Control Theory and Informatics
ISSN 2224-5774 (Paper) ISSN 2225-0492 (Online)
Vol.6, No.3, 2016

Stability of Linear Multiple Different Order Caputo Fractional System

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Abstract
In this paper, we introduce a new equivalent system to the higher order Caputo fractional system (CFS) . This equivalent system has multiple order Caputo fractional derivatives (CFDs). These CFDs are lying between zero and one. As well as, we find the fundamental solution for linear CFS with multiple order CFDs. Also, we introduce new criteria of studying the stability (asymptotic stability) of the linear CFS with multiple order CFDs. These criteria can be applied in three cases: the first, all CFDs is lying between zero and one. The second, all CFDs are lying between one and two. Finally, some of CFDs are lying between zero and one, and the rest of these derivatives are lying between one and two. The criteria are depending on the position of eigenvalues of the matrix system in the complex plane. These criteria are considered as a generalized of the classical criteria which is used to study the stability of linear first ODEs. Also, these criteria are considered as generalized of the criteria which used to study the stability same order CFS in case when all CFDs lying between zero and one, also in case when all CFDs lying between one and two. Several examples are given to show the behavior of the solution near the equilibrium point.
Keywords: Caputo fractional derivatives; Linear Caputo fractional system ; Fundamental solution Stability analysis.

1. Introduction

The calculus of integrals and derivatives of any arbitrary real or complex order is called fractional calculus. The topic of fractional calculus has been known since the expansion of the classical integer order calculus with the initial works being related with Leibniz and L'Hospital. All old reference points out the foundation of fractional derivative is back to correspondence letters between Leibniz and L'Hospital in 1695 where half-order derivative was mentioned [1, 2, 3].

Differential equations content fractional derivative are called Fractional differential equations (FDEs). FDEs have been used to describe many real worlds modeling like, damping laws, fluid mechanics, rheology, physics, mathematical biology, diffusion processes, electrochemistry, and so on. For example, the diffusion of heat through a semi- infinite solid can be described by FOS [3]. Some results related to the existence and uniqueness solutions for FDEs may be found in the books by Podlubny [3] , Kilbas et al. [1] and Samko et al.[2].

In recent decades, the study of stability analysis for FDEs became a fruitful spot for many researchers. In fact, this subject is more complex than the ODEs since fractional derivative is represented as integro-differential, so it is nonlocal and has weakly singular kernels. Matignon [4, 5], in has PhD thesis, was the first person how introduced some stability results related to a restrictive modeling of FDEs. There are many important result related on linear FDEs with CFDs order α , where $0 < \alpha < 1$ .After that, Qian et al. [6] investigated the linear FDEs with Riemann-Liouville derivative and the same fractional order α , where $0 < \alpha < 1$. After that, many researchers have been investigated stability of linear and nonlinear FDEs with $0 < \alpha < 1$ [6]-[18]. W. H. Deng et al. [19, 20] studied the stability and asymptotical stability for linear (linear time delay) fractional system with multi-order (rational) CFD. However, many stability methods for linear FDEs systems like, frequency domain methods, Linear Matrix Inequalities and conversion methods have emerged in progression [18]. In Fact, the stability results of FDEs are applicable many in physical systems, see, Ahn and Chen [7], Ahmed, EI-Saka, and EI-Saka [22], Li, Chen, and Podlubny [21], Li and Zhang [9], Miller and Ross [10], Odibat [13], Radwan, Soliman, Elwakil, and Sedeek [11], Sabatier, Moze, and Farges [18] , Samko et al. [2],Wen,Wu, and Lu, [16] . On the other hand, there are many FDEs have fractional orders not lying in $(0,1)$. In fact, there are many physical systems can be described by FDEs fractional orders lying in $(1,2)$ [1]-[3]. The study of stability (asymptotic stability) of many kind of linear FDEs with order $(1,2)$ have been done by many authors [12, 18, 8, 23]. While, H.S. Ahn et al. [15] was introduced the necessary and sufficient condition for commensurate order fractional linear system. As well as, H.S. Ahn et al. [25] give new criteria of studying the robust stability of uncertain linear time invariant fractional system.

The fundamental solution and the stability analysis of the linear FDEs system with constant coefficients are studied in two cases: the first when the system has the same order fractional derivative while the second when the systems have multiple rational orders [13, 19, 26]. In [26], the rational order CFS is transformed into an equivalent system of the same order CFDs. These CFDs are lying between zero and one. However, the order of CFDs in many fractional dynamical systems is not always rational order. The motivation of this paper is introducing a new equivalent system to the higher order CFS. Furthermore, since the equivalent system in the proposed method has multiple different orders CFDs and there is no method enable us to solve such systems, we introduce a method to find the fundamental solution for linear CFS with multiple different orders CFDs. As well as, we introduce new criteria of studying the stability (asymptotic stability) of the linear CFS with multiple order CFDs.

2. Preliminaries

This section is devoted to the definition Caputo fractional derivatives and their properties.

Definition 2.1 [1]-[2]

The Riemann–Liouville fractional integral of function $x(t)$ is defined as

$$D_t^{-\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau)\, d\tau. \qquad (2.1)$$

Where $\alpha > 0,\ t > 0, \Gamma(.)$ is the Gamma function.

Definition2.2 (**Caputo derivative**) [1]-[3]

In the fractional calculus the Caputo derivative is defined

$$ {}_{t_o}^{C}D_t^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_o}^{t} \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \quad ,(\alpha \in (n-1, n]) \qquad (2.2)$$

Where $x(t)$ is an arbitrary differentiable function, $n \in \mathrm{N}$ and ${}_{t_o}^{C}D_t^{\alpha}$ is Caputo fractional derivative of order $n-1 < \alpha < n$, and $\Gamma(.)$ denotes the Gamma function.

Definition 2.3 [1, 3]

The one-parameter and two –parameter Mittag-Leffler functions are defined as, respectively

$$E_{\alpha}(\ t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+1)}\ , \quad (\alpha > 0), \qquad (2.3)$$

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+\beta)}\ , \quad (\alpha > 0, \beta > 0), \qquad (2.4)$$

Property 2.4 [3]

The formula for the Laplace transform of the Caputo fractional derivative is given as

$$\int_0^{\infty} e^{-st} \left\{ {}_0^{C}D_t^{\alpha}\, x(t) \right\} dt = s^{\alpha} X(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} x^{(k)}(0), \qquad (2.5)$$

where $n-1 < \alpha < n$, and $n \in Z^{+}$.

Property 2.5: [1, 3]

The Laplace transform of two –parameter Mittag-Leffler function is given as

$$L\{ t^{\beta-1} E_{\alpha,\beta}(-\lambda t^{\alpha}) \} = \frac{s^{\alpha-\beta}}{s^{\alpha}+\lambda}\ , \quad (R(s) > |\lambda|^{\frac{1}{\alpha}}), \qquad (2.6)$$

where $t \geq 0, R(s)$ denotes the real part of $s, \lambda \in \mathrm{R}$.

