x_1 + B_1 u + \left(I - B_1 B_1^{\dagger}\right) C_1 v + \left(I - B_1 B_1^{\dagger}\right) C_1 (f - v) + B_1 B_1^{\dagger} C_1 f \right] \\
&= V_{x_1}^T \left[(A_1 + A_2 L)x_1 + B_1 u + \left(I - B_1 B_1^{\dagger}\right) C_1 v \right] + V_{x_1}^T \left(I - B_1 B_1^{\dagger}\right) C_1 (f - v) + V_{x_1}^T B_1 B_1^{\dagger} C_1 f
\end{aligned}$$

Substitution (28a) , (28b) and (28c), yields

$$\begin{aligned}
\dot{V}(x_1) &= -f_{max}^2(x) - \rho^2 g_{max}^2(x) - \beta^2 x_1^T Q_1 x_1 - \mu^T R \mu - \rho^2 v^T W v - 2\mu^T R B_1^{\dagger} C_1 f - 2\rho^2 v^T W (f - v) \\
&= -f_{max}^2(x) - \rho^2 g_{max}^2(x) - \beta^2 x_1^T Q_1 x_1 - \mu^T R \mu - \rho^2 v^T W v - 2\mu^T R B_1^{\dagger} C_1 f - 2\rho^2 v^T W f + 2\rho^2 v^T W v \\
&= -f_{max}^2(x) - \rho^2 g_{max}^2(x) - \beta^2 x_1^T Q_1 x_1 - \mu^T R \mu + \rho^2 v^T W v - 2\mu^T R B_1^{\dagger} C_1 f - 2\rho^2 v^T W f + \left(B_1^{\dagger} C_1 f\right)^T R B_1^{\dagger} C_1 f \\
&\qquad - \left(B_1^{\dagger} C_1 f\right)^T R B_1^{\dagger} C_1 f \\
&= -f_{max}^2(x) - \rho^2 g_{max}^2(x) - \beta^2 x_1^T Q_1 x_1 + \left(B_1^{\dagger} C_1 f\right)^T R B_1^{\dagger} C_1 f + \rho^2 v^T W v - 2\rho^2 v^T W f \\
&\qquad - \left(u + B_1^{\dagger} C_1 f\right)^T R \left(u + B_1^{\dagger} C_1 f\right) \\
&= -f_{max}^2(x) - \rho^2 g_{max}^2(x) - \beta^2 \|x_1\|_{Q_1}^2 + \left\|B_1^{\dagger} C_1 f\right\|_R^2 - 2\rho^2 v^T W f - \left\|u + B_1^{\dagger} C_1 f\right\|_R^2 + \rho^2 \|v\|_W^2
\end{aligned}$$

Since $\rho^2\|v-f\|_W^2 = \rho^2(\|v\|_W^2 - 2\rho^2 v^T W f + \|f\|_W^2)$

$\geq -2\rho^2 v^T W f$

Therefore, $-2\rho^2 v^T W f \leq \rho^2\|v-f\|_W^2 \leq \rho^2\|v\|_W^2 + \rho^2\|f\|_W^2$

$\dot{V}(x_1) = -f_{max}^2(x) - \rho^2 g_{max}^2(x) - \beta^2\|x_1\|_{Q_1}^2 + \left\|B_1^\dagger C_1 f\right\|_R^2 + \rho^2\|v\|_W^2$

$+\rho^2\|f\|_W^2 - \left\|u + B_1^\dagger C_1 f\right\|_R^2 + \rho^2\|v\|_W^2$

$= -f_{max}^2(x) - \rho^2 g_{max}^2(x) - \beta^2\|x_1\|_{Q_1}^2 + \left\|B_1^\dagger C_1 f\right\|_R^2 + \rho^2\|v\|_W^2 + \rho^2\|f\|_W^2 - \left\|u + B_1^\dagger C_1 f\right\|_R^2 + \rho^2\|v\|_W^2$

$= -\left(f_{max}^2(x) - \left\|B_1^\dagger C_1 f\right\|_R^2\right) - \rho^2(g_{max}^2(x) - \|f\|_W^2) - \beta^2\|x_1\|_{Q_1}^2 + +2\rho^2\|v\|_W^2 - \left\|u + B_1^\dagger C_1 f\right\|_R^2$

