A Probabilistic Monte Carlo model for pricing discrete barrier options

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Abstract

We present an original Probabilistic Monte Carlo (PMC) model for pricing European discrete barrier options. Based on Monte Carlo simulation, the PMC model computes the probability of not crossing the barrier for knock-out options and crossing the barrier for knock-in options. This probability is then multiplied by an average sample discounted payoff of a plain vanilla option that has the same inputs as the barrier option but without barrier and to which we have applied a filter. We test the consistency of our model with an analytical solution (Merton 1973 and Reiner & Rubinstein 1991) adjusted for discretization by Broadie *et al.* (1997) and a naïve numerical model using Monte Carlo simulation presented by Clewlow & Strickland (2000). We show that the PMC model accurately price barrier equity options. Market participants in need of selecting a reliable and simple numerical method for pricing discrete barrier options will find our paper appealing. Moreover, the idea behind the method is so elementary that it can be applied to the pricing of complex derivatives involving barriers, easing the valuation step significantly.

Keywords: Monte Carlo Simulation, Option Pricing; Discrete Barrier Options

1. Introduction

Barrier options are cheaper than plain-vanilla options but have a higher risk of loss due to their barrier(s). With a cheap premium, barrier options have been attractive and traded over the counter since 1967 (Haug 2006). Merton (1973) has pioneered their pricing when they are monitored in continuous time. However, the most common traded barrier options are monitored in discrete time and their pricing is more challenging. A few solutions are analytical with a correction for continuity; the most popular solutions are numerical with lattices or Monte Carlo (MC) simulation. Nonetheless, Ahn *et al.* (1999) pointed out that lattice-based solutions inaccurately value discretely monitored barrier options. Our paper presents an original Probabilistic Monte Carlo (PMC) model for pricing European discrete barrier equity options. Based on MC simulation, the PMC model computes the probability of not crossing the barrier for knock-out options and crossing the barrier for knock-in options. This probability is then multiplied by an average sample discounted payoff of a plain vanilla option that has the same inputs as the barrier option but without barrier and to which we have applied a filter. We test the consistency of our model with the pricing of a European up-and-out discrete barrier equity option obtained with an analytical solution (Merton 1973, Reiner and Rubinstein 1991) adjusted for discretization by Broadie *et al.* (1997) and a naïve MC simulation presented by Clewlow and Strickland (2000). We apply the PMC model to the computation of 4 different types of single barrier options: up-and-out, up-and-in, down-and-in, and down-and-out. Section 2 will review the literature concerning the pricing of barrier options. Section 3 will present the methodology in five steps. Section 4 will present the results and section 5 will wrap up our findings.

2. Literature review

Barrier options have been very common on the over-the-counter market since the late 1980s, the main reason being that holders pay lower premiums than for plain-vanilla options. We may find barrier options traded on Exchanges (Easton *et al.* 2004) but they are no so common. They appeared first in 1991 in the U.S. on the CBOE (Chicago Board Options Exchange) and on AMEX (American Exchange) but with a limited success. They were introduced on ASX (Australian Stock Exchange) in 1998, having a better success on the Australian market.

The valuation of simple barrier options relies on closed-form solutions and numerical solutions. Merton (1973), Cox & Rubinstein (1985), and Rubinstein & Reiner (1991) proposed closed-form solutions. These analytical

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solutions assume that the barrier is monitored continuously but cannot be applied to barrier options where the crossing of the barrier is monitored discretely such as, for example, the options traded on ASX.

Numerical solutions use either lattice or MC simulation. Cox and Rubinstein (1985), Hudson (1992), Boyle and Lau (1994), Derman *et al*. (1995), Kat & Verdonk (1995), Ritchken (1995), and Cheuk and Vorst (1996) promoted lattice solutions. Derivatives week (1995) raises the problem of the lack of models to price discrete barrier options since many traded barrier options have a discrete monitoring, typically daily closing. Ahn *et al.* (1999) pointed out that lattice-based solutions inaccurately value discretely monitored barrier options. 'The cause is often non-linearity or discontinuity in the option payoff that occurs only in a small region'. These authors propose an adaptive mesh model that 'constructs small sections of fine high-resolution mesh in the critical areas and grafts them onto a base lattice with coarser time and price steps elsewhere.'