Lemma 2.6 [3]

If $\alpha \in (0,2)$, β is an arbitrary real number, μ satisfies $\frac{\alpha\pi}{2} < \mu < \min\{\pi, \pi\alpha\}$, and C_1, C_2 are real constants, then

$$\left| E_{\alpha,\beta}(z) \right| \leq C_1 \left(1+|z|\right)^{(1-\beta)/\alpha} \exp\left(\mathrm{Re}(z^{1/\alpha})\right) + \frac{C_2}{1+|z|}, \qquad (2.7)$$

Where $|\arg(z)| \leq \mu, |z| \geq 0$.

Lemma 2.7 [14]

The following properties hold.

(i) There exist finite real constants $M_1, M_2 \geq 1$ such that for any $\alpha \in (0,1)$,

$$E_{\alpha,1}(At^{\alpha}) \leq M_1 \left\|e^{At}\right\|, \quad (2.8)$$

$$E_{\alpha,\alpha}(At^{\alpha}) \leq M_2 \left\|e^{At}\right\|, \quad (2.9)$$

where A denotes matrix, $\|.\|$ denotes any vector or induced matrix norm.

(ii) If $\alpha \geq 1$, then for $\beta = 1, 2, \alpha$

$$E_{\alpha,\beta}(At^{\alpha}) \leq \left\|e^{At}\right\|. \quad (2.10)$$

<u>Theorem 2.8</u> [24]

If $x(t) \in C^1[0,T]$ for some $T > 0$, then

$${}_0^C D_t^{\alpha_2}\, {}_0^C D_t^{\alpha_1} x(t) = {}_0^C D_t^{\alpha_1}\, {}_0^C D_t^{\alpha_2} x(t) = {}_0^C D_t^{\alpha_1+\alpha_2} x(t), \quad t \in [0,T],$$

Where $\alpha_1, \alpha_2 \in \mathbb{R}^+$ and $\alpha_1 + \alpha_2 \leq 1$.

<u>Theorem 2.9</u> [24]

If $x(t) \in C^m[0,T]$ for $T > 0$, then ${}_0^C D_t^{\alpha} x(t) = {}_0^C D_t^{\alpha_n}\, {}_0^C D_t^{\alpha_{n-1}} \ldots {}_0^C D_t^{\alpha_2}\, {}_0^C D_t^{\alpha_1} x(t), \ \forall t \in [0,T]$ where

$\alpha = \sum_{i=1}^{n} \alpha_i, \alpha_i \in (0,1], m-1 < \alpha < m \in Z^+$ and there exist $i_k < n$ such that $\sum_{j=1}^{i_k} \alpha_j = k$, and $k = 1, 2, \ldots, m-1$.

<u>Theorem 2.10:</u> [4, 5]

Let A be $n \times n$ real matrix. Then a necessary and sufficient condition for the asymptotical stability of

$${}_0^C D_t^{\alpha} x(t) = Ax(t)$$

is $\left|\arg(spec(A))\right| > \frac{\alpha\pi}{2}$, where $\alpha \in (0,1)$ and $spec(A)$ is the spectrum (set of all eigenvalues) of A.

<u>Theorem 2.11</u>[18]

The system ${}_0^C D_t^{\alpha} x(t) = Ax(t)$ is asymptotically stable if the following condition is satisfy $\left|\arg(eig(A))\right| > \frac{\alpha\pi}{2}$,

Where $\alpha \in (0,2)$ and $eig(A)$ are the eigenvalues of A.

3. Equivalent system for high order autonomous CFS:

Consider the high order autonomous CFS

$${}_0^C D_t^{\alpha} x(t) = f(x(t)), \quad (3.1)$$ where $m-1 < \alpha < m$, $\alpha = \sum_{i=1}^{N} \alpha_i$,

$0 < \alpha_i \leq 1$ and $\sum_{i=1}^{N_k} \alpha_i = k, \ k = 1, 2, \ldots m-1$. with initial conditions

$$x^{(k)}(0) = x_k, \quad k = 0, 1, \ldots, m-1. \quad (3.2)$$

<u>Theorem 3.1</u> The high order autonomous CFS (3.1) with the initial conditions (3.2) is equivalent to a system of FDEs with derivative order $0 < \alpha_i \leq 1$, $i = 1, 2, \ldots, N$.

Proof:

Let $y_1(t) = x(t)$ and differentiate ${}_0^C D_t^{\alpha_1}$ to both sides, one can have

$${}_0^C D_t^{\alpha_1} y_1(t) = {}_0^C D_t^{\alpha_1} x(t).$$

Then, let $y_2(t) = {}_0^C D_t^{\alpha_1} x(t)$ and differentiate ${}_0^C D_t^{\alpha_2}$ to both sides, one can get

$${}_0^C D_t^{\alpha_2} y_2(t) = {}_0^C D_t^{\alpha_2}\, {}_0^C D_t^{\alpha_1} x(t) = {}_0^C D_t^{\alpha_2+\alpha_1} x(t).$$

Again, let $y_3(t) = {}_0^C D_t^{\alpha_2+\alpha_1} x(t)$ and differentiate ${}_0^C D_t^{\alpha_3}$ to both sides, one can get

$${}_0^C D_t^{\alpha_3} y_3(t) = {}_0^C D_t^{\alpha_3}\, {}_0^C D_t^{\alpha_2+\alpha_1} x(t) = {}_0^C D_t^{\alpha_3+\alpha_2+\alpha_1} x(t).$$

and so on, let $y_N(t) = {}_0^C D_t^{\sum_{i=1}^{N-1}\alpha_i} x(t)$ and differentiate ${}_0^C D_t^{\alpha_N}$ to both sides, one can get

${}_0^C D_t^{\alpha_N} y_N(t) = {}_0^C D_t^{\alpha_N}\, {}_0^C D_t^{\sum_{i=1}^{N-1}\alpha_i} x(t) = {}_0^C D_t^{\alpha} x(t)$.