$\leq -\beta^2\|x_1\|_{Q_1}^2 + 2\rho^2\|v\|_W^2 + \acute{\beta}^2\|x_1\|_{Q_1}^2 - \acute{\beta}^2\|x_1\|_{Q_1}^2$

$\leq 2\rho^2\|v\|_W^2 - \acute{\beta}^2\|x_1\|_{Q_1}^2 - \left(\beta^2 - \acute{\beta}^2\right)\|x_1\|_{Q_1}^2$

But $2\rho^2\|v\|_W^2 \leq \acute{\beta}^2\|x_1\|_{Q_1}^2 \Rightarrow 2\rho^2\|v\|_W^2 - \acute{\beta}^2\|x_1\|_{Q_1}^2 \leq 0$ and $\left|\acute{\beta}\right| \leq |\beta| \Rightarrow \beta^2 - \acute{\beta}^2 \geq 0$. Therefore $\dot{V}(x_1) \leq 0$

Thus, the condition of Lyapunov stability is satisfied .

Consequently, there exists a neighborhood $N_C = \left\{x_1 \in \widetilde{\Omega}_x \,;\, \|x_1\|_{Q_1} < C\right\}$ for some $C > 0$. Such that if $x_1(t)$ enters N_C then $\lim_{t\to\infty}\|x_1(t)\|_{Q_1} = 0$.

But $x_1(t)$ cannot remains forever outside N_C, otherwise $\|x_1(t)\|_{Q_1} > C$ for all $t > 0$, therefore

$V(x_1(t)) - V(x_1(0)) = \int_0^t \dot{V}(x_1(s))ds$

$\leq -\int_0^t \|x_1(s)\|_{Q_1}^2 ds$

$\leq -\int_0^t C^2 ds$

$= -C^2 t$

$V(x_1(t)) \leq V(x_1(0)) - C^2 t$

Letting $\to \infty$, we have $V(x_1(t)) \to -\infty$ which contradicts the fact that $V(x_1(t)) > 0$ for all $x_1 \in \widetilde{\Omega}_x$. Therefore $\lim_{t\to\infty}\|x_1(t)\|_{Q_1} = 0$. But $x_2(t) = Lx_1(t)$ such that $x(t) = (x_1(t), x_2(t)) \in \widetilde{\Omega}$.

Then

$\lim_{t\to\infty}\|x_2(t)\| = \left\|\lim_{t\to\infty} x_2(t)\right\|$

$= \|\lim_{t\to\infty} Lx_1(t)\|,\ x_1 \in \widetilde{\Omega}_x$

$= \|L \lim_{t\to\infty} x_1(t)\|,\ x_1 \in \widetilde{\Omega}_x$

$= 0$

So $\lim_{t\to\infty}\|x(t)\| = \|\lim_{t\to\infty}(x_1(t), x_2(t))\|$

$= \|(0,0)\|$

$= 0.$ For all $x(t) \in \widetilde{\Omega}$ ■

6 Illustration

Consider the robust descriptor system

$$E\dot{x} = Ax + Bu + Cf(x)$$

Where $x^T = (x_1, x_2, x_3, x_4)$ and

$E = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 3 & 3 & 3 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$,

$C = \begin{pmatrix} 0.6 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $f(x) = 5p_1\, x_1 \cos\left(\frac{1}{x_2 + p_2}\right) + 5p_3 x_2 \sin(p_4 x_1 x_2),\ x_2 \neq -p_2,$

$p_1 \in [-0.2\,, 0.2],\ \ p_2 \in [-10\,, 100],\ \ p_3 \in [0\,, 0.2],\ \ p_4 \in [-100\,, 0].$

Since $E = 2 = n_0$, then E is singular matrix.

Using the transformation matrices $\mathbb{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\mathbb{Q} = I_4$ and putting

$\mathbb{P}^{-1}x = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$, $\quad Y_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\quad Y_2 = \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}$, the system above can be transformed to

$$\dot{Y}_1 = A_1 Y_1 + A_2 Y_2 + B_1 u + C_1\left(5p_1 y_2 \cos\left(\frac{1}{y_1 + p_2}\right) + 5p_3 y_1 \sin(p_4 y_2 y_1)\right)$$

$$0 = A_3 Y_1 + A_4 Y_2$$

Where $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A_3 = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C_1 = \begin{pmatrix} 0.2 \\ 0 \end{pmatrix}$.