So , we have the following system

$$\begin{aligned}
{}_0^C D_t^{\alpha_1} y_1(t) &= y_2(t),\\
{}_0^C D_t^{\alpha_2} y_2(t) &= y_3(t),\\
{}_0^C D_t^{\alpha_3} y_3(t) &= y_4(t),\\
&\vdots\\
{}_0^C D_t^{\alpha_{N-1}} y_{N-1}(t) &= y_N(t),\\
{}_0^C D_t^{\alpha_N} y_N(t) &= f(y_1)
\end{aligned} \tag{3.3}$$

The Laplace transform to both sides of system (3.3) gives

$$\begin{aligned}
s^{\alpha_1}\overline{y}_1(s) - s^{\alpha_1-1}y_1(0) &= \overline{y}_2(s),\\
s^{\alpha_2}\overline{y}_2(s) - s^{\alpha_2-1}y_2(0) &= \overline{y}_3(s),\\
s^{\alpha_3}\overline{y}_3(s) - s^{\alpha_3-1}y_3(0) &= \overline{y}_4(s),\\
&\vdots\\
s^{\alpha_{N-1}}\overline{y}_{N-1}(s) - s^{\alpha_{N-1}-1}y_{N-1}(0) &= \overline{y}_N(s),\\
s^{\alpha_N}\overline{y}_N(s) - s^{\alpha_N-1}y_N(0) &= \overline{f}(s)
\end{aligned} \tag{3.4}$$

Choose the initial condition of system (3.1.3) as follows

$$\begin{aligned}
&y_{N_k+1}(0) = x^{(k)}(0),\ y_{N_k+2}(0) = 0, \dots, y_{N_k}(0) = 0, \quad k = 1, 2, \dots, m-1\\
&y_{N_{m-1}+1}(0) = x^{(m-1)}(0),\ y_{N_{m-1}+2}(0) = 0, \dots
\end{aligned} \tag{3.5}$$

By using the initial conditions (3.5) and backward substitution of system (3.4) one can have

$$s^{\alpha}\overline{y}_1(s) - s^{\alpha-1}x(0) - s^{\alpha-2}x'(0) - \cdots - s^{\alpha-m}x^{(m-1)}(0) = \overline{f}(s) \tag{3.6}$$

In fact, Eq. (3.6) is exactly the Laplace transform of system (3.1) with the initial conditions (3.2). So that the system (3.3) with the initial conditions (3.5) is equivalent to system (3.1) with the initial conditions (3.2).

Now, Consider the multi high order autonomous CFS

$${}_0^C D_t^{\alpha} x(t) + a_1\, {}_0^C D_t^{\alpha-\alpha_N} x(t) + a_2\, {}_0^C D_t^{\alpha-\alpha_{N-1}-\alpha_N} x(t) + a_3\, {}_0^C D_t^{\alpha-\alpha_{N-2}-\alpha_{N-1}-\alpha_N} x(t) + \cdots + a_N x(t) = f(x(t)), \tag{3.7}$$

where $m-1 < \alpha < m$, $\alpha = \sum_{i=1}^{N}\alpha_i$, $0 < \alpha_i \le 1$ and $\sum_{i=1}^{N_k}\alpha_i = k,\ k = 1, 2, \dots m-1$ .with initial conditions (3.2).

Theorem 3.2 The multi high order autonomous CFS (3.7) with the initial conditions (3.2) is equivalent to

$$\begin{aligned}
{}_0^C D_t^{\alpha_1} y_1(t) &= y_2(t),\\
{}_0^C D_t^{\alpha_2} y_2(t) &= y_3(t),\\
{}_0^C D_t^{\alpha_3} y_3(t) &= y_4(t),\\
&\vdots\\
{}_0^C D_t^{\alpha_{N-1}} y_{N-1}(t) &= y_N(t),\\
{}_0^C D_t^{\alpha_N} y_N(t) &= -a_N y_1(t) - a_{N-1}y_2(t) - a_{N-2}y_3(t) - \cdots - a_1 y_{N-1}(t) + f(y_1)
\end{aligned} \tag{3.8}$$

with the initial conditions (3.1.5).

4. General solution of multiple different orders linear CFS:

In this section, we adopt the general solution of multiple different orders linear CFS in the form

$${}_0^C D_t^{\overline{\alpha}} x(t) = Ax(t), \quad x(t) \in R^n \tag{4.1}$$

where A is $n\times n$ matrix , $\overline{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_n)$, and $0<\alpha_k<1$, $k=1,2,\dots,n$
with initial conditions

$$x_k(0)=x_k,\quad k=1,2,\dots,n \tag{4.2}$$

First, assume the matrix A has different eigenvalues, that is, there exist invertible matrix P such that $A=P\Lambda P^{-1}$ where $\Lambda=diag(\lambda_1,\lambda_2,\dots,\lambda_n)$.

Assume $y(t)=P^{-1}x(t)$ then the system (4.1) become

$${}_0^CD_t^{\overline{\alpha}}\ y(t)=\Lambda\, y(t) \tag{4.3}$$

And the initial condition (4.2) become

$$y(0)=P^{-1}x(0) \tag{4.4}$$

By taking Laplace transform to both sides of (4.3), one can have

$$s^{\alpha_k}\overline{y}_k(s)-s^{\alpha_k-1}y_k(0)=\lambda_k\,\overline{y}_k(s),\quad k=1,2,\dots,n \tag{4.5}$$

$$\overline{y}_k(s)=\frac{s^{\alpha_k-1}}{(s^{\alpha_k}-\lambda_k)}y_k(0),\quad k=1,2,\dots,n \tag{4.6}$$

By taking inverse Laplace transform to both sides of (46), we have

$$y_k(t)=E_{\alpha_k,1}(\lambda_k t^{\alpha_k})y_k(0),\quad k=1,2,\dots,n \tag{4.7}$$

$$y(t)=\begin{pmatrix} E_{\alpha_1,1}(\lambda_1 t^{\alpha_1}) & 0 & \cdots & 0 \\ 0 & E_{\alpha_2,1}(\lambda_2 t^{\alpha_2}) & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & E_{\alpha_n,1}(\lambda_n t^{\alpha_n}) \end{pmatrix}\begin{pmatrix} y_1(0) \\ y_2(0) \\ \vdots \\ y_{n-1}(0) \\ y_n(0) \end{pmatrix} \tag{4.8}$$

So that the solution of (4.1) with the initial conditions (4.2) is

$$x(t)=P\begin{pmatrix} E_{\alpha_1,1}(\lambda_1 t^{\alpha_1}) & 0 & \cdots & 0 \\ 0 & E_{\alpha_2,1}(\lambda_2 t^{\alpha_2}) & \cdots & 0 \\ \vdots & \cdots & \ddots\ \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & E_{\alpha_n,1}(\lambda_n t^{\alpha_n}) \end{pmatrix}P^{-1}x(0) \tag{4.9}$$

Second, if the square matrix A have repeated eigenvales then we have invertible matrix P such that $PA\,P^{-1}=diag\,(J_1,J_2,\dots,J_l)$

Assume $y(t)=P^{-1}x(t)$ then the system (4.1) become

$$\begin{pmatrix} {}_0^CD_t^{\alpha_1}y_1(t) \\ {}_0^CD_t^{\alpha_2}y_2(t) \\ \vdots \\ {}_0^CD_t^{\alpha_n}y_n(t) \end{pmatrix}=\begin{pmatrix} J_{n_1} & 0 & \cdots & 0 \\ 0 & J_{n_2} & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{n_l} \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} \tag{4.10}$$

where $n=n_1+n_2+\dots+n_l$

In fact, the system (4.10) can be divided to l subsystems as follows

$$\begin{pmatrix} {}_0^C D_t^{\alpha_1} y_1(t) \\ {}_0^C D_t^{\alpha_2} y_2(t) \\ \vdots \\ {}_0^C D_t^{\alpha_{n_1}} y_{n_1}(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 1 & \cdots & 0 \\ 0 & \lambda_1 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_1 \end{pmatrix}_{n_1\times n_1} \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{n_1}(t) \end{pmatrix} \tag{4.11}$$

$$\begin{pmatrix} {}_0^C D_t^{\alpha_{n_1+1}} y_{n_1+1}(t) \\ {}_0^C D_t^{\alpha_{n_1+2}} y_{n_1+2}(t) \\ \vdots \\ {}_0^C D_t^{\alpha_{n_1+n_2}} y_{n_1+n_2}(t) \end{pmatrix} = \begin{pmatrix} \lambda_2 & 1 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_2 \end{pmatrix}_{n_2\times n_2} \begin{pmatrix} y_{n_1+1}(t) \\ y_{n_1+2}(t) \\ \vdots \\ y_{n_1+n_2}(t) \end{pmatrix} \tag{4.12}$$

$$\vdots$$

$$\begin{pmatrix} {}_0^C D_t^{\alpha_{n-n_l+1}} y_{n-n_l+1}(t) \\ {}_0^C D_t^{\alpha_{n-n_l+2}} y_{n-n_l+2}(t) \\ \vdots \\ {}_0^C D_t^{\alpha_n} y_n(t) \end{pmatrix} = \begin{pmatrix} \lambda_l & 1 & \cdots & 0 \\ 0 & \lambda_l & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_l \end{pmatrix}_{n_l\times n_l} \begin{pmatrix} y_{n-n_l+1}(t) \\ y_{n-n_l+2}(t) \\ \vdots \\ y_n(t) \end{pmatrix} \tag{4.13}$$

First, we find the solution of (4.11), by using Laplace transform one can get

$$\begin{pmatrix} s^{\alpha_1}\overline{y}_1(s) - s^{\alpha_1-1}y_1(0) \\ s^{\alpha_2}\overline{y}_2(s) - s^{\alpha_2-1}y_2(0) \\ \vdots \\ s^{\alpha_{n_1}}\overline{y}_{n_1}(s) - s^{\alpha_{n_1}-1}y_{n_1}(0) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 1 & \cdots & 0 \\ 0 & \lambda_1 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_1 \end{pmatrix}_{n_1\times n_1} \begin{pmatrix} \overline{y}_1(s) \\ \overline{y}_2(s) \\ \vdots \\ \overline{y}_{n_1}(s) \end{pmatrix} \tag{4.14}$$

by solving (4.14) backward, one can have

$$\overline{y}_{n_1}(s) = \frac{s^{\alpha_{n_1}-1}}{(s^{\alpha_{n_1}} - \lambda_1)} y_{n_1}(0)$$

$$\overline{y}_{n_1-1}(s) = \frac{s^{\alpha_{n_1-1}-1}}{(s^{\alpha_{n_1-1}} - \lambda_1)} y_{n_1-1}(0) + \frac{s^{\alpha_{n_1}-1}}{(s^{\alpha_{n_1-1}} - \lambda_1)(s^{\alpha_{n_1}} - \lambda_1)} y_{n_1}(0)$$

$$\overline{y}_{n_1-2}(s) = \frac{s^{\alpha_{n_1-2}-1}}{(s^{\alpha_{n_1-2}} - \lambda_1)} y_{n_1-2}(0) + \frac{s^{\alpha_{n_1-1}-1}}{(s^{\alpha_{n_1-2}} - \lambda_1)(s^{\alpha_{n_1-1}} - \lambda_1)} y_{n_1-1}(0) + \frac{s^{\alpha_{n_1}-1}}{(s^{\alpha_{n_1-2}} - \lambda_1)(s^{\alpha_{n_1-1}} - \lambda_1)(s^{\alpha_{n_1}} - \lambda_1)} y_{n_1}(0)$$

$$\vdots$$

$$\overline{y}_1(s) = \frac{s^{\alpha_1-1}}{(s^{\alpha_1} - \lambda_1)} y_1(0) + \frac{s^{\alpha_2-1}}{(s^{\alpha_1} - \lambda_1)(s^{\alpha_2} - \lambda_1)} y_2(0) + \cdots + \frac{s^{\alpha_{n_1}-1}}{(s^{\alpha_1} - \lambda_1)(s^{\alpha_2} - \lambda_1)\cdots(s^{\alpha_{n_1}} - \lambda_1)} y_{n_1}(0)$$

By taking Laplace inverse to the above equations, we have

$$y_{n_1}(t) = E_{\alpha_{n_1},1}(\lambda_1 t^{\alpha_{n_1}}) y_{n_1}(0)$$

$$y_{n_1-1}(t) = E_{\alpha_{n_1-1},1}(-\lambda_1 t^{\alpha_{n_1-1}}) y_{n_1-1}(0) + t^{\alpha_{n_1-1}-1} E_{\alpha_{n_1-1},\alpha_{n_1-1}}(-\lambda_1 t^{\alpha_{n_1-1}}) * E_{\alpha_{n_1},1}(-\lambda_1 t^{\alpha_{n_1}}) y_{n_1}(0)$$

$$\vdots \tag{4.15}$$

$$y_1(t) = E_{\alpha_1,1}(\lambda_1 t^{\alpha_1}) y_1(0) + \cdots + t^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1 t^{\alpha_1}) * \cdots * t^{\alpha_{n_1-1}-1} E_{\alpha_{n_1-1},\alpha_{n_1-1}}(\lambda_1 t^{\alpha_{n_1-1}}) * E_{\alpha_{n_1},1}(\lambda_1 t^{\alpha_{n_1}}) y_{n_1}(0)$$

5. Stability analysis of multiple different orders linear CFS:

In this section, we study the stability of the multiple different orders linear CFS in the form

$${}_0^C D_t^{\overline{\alpha}}\, x(t) = Ax(t), \quad x(t) \in R^n \tag{5.1}$$

Where A is $n\times n$ matrix, $\bar{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_n)$, and $0<\alpha_k<1$, $k=1,2,\dots,n$
with initial conditions

$x_k(0)=x_k,\quad k=1,2,\dots,n$ (5.2)

Theorem 5.1 If A is $n\times n$ matrix has different eigenvalues. Then the zero solution of (5.1)-(5.2) is stable if and only if $|\arg(\lambda_i)|\geq \text{Max}\left\{\frac{\alpha_1\pi}{2},\frac{\alpha_2\pi}{2},\dots,\frac{\alpha_n\pi}{2}\right\}$, $i=1,2,\dots,n$ and asymptotically stable if and only if $|\arg(\lambda_i)|>\text{Max}\left\{\frac{\alpha_1\pi}{2},\frac{\alpha_2\pi}{2},\dots,\frac{\alpha_n\pi}{2}\right\}$, $i=1,2,\dots,n$.