Clearly $\|f\|_R^2 \leq y_1^2 + y_2^2 = (y_1, y_2)^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Y_1^T Y_1 = g_{max}^2(y)$.

$$B_1^{\dagger} = (B_1^T B_1)^{-1} B_1^T = \left[(0 \quad 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]^{-1} (0 \quad 1) = (0 \quad 1)\ .$$

$$B_1^{\dagger} C f = (0 \quad 1)\begin{pmatrix} 0.2 \\ 0 \end{pmatrix} f = 0.f = 0\ .$$

$$\left\|B_1^{\dagger} C f\right\|_R^2 = 0 = f_{max}^2(y)\ .$$

$$I - B_1 B_1^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0 \quad 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$(I - B_1 B_1^{\dagger})C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0.2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0 \end{pmatrix}.$$

Here, the consistency space is $\widetilde{\Omega} = \{Y_1 \in \widetilde{\Omega}_y = \mathbb{R}^2, Y_2 \in \mathbb{R}^2 \,|\, A_3 Y_1 + A_4 Y_2 = 0\}$

Since $A_4 = I$ is invertible, then

$\widetilde{\Omega} = \{Y_1 \in \widetilde{\Omega}_y = \mathbb{R}^2, Y_2 = -A_4^{-1} A_3 Y_1\}$.

$\widetilde{\Omega} = \left\{(y_1, y_2)^T \in \widetilde{\Omega}_y = \mathbb{R}^2, (y_3, y_4)^T = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\}$

$\widetilde{\Omega} = \{(y_1, y_2)^T \in \widetilde{\Omega}_y = \mathbb{R}^2, y_3 = y_1\,, y_4 = -2y_1 + y_2\}$

Therefore, the initial condition $(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0})$ is consistent iff

$$y_{3,0} = y_{1,0} \text{ and } y_{4,0} = -2y_{1,0} + y_{2,0} \text{ for a given } (y_{1,0}, y_{2,0}).$$

Let $= \beta = 1$. Then the corresponding optimal control problem is as follow, for the nominal system

$$\dot{Y}_1 = (A_1 - A_2 A_3) Y_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0.2 \\ 0 \end{pmatrix} v$$

$$\dot{Y}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0.2 \\ 0 \end{pmatrix} v$$

Find a feedback control law (μ, v) that minimize the cost function

$$\int_0^\infty (g_{max}^2(y) + Y_1^T Y_1 + \mu^T \mu + v^T v)dt = \int_0^\infty (2Y_1^T Y_1 + \mu^T \mu + v^T v)dt$$

It is easy to show that the solution of this optimal control problem is

$$\mu = -1.2906y_1 - 2.1247y_2\ ,\ \ v = -0.5783y_1 - 0.2581y_2$$

Or $\mu = -1.2906x_2 - 2.1247x_1\ ,\ \ v = -0.5783x_2 - 0.2581x_1$

Since the condition $2\rho^2\|v\|_W^2 \leq \acute{\beta}^2\|x_1\|_{Q_1}^2$ is satisfied , then

$\mu = -1.2906y_1 - 2.1247y_{x2} \Rightarrow \mu = -1.2906x_2 - 2.1247x_1$ is the optimal control.

By theorem 2, this is a solution to the original robust control system.

The solutions of the optimal control problem and robust control of the equivalent system are shown in the following figures:

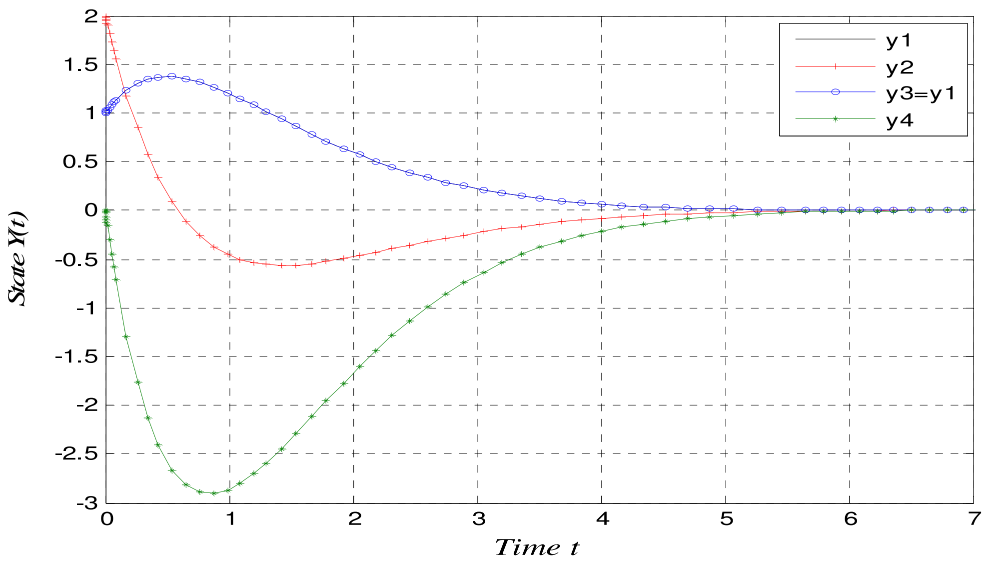

Figure (1) Optimal solution represents (y_1, y_2, y_3, y_4) *with* $(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0}) = (1,2,1,0) \in w_k$

It is very important to notice that in contracts to the state-space (o.d.e) systems, when the eigenvalues of the nominal system have negative real part then the system is stable for all initial condition. While in descriptor systems the initial condition divided into two parts, the first one concerns the dynamic (o.d.e) and the second part is the algebraic equation which is called the consistent initial conditions. And these initial conditions effect the system stability even when the spectrum of the dynamic system lie in the left half of $\mathbb{C}$ as one can see the figure (2), the dynamic state space and the non dynamic state space vector are far from the equilibrium point (0,0) when choosing the initial condition out of the consistency region.

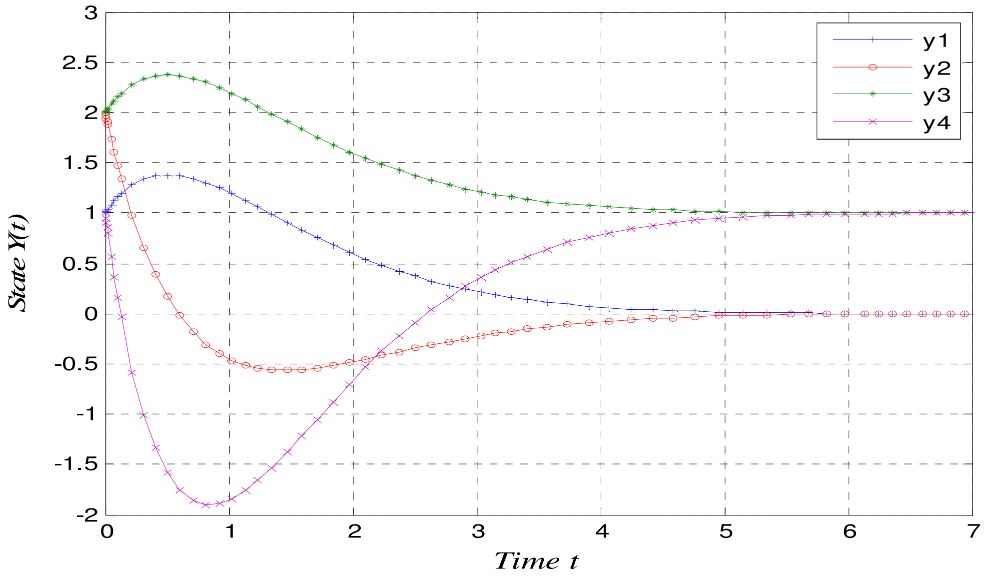

Figure (2) Optimal solution represents (y_1, y_2, y_3, y_4) *with* $\left(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0}\right) = (1,2,2,1) \notin w_k$