Proof: Since the square matrix A have different eigenvalues it is always possible to find invertible matrix P such that $P^{-1}A\,P=diag\,(\lambda_1,\lambda_2,\dots,\lambda_n)=\Lambda$. Let $y(t)=P^{-1}x(t)$ then we have

$${}_0^CD_t^{\bar{\alpha}}\; y(t)=\Lambda y(t) \tag{5.3}$$

$$y(0)=P^{-1}x(0) \tag{5.4}$$

We can rewrite the (5.3) system as

$${}_0^CD_t^{\alpha_k}\; y_k(t)=\lambda_k y_k(t)\;,\quad k=1,2,\dots,n \tag{5.5}$$

Each equation in system (5.3) can be study independently, so that the zero solution of the kth equation is stable (asymptotically stable) if and only if $|\arg(\lambda_k)|\geq\frac{\alpha_k\pi}{2}$ ($|\arg(\lambda_k)|>\frac{\alpha_k\pi}{2}$) for all $k=1,2,\dots,n$.

So, the zero solution of system (5.3)-(5.4) is stable (asymptotically stable) if and only if

$$|\arg(\lambda_i)|\geq \text{Max}\left\{\frac{\alpha_1\pi}{2},\frac{\alpha_2\pi}{2},\dots,\frac{\alpha_n\pi}{2}\right\},\; i=1,2,\dots,n \tag{5.6}$$

$$(|\arg(\lambda_i)|>\text{Max}\left\{\frac{\alpha_1\pi}{2},\frac{\alpha_2\pi}{2},\dots,\frac{\alpha_n\pi}{2}\right\},\; i=1,2,\dots,n) \tag{5.7}$$

Since the system (5.3)-(5.4) is equivalent to the system (5.1)-(5.2), one can have directly, that the system (5.1)-(5.2) is stable (asymptotically stable) if and only if the condition (5.6) (the condition (5.7)) is held.

Now, consider the following multiple different orders linear CFS

$${}_0^CD_t^{\bar{\alpha}}\; x(t)=Ax(t),\;\; x(t)\in R^n \tag{5.8}$$

where $A=P\;diag\,(J_1,J_2,\dots,J_l)P^{-1}$ is $n\times n$ matrix, $n=n_1+n_2+\cdots+n_l$ and $\bar{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_n)$, $\alpha_{n_k+1}=\alpha_{n_k+2}=\cdots=\alpha_{n_k+n_{k+1}}$, $n_0=1$, $k=1,2,\dots,l$, and $0<\alpha_k<1$, $k=1,2,\dots,n$ with initial conditions

$$x_k(0)=x_k,\quad k=1,2,\dots,n\;. \tag{5.9}$$

Theorem 5.2 The zero solution of (5.8)-(5.9) is stable if and only if

$|\arg(\lambda_i)|\geq \text{Max}\left\{\frac{\alpha_1\pi}{2},\frac{\alpha_2\pi}{2},\dots,\frac{\alpha_n\pi}{2}\right\}$, $i=1,2,\dots,n$ and asymptotically stable if and only if

$|\arg(\lambda_i)|>\text{Max}\left\{\frac{\alpha_1\pi}{2},\frac{\alpha_2\pi}{2},\dots,\frac{\alpha_n\pi}{2}\right\}$, $i=1,2,\dots,n$

Proof: Since the square matrix A have different eigenvalues it is always possible to find invertible matrix P such that $A=P\,diag\,(J_1,J_2,\dots,J_l)P^{-1}$. Let $y(t)=P^{-1}x(t)$ then we have

$${}_0^CD_t^{\bar{\alpha}}\; y(t)=diag\,(J_1,J_2,\dots,J_l)\,y(t) \tag{5.10}$$

With $y(0)=P^{-1}x(0)$ (5.11)

We can rewrite the above system as

$$\begin{pmatrix} {}_0^CD_t^{\alpha_1} y_1(t) \\ {}_0^CD_t^{\alpha_1} y_2(t) \\ \vdots \\ {}_0^CD_t^{\alpha_1} y_{n_1}(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 1 & \cdots & 0 \\ 0 & \lambda_1 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_1 \end{pmatrix}_{n_1\times n_1} \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{n_1}(t) \end{pmatrix} \qquad (5.12_1)$$

$$\begin{pmatrix} {}_0^CD_t^{\alpha_{n_1+1}} y_{n_1+1}(t) \\ {}_0^CD_t^{\alpha_{n_1+1}} y_{n_1+2}(t) \\ \vdots \\ {}_0^CD_t^{\alpha_{n_1+1}} y_{n_1+n_2}(t) \end{pmatrix} = \begin{pmatrix} \lambda_2 & 1 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_2 \end{pmatrix}_{n_2\times n_2} \begin{pmatrix} y_{n_1+1}(t) \\ y_{n_1+2}(t) \\ \vdots \\ y_{n_1+n_2}(t) \end{pmatrix} \qquad (5.12_2)$$

$$\vdots$$

$$\begin{pmatrix} {}_0^CD_t^{\alpha_n} y_{n-n_l+1}(t) \\ {}_0^CD_t^{\alpha_n} y_{n-n_l+2}(t) \\ \vdots \\ {}_0^CD_t^{\alpha_n} y_n(t) \end{pmatrix} = \begin{pmatrix} \lambda_l & 1 & \cdots & 0 \\ 0 & \lambda_l & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_l \end{pmatrix}_{n_l\times n_l} \begin{pmatrix} y_{n-n_l+1}(t) \\ y_{n-n_l+2}(t) \\ \vdots \\ y_n(t) \end{pmatrix} \qquad (5.12_l)$$

The above subsystems have same order fractional system and we can study the stability of each subsystems independently, so that the zero solution of the k[th] equation is stable (asymptotically stable) if and only if $\left|\arg(\lambda_{n_{k-1}+1})\right| \ge \frac{\alpha_{n_{k-1}+1}\pi}{2}$ ($\left|\arg(\lambda_{n_{k-1}+1})\right| > \frac{\alpha_{n_{k-1}+1}\pi}{2}$) for all $n_0 = 0,\ k = 1,2,\ldots,l$.