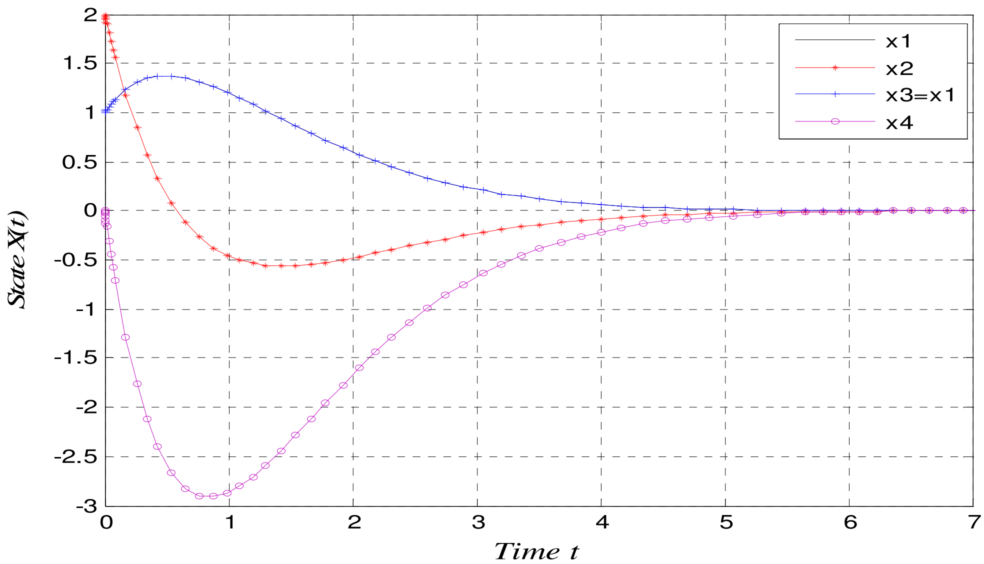

Figure (3) The nominal system represents (x_1, x_2, x_3, x_4) *with* $\left(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}\right) = (1,2,1,0)$

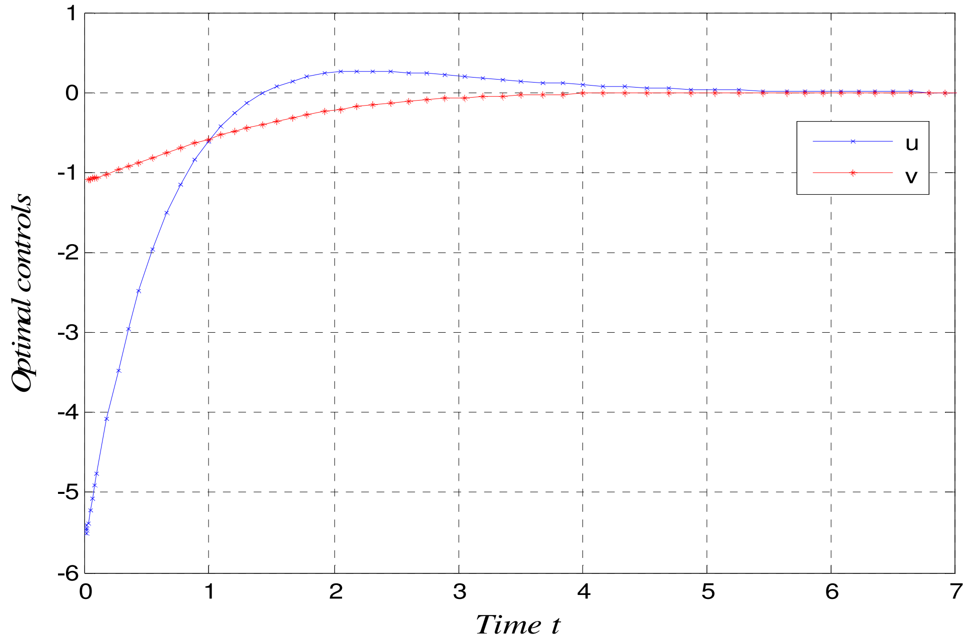

Figure (4) The optimal controls $u(t)$ *and* $v(t)$

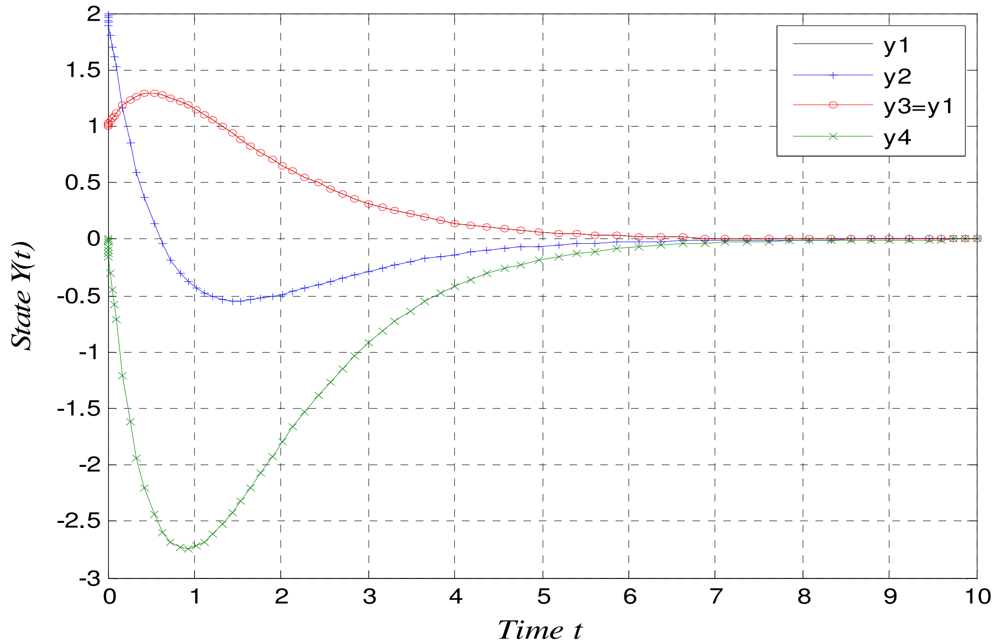

Figure (5) Robust solution represents (y_1, y_2, y_3, y_4) *with* $\left(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0}\right) = (1,2,1,0) \in w_k$,

$p_1 = -0.2,\ p_2 = -10,\ p_3 = 0,\ p_4 = -100$

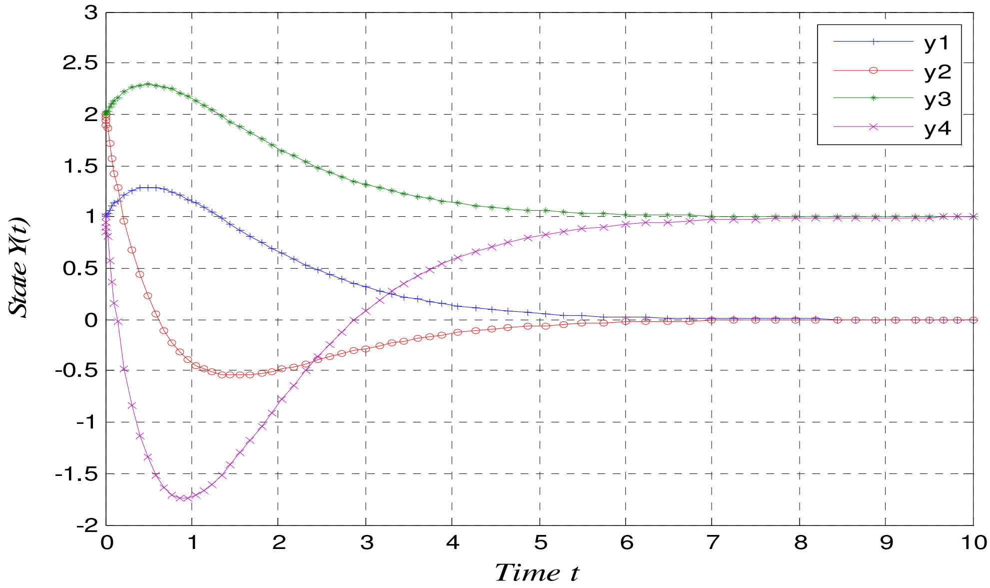

Figure (6) Robust solution represents (y_1, y_2, y_3, y_4) *with* $(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0}) = (1,2,2,1) \notin w_k$,