So, the zero solution of system (5.12) is stable (asymptotically stable) if and only if

$$\left|\arg(\lambda_{n_{k-1}+1})\right| \ge \text{Max}\left\{\frac{\alpha_{n_0+1}\pi}{2}, \frac{\alpha_{n_1+1}\pi}{2}, \ldots, \frac{\alpha_{n_{l-1}+1}\pi}{2}\right\},\ k = 1,2,\ldots,l, n_0 = 0, \qquad (5.13)$$

$$(\left|\arg(\lambda_{n_{k-1}+1})\right| > \text{Max}\left\{\frac{\alpha_{n_0+1}\pi}{2}, \frac{\alpha_{n_1+1}\pi}{2}, \ldots, \frac{\alpha_{n_{l-1}+1}\pi}{2}\right\},\ k = 1,2,\ldots,l, n_0 = 0,). \qquad (5.14)$$

Since the system (5.8)-(5.9) is equivalent to the system (5.10)-(5.11), one can have directly, that the system (5.8)-(5.9) is stable (asymptotically stable) if and only if the condition (5.6) (the condition (5.7)) is held.

Consider the following multiple deferent orders linear CFS

$${}_0^CD_t^{\overline{\alpha}}\ x(t) = Ax(t),\quad x(t) \in R^n \qquad (5.15)$$

where A is $n \times n$ matrix has different eigenvalues, $\overline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and $1 < \alpha_k < 2,\ k = 1,2,\ldots,n$ with initial conditions

$$x_k(0) = x_{1\,k}, x_k'(0) = x_{2\,k} \quad k = 1,2,\ldots,n \qquad (5.16)$$

Theorem 5.3 The zero solution of (5.15)-(5.16) is stable if and only if

$\left|\arg(\lambda_i)\right| \ge \text{Max}\left\{\frac{\alpha_1\pi}{2}, \frac{\alpha_2\pi}{2}, \ldots, \frac{\alpha_n\pi}{2}\right\},\ i = 1,2,\ldots,n$ and asymptotically stable if and only if

$\left|\arg(\lambda_i)\right| > \text{Max}\left\{\frac{\alpha_1\pi}{2}, \frac{\alpha_2\pi}{2}, \ldots, \frac{\alpha_n\pi}{2}\right\},\ i = 1,2,\ldots,n$.

We will omit the proof since it is similar to the proof of theorem 5.1.

Now, Consider the following multiple different orders linear CFS

$${}_0^CD_t^{\overline{\alpha}}\ x(t) = Ax(t),\quad x(t) \in R^n \qquad (5.17)$$

where $A = P\ diag\,(J_1, J_2, \dots, J_l)P^{-1}$ is $n\times n$ matrix, $n = n_1 + n_2 + \cdots + n_l$ and $\overline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_{n_k+1} = \alpha_{n_k+2} = \cdots = \alpha_{n_k+n_{k+1}}$, $n_0 = 0$, $k = 1,2,\dots,l$, and $1 < \alpha_k < 2$, $k = 1,2,\dots,n$ with initial conditions

$$x_k(0) = x_{1k}, x_k'(0) = x_{2k} \quad k = 1,2,\dots,n \tag{5.18}$$

Theorem 5.4 The zero solution of (5.17)-(5.18) is stable if and only if

$\left|\arg(\lambda_i)\right| \geq \text{Max}\left\{\frac{\alpha_1\pi}{2}, \frac{\alpha_2\pi}{2}, \dots, \frac{\alpha_n\pi}{2}\right\}$, $i = 1,2,\dots,n$ and asymptotically stable if and only if

$\left|\arg(\lambda_i)\right| > \text{Max}\left\{\frac{\alpha_1\pi}{2}, \frac{\alpha_2\pi}{2}, \dots, \frac{\alpha_n\pi}{2}\right\}$, $i = 1,2,\dots,n$.

We will omit the proof since it is similar to the proof of theorem 5.2.

Consider the following multiple deferent orders linear CFS

$${}_0^C D_t^{\overline{\alpha}}\, x(t) = Ax(t),\quad x(t) \in R^n \tag{5.20}$$

where A is $n\times n$ matrix has different eigenvalues, $\overline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, and $0 < \alpha_k < 1$, $k = 1,2,\dots,n_1$, $1 < \alpha_{n_1+k} < 2$, $k = 1,2,\dots,n_2$, and $n = n_1 + n_2$ with initial conditions

$$x_k(0) = x_{1k}, k = 1,2,\dots,n_1 \text{ and } x_k(0) = x_{1k}, x_k'(0) = x_{2k} \quad k = 1,2,\dots,n_2. \tag{5.21}$$

Theorem 5.5 The zero solution of (5.20)-(5.21) is stable if and only if

$\left|\arg(\lambda_k)\right| \geq \text{Max}\left\{\frac{\alpha_{n_1+1}\pi}{2}, \frac{\alpha_{n_1+1}\pi}{2}, \dots, \frac{\alpha_n\pi}{2}\right\}$, $k = 1,2,\dots,n$ and asymptotically stable if and only if

$\left|\arg(\lambda_k)\right| > \text{Max}\left\{\frac{\alpha_{n_1+1}\pi}{2}, \frac{\alpha_{n_1+1}\pi}{2}, \dots, \frac{\alpha_n\pi}{2}\right\}$, $k = 1,2,\dots,n$.

We will omit the proof since it is similar to the proof of theorem 5.1 and theorem 5.2.
Consider the following multiple different orders linear CFS

$${}_0^C D_t^{\overline{\alpha}}\, x(t) = Ax(t),\quad x(t) \in R^n \tag{5.22}$$

where $A = P\ diag\,(J_1, J_2, \dots, J_l)P^{-1}$ is $n\times n$ matrix, $n = n_1 + n_2 + \cdots + n_l$ and $\overline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_{n_k+1} = \alpha_{n_k+2} = \cdots = \alpha_{n_k+n_{k+1}}$, $n_0 = 0$, $k = 1,2,\dots,l$, and $0 < \alpha_{n_k} < 1$, $k = 1,2,\dots,m$, $1 < \alpha_{n_{l-k}} < 2$, $k = 0,1,2,\dots,m-1$ where m an integer number such that $0 < m < l$. with the initial conditions

$$x_k(0) = x_{1k}, k = 1,2,\dots,n_1 + n_2 + \cdots + n_m \tag{5.23}$$

$$\text{and}\quad x_k(0) = x_{1k}, x_k'(0) = x_{2k} \quad k = 1,2,\dots,n_{m+1} + n_{m+1} + \cdots + n_l\ . \tag{5.24}$$

Theorem 5.6 The zero solution of (5.22)-(5.24) is stable if and only if

$\left|\arg(\lambda_k)\right| \geq \text{Max}\left\{\frac{\alpha_{n_{m+1}}\pi}{2}, \frac{\alpha_{n_{m+2}}\pi}{2}, \dots, \frac{\alpha_{n_l}\pi}{2}\right\}$, $k = 1,2,\dots,n$ and asymptotically stable if and only if

$\left|\arg(\lambda_k)\right| > \text{Max}\left\{\frac{\alpha_{n_{m+1}}\pi}{2}, \frac{\alpha_{n_{m+2}}\pi}{2}, \dots, \frac{\alpha_{n_l}\pi}{2}\right\}$, $k = 1,2,\dots,n$.