$p_1 = -0.2,\ p_2 = -10,\ p_3 = 0,\ p_4 = -100$

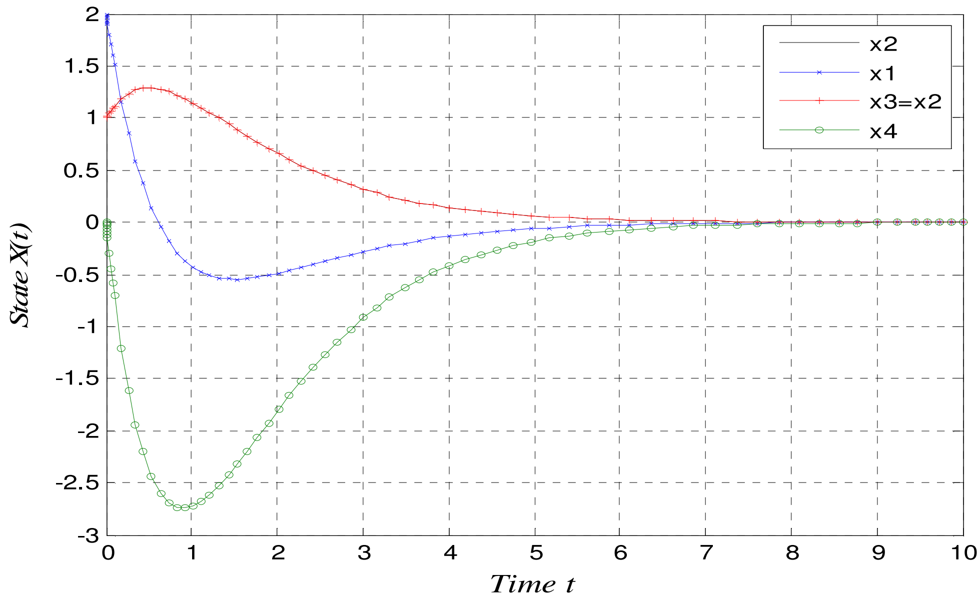

Figure (7) Robust solution represents (x_1, x_2, x_3, x_4) *with* $(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) = (2,1,1,0)$,

$p_1 = -0.2,\ p_2 = -10,\ p_3 = 0,\ p_4 = -100$

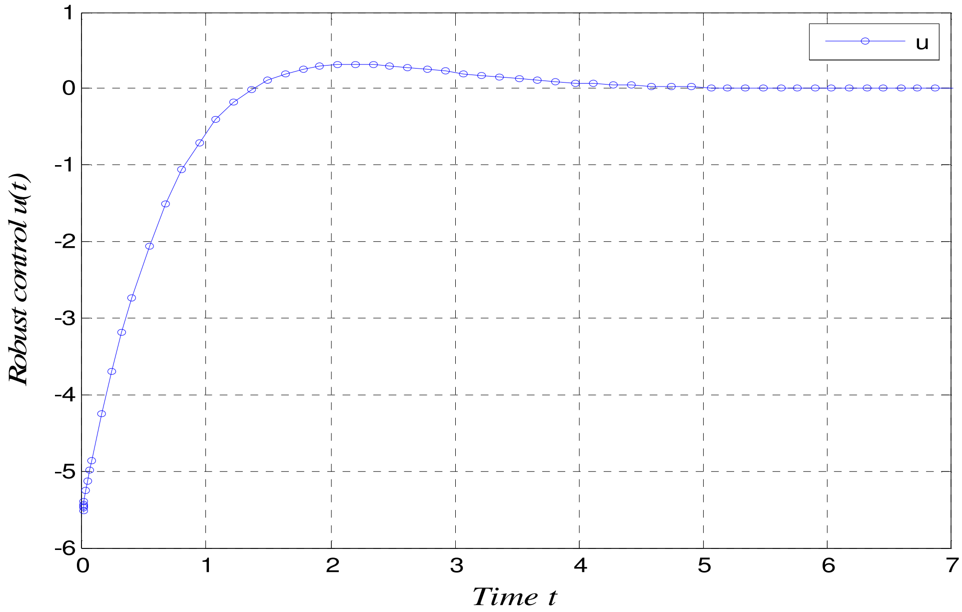

Figure (8) The robust control with

$(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0}) = (1,2,1,0) \in w_k$,

$p_1 = -0.2,\ p_2 = -10,\ p_3 = 1,\ p_4 = -100$

7 Conclusions

From this work, we can conclude the following points:

3. The solvability and Stabilizability of the robust control problem of some non-linear semi-explicit descriptor uncertain systems without matching condition is discussed via an optimal control approach in the sense that, the solution of an equivalent optimal control problem to the uncertain nonlinear descriptor system, is the solution to the given descriptor one with matching condition.
4. This novel approach is very applicable for a large class of systems and make the original problems tractable and easy for point of applications.

8 future work

The solution of the robust control problem of some non-linear semi-explicit descriptor uncertain systems without matching condition and non-linear algebraic equation have been considered for publication:

References

Ayman A.S. Robust stabilization of singular non-bilinear dynamical control systems with parametric uncertainty. *M.Sc. thesis*, College of Science, AL-Mustansiriyah University (2012).

Brenan, K. E., Campbell, S., and Petzold, L. R. Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations. *SIAM*, New York, (1996).

Campbell S.L. Singular systems of differential equations, volume 1. *Pitman London*, (1980).

Dai, L. Singular Control Systems. *Lecture Notes in Control and Information Sciences. Springer-Verlag*, Berlin, (1989).

Danielle C. Tarraf and H. Harry A. On the Nature and Stability of Differential-Algebraic Systems. *Proceedings of the American control conference*, Anchorage, A K May 8-10 (2002).

Debeljkovic D. and Buzurovic I. Lyapunov stability of linear continuous singular systems: an overview. *International journal of information and systems sciences computing and Information* .Volume 7, Number 2-3, Pages 247–268. (2011).

FENG LIN. ROBERT, D. BRANDT, JING SUN. Robust control of nonlinear systems: compensation for uncertainty. *INT.J. Control*, VOL.56, No.6, 1453-1459, (1992).

FENG LIN. An optimal control approach to robust control design. *INT.J. Control*, VOL.73, No.3, 177-186, (2000).

Kunkel, P. and Mehrmann, V. Analysis of over- and underdetermined nonlinear differential-algebraic systems with application to nonlinear control problems. *Mathematics of Control, Signals, and Systems*, 14:233–256, (2001).

Kunkel, P. and Mehrmann, V. Index reduction for differential-algebraic equations by minimal extension. *Zeitschrift für Angewandte Mathematik und Mechanik*, VOL.84, No.9:579–597, (2004).

Kunkel, P. and Mehrmann, V. Canonical forms for linear differential-algebraic equations with variable coefficients. *Journal of Computational and Applied Mathematics* 56 , 22.5-251 (1994).

Kunkel, P. and Mehrmann, V. Regular solution of nonlinear differential-algebraic equations and their determination. *Deutsche Foreschungsgemein schaft, Research grant Me* 790/5-2 Differentiell-algebraische Gleichungen. (1995).

Michael M. Stability criteria for nonlinear fully implicit differential-algebraic systems. *PhD thesis*, Universität zu Köln, (2011).

Nikuie M., Mirnia M.K. and Mahmoudi Y. Some results about the index of matrix and Drazin inverse. *Mathematical Sciences*, Vol. 4, No. 3 (2010) 283-294.

Radhi A. Zaboon, Sabeeh L. Jasim. Optimal Control Equivalent Approach to Non-Linear Uncertain Descriptor Systems with Matching Condition. *Control theory and informatics*. Vol.6, No.3. 2016, pp 19-29.

Radhi A. Zaboon, Hadi Jaber Mustafa, Ghaswan M. J. AL-Hashimi, Robust control of nonlinear dynamical control system with matching condition*, Journal of college of education* No. 3,Vol. 28, (2008),pp.291-308, Iraq.

Sharvan S.S. Stability results for constrained dynamical systems. *PhD thesis, Hamilton Institute National University of Ireland* Maynooth Ollscoil na hÉireann, Má Nuad, (2012).

SUN Liying and WANG Yuzhen, Robust Stabilization of a Class of Nonlinear Descriptor Systems, *proceedings of the 31st Chinese control conference* July 25-27 2012, Hefei, China.

Tadeusz K., Positivity of descriptor linear systems with regular pencils. *Archives Of Electrical Engineering*, VOL. 61(1), pp. 101-113 (2012).

Xiaoming Su and Yali Zhi, Sufficient Conditions of Asymptotic Stability of the Time-Varying Descriptor Systems, *Hindawi Publishing Corporation International Journal of Engineering Mathematics* Volume 2013, Article ID 920851, 4 pages.