We will omit the proof since it is similar to the proof of theorem 5.1 and theorem 5.2.

6. Illustrated examples:

This section is devoted to provide several illustrate examples to find the fundamental solution and studying the stability analysis of multiple deferent orders linear CFS**.**

Example 6.1 Find the fundamental solution and discuss the stability of the following fractional system

$$\begin{pmatrix} {}_0^cD_t^\alpha\, x_1(t) \\ {}_0^cD_t^\beta\, x_2(t) \\ {}_0^cD_t^\gamma\, x_3(t) \end{pmatrix} = \begin{pmatrix} -14 & -8 & -5 \\ 17 & 9 & 7 \\ 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix},$$ for any initial conditions $x_1(0)$, $x_2(0)$ and $x_3(0)$ with $0<\alpha<1, 0<\beta<1$ and $0<\gamma<1$.

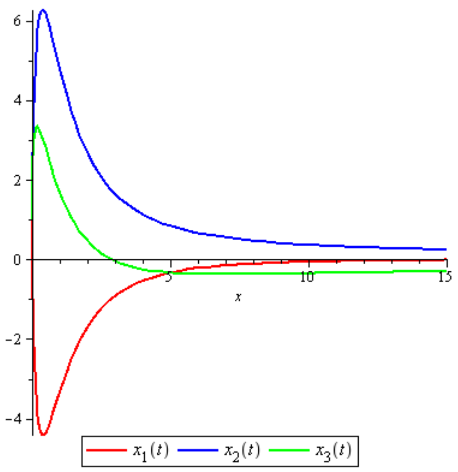
The eigenvalues of the system matrix are $\lambda_1=-1, \lambda_2=-2,$ and $\lambda_3=-3.$ So, we have $|\arg(\lambda_k)|=\pi > \text{Max}\left\{\frac{\alpha\pi}{2},\frac{\beta\pi}{2},\frac{\gamma\pi}{2}\right\}$ for all $0<\alpha<1, 0<\beta<1$,and $0<\gamma<1$. Clearly, from the above inequality the zero solution of this system is asymptotically stable. The solution of the given system is

$$x_1(t) = \left(-7x_1(0)-5x_2(0)-3x_3(0)\right)E_{\alpha,1}(-t^\alpha)+\left(3x_1(0)+2x_2(0)+x_3(0)\right)E_{\beta,1}(-2t^\beta)+ \left(5x_1(0)+3x_2(0)+2x_3(0)\right)E_{\gamma,1}(-3t^\gamma)$$

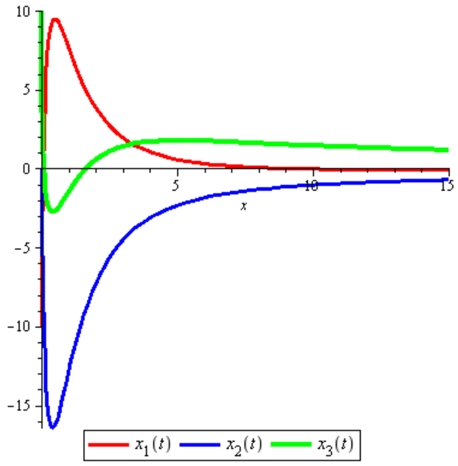
$$x_2(t) = \left(7x_1(0)+5x_2(0)+3x_3(0)\right)E_{\alpha,1}(-t^\alpha)+\left(3x_1(0)+2x_2(0)+x_3(0)\right)E_{\beta,1}(-2t^\beta)+ \left(-10x_1(0)-6x_2(0)-4x_3(0)\right)E_{\gamma,1}(-3t^\gamma)$$

$$x_3(t) = \left(7x_1(0)+5x_2(0)+3x_3(0)\right)E_{\alpha,1}(-t^\alpha)+\left(-12x_1(0)-8x_2(0)-4x_3(0)\right)E_{\beta,1}(-2t^\beta)+ \left(5x_1(0)+3x_2(0)+2x_3(0)\right)E_{\gamma,1}(-3t^\gamma)$$

In the following figures, we will show the behaviors the solution in case $\alpha=\frac{\sqrt{3}}{2}$, $\beta=\frac{1}{\sqrt{2}}$ and $\gamma=\frac{\sqrt{6}}{3}$ for different initial conditions.



The solution with the initial conditions
$x_1(0)=1,\ x_2(0)=1,\ x_2(0)=1$



The solution with the initial conditions
$x_1(0) = -10,\ x_2(0) = 0,\ x_2(0) = 10$

Example 6.2 Find the fundamental solution and discuss the stability of the following fractional system

$$\begin{pmatrix} {}_0^C D_t^{\alpha} x_1(t) \\ {}_0^C D_t^{\alpha} x_2(t) \\ {}_0^C D_t^{\beta} x_3(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix},$$ for any initial conditions $x_1(0),\ x_2(0)$ and $x_3(0)$, Also, $0 < \alpha < 1$,and $0 < \beta < 1$.

The eigenvalues of the system matrix are $\lambda_1 = -1, \lambda_2 = -1$, and $\lambda_3 = -2$

So, we have $|\arg(\lambda_k)| = \pi >$ Max$\left\{\frac{\alpha\pi}{2}, \frac{\beta\pi}{2}\right\}$for all $0 < \alpha < 1$,and $0 < \beta < 1$.

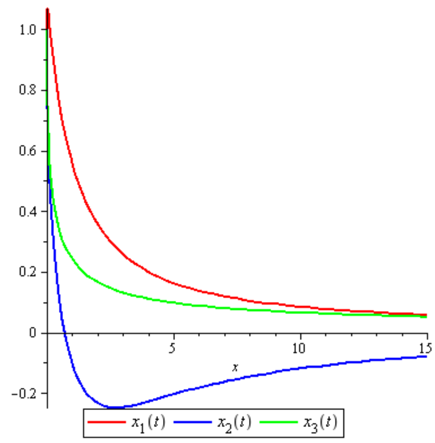
Clearly, from the above inequality the zero solution of this system is asymptotically stable. The solution of the given system is

$$x_1(t) = (x_1(0) + x_3(0))E_{\alpha,1}(-t^{\alpha}) - x_3(0)E_{\beta,1}(-2t^{\beta}),$$

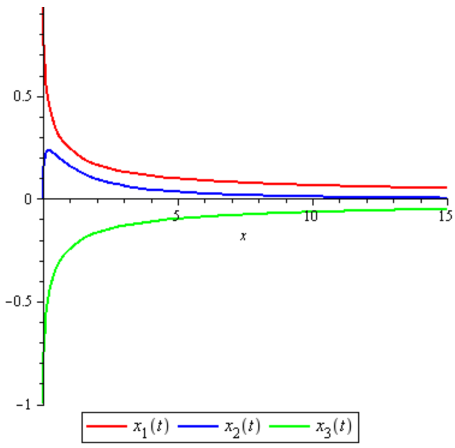
$$x_2(t) = (x_2(0) - x_3(0))E_{\alpha,1}(-t^{\alpha}) + (x_1(0) + x_3(0))\sum_{i=0}^{\infty} \frac{C_2^i (-1)^{(i-1)} t^{\alpha i}}{\Gamma(i\alpha + 1)} + x_3(0)E_{\beta,1}(-2t^{\beta})$$

$$x_3(t) = x_3(0)E_{\beta,1}(-2t^{\beta})$$

In the following figures, we will show the behaviors the solution in case $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = \frac{1}{\sqrt{3}}$ for different initial conditions.



The solution with the initial conditions
$x_1(0)=1,\ x_2(0)=1,\ x_2(0)=1$



The solution with the initial conditions
$x_1(0)=1,\ x_2(0)=0,\ x_2(0)=-1$

Example 6.3 Find the fundamental solution and discuss the stability of the following fractional system

$$\begin{pmatrix} {}_0^C D_t^{\alpha}\, x_1(t) \\ {}_0^C D_t^{\beta}\, x_2(t) \\ {}_0^C D_t^{\gamma}\, x_3(t) \end{pmatrix} = \begin{pmatrix} -14 & -8 & -5 \\ 17 & 9 & 7 \\ 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix},$$

for any initial conditions $x_1(0), x_1'(0),\ \ x_2(0), x_2'(0)$, and $x_3(0)$ with $1<\alpha<2, 1<\beta<2$ and $0<\gamma<1$.

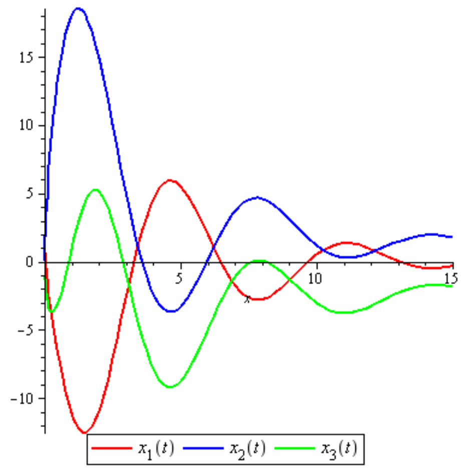
The eigenvalues of the system matrix are $\lambda_1=-1, \lambda_2=-2$, and $\lambda_3=-3$. So, we have $|\arg(\lambda_k)| = \pi > \text{Max}\left\{\frac{\alpha\pi}{2}, \frac{\beta\pi}{2}\right\}$ for all $1<\alpha<2$ and $1<\beta<2$. Clearly, from the above inequality the zero solution of this system is asymptotically stable. The solution of the given system is

$$\begin{aligned} x_1(t) = {} & \left(-7x_1(0)-5x_2(0)-3x_3(0)\right)E_{\alpha,1}(-t^{\alpha}) + \left(3x_1(0)+2x_2(0)+x_3(0)\right)E_{\beta,1}(-2t^{\beta}) + \\ & \left(5x_1(0)+3x_2(0)+2x_3(0)\right)E_{\gamma,1}(-3t^{\gamma}) - t(7x_1'(0)+5x_2'(0))E_{\alpha,2}(-t^{\alpha}) + \\ & t(3x_1'(0)+2x_2'(0))E_{\beta,2}(-2t^{\beta}) \end{aligned}$$

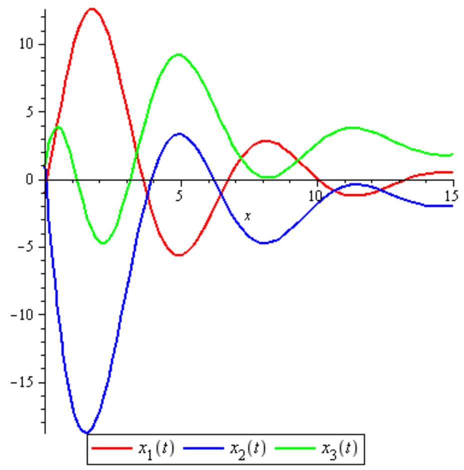
$$x_2(t) = \left(7x_1(0)+5x_2(0)+3x_3(0)\right)E_{\alpha,1}(-t^{\alpha}) + \left(3x_1(0)+2x_2(0)+x_3(0)\right)E_{\beta,1}(-2t^{\beta}) + \left(-10x_1(0)-6x_2(0)-4x_3(0)\right)E_{\gamma,1}(-3t^{\gamma}) + t\left(7x_1'(0)+5x_2'(0)\right)E_{\alpha,2}(-t^{\alpha}) + t\left(3x_1'(0)+2x_2'(0)\right)E_{\beta,2}(-2t^{\beta})$$

$$x_3(t) = \left(7x_1(0)+5x_2(0)+3x_3(0)\right)E_{\alpha,1}(-t^{\alpha}) + \left(-12x_1(0)-8x_2(0)-4x_3(0)\right)E_{\beta,1}(-2t^{\beta}) + \left(5x_1(0)+3x_2(0)+2x_3(0)\right)E_{\gamma,1}(-3t^{\gamma}) + t\left(7x_1'(0)+5x_2'(0)\right)E_{\alpha,2}(-t^{\alpha}) + t\left(-12x_1'(0)-8x_2'(0)\right)E_{\beta,2}(-2t^{\beta})$$

In the following figures, we will show the behaviors the solution in case $\alpha = \sqrt{3}$, $\beta = \sqrt{2}$ and $\gamma = \frac{1}{\sqrt{2}}$ for different initial conditions.



The solution with the initial conditions
$x_1(0)=1, x_1'(0)=1\, x_2(0)=0, x_2'(0)=2,\ x_3(0)=-1$



The solution with the initial conditions
$x_1(0)=-1, x_1'(0)=-1\, x_2(0)=1, x_2'(0)=-2,\ x_3(0)=1$

7. Conclusions:

In this paper, a new equivalent system to the higher order (CFS) has been introduced. This equivalent system has multiple different orders (CFDs). The fundamental solution for linear CFS with multiple different orders CFDs is derived. New criteria for studying the stability (asymptotic stability) of the linear multiple different orders CFDs are investigated. These criteria are depending on the position of eigenvalues of the matrix system in the complex plane. As well as, these criteria are considered as a generalized of the classical criteria which is used to study the stability of linear first ODEs. Also, these criteria are considered as generalized of the criteria theorem (2.10) and theorem (2.11). To show the behavior of the solution near the equilibrium, several examples are illustrated.

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