Broadie *et al.* (1997) showed 'that discrete barrier options can be priced with remarkable accuracy using continuous barrier formulas by applying a simple continuity correction to the barrier'. The approximation is remarkably accurate except in extremes cases with H very close to S0. While very convenient in practice, this analytical approximation is limited to single-barrier options and to the geometric Brownian motion process. Howison & Steinberg (2007) extended the applications of the 'continuity correction' presented by Broadie et al. (1997) to a wide variety of cases, using a matched asymptotic expansions approach.

In the 2000s, authors promoted pure jump and jump-diffusion asset pricing models based on Lévy processes. Boyarchenko & Levendorskii (2002) derived explicit formulas for barrier options of European type and touch-and-out options assuming that under a chosen equivalent martingale measure the stock returns follow a Lévy process. Petrella & Kou (2004) provided a comprehensive study of discrete single-barrier options in Merton's and Kou's jump-diffusion models in this framework based on the Spitzer's identity. Feng & Linetsky (2008) presented a Lévy process-based models solution to price discretely monitored single- and double-barrier options. 'The method involves a sequential evaluation of Hilbert transforms of the product of the Fourier transform of the value function at the previous barrier monitoring date and the characteristic function of the (Esscher transformed) Levy process.' Besides the promoters of the Lévy process, other authors proposed the application of regime-switching models to investigate option valuation problems, especially barrier options. The basic idea of regime-switching models is to allow the model parameters to change overtime according to a state process, which is usually modeled as a Markov chain. A key advantage of regime-switching models is to incorporate the impact of structural changes in economic conditions on the price dynamics. Guo (2001) applied a regime-switching model to the pricing of options. Elliott *et al.* (2014) presented a solution for pricing 'both European-style and American-style barrier options in a Markovian, regime-switching, Black-Scholes-Merton economy, where the price process of an underlying risky asset is governed by a Markovian, regime-switching, geometric Brownian motion.'

Finally, the trend of the 2010s has been to apply the SABR (Stochastic, Alpha, Beta, Rho) stochastic volatility model to the pricing of options, particularly barrier options. The SABR model attempts to capture the volatility smile. Tian *et al.* (2012) priced barrier and American options by the least squares MC method under the SABR model. Shiraya *et al.* (2012) provided a numerical model for pricing double barrier call options with discrete monitoring under Heston and λ-SABR models.

3. Methodology

We price a 6-month European up-and-out discrete barrier equity option with the Probabilistic Monte Carlo (PMC) model.

Our benchmarks are the analytical solution of Merton (1973) and Reiner & Rubinstein (1991) and a naïve numerical model using MC simulation presented by Clewlow & Strickland (2000). Our methodology presents 5 steps.

3.1 Step 1

By simulating the stock price using the following Brownian motion:

$$S_{t+1} = S_t . e^{\left(\left(r-\delta-0.5\sigma^2\right)dt+\sigma\sqrt{dt}.\varepsilon\right)} \quad (1),$$

we price a plain vanilla European call option with the exact same parameters as the barrier option but without barrier. The algorithm is borrowed from Clewlow & Strickland (2000). For *N* trajectories of the stock prices simulated over the life of the option with a maturity of T years (in our example T= 0.5 year = 125 steps, assuming a year equal to 250 business days), we count the number of times the stock price has reached maturity of the option without hitting the barrier, i.e. the probability P that the option is not knocked out during its life.

3.2 Step 2

We filter $c_{Ti} = (S_{Ti}-X)^+$, the option value of a plain vanilla call option at maturity of the ith trajectory, S_{Ti} being the stock price at maturity of the option and X the strike price: given H the up-and-out-barrier, if $(S_{Ti}-X)^+ \geq (H-X)$ then $c_{Ti} = 0$, since c_{Ti} cannot have a value equal to or higher than $(H-X)$ otherwise the barrier is activated $(S_{Ti} \geq H)$ and the option is knocked out.

3.3 Step 3

Simulating N trajectories of the stock price and applying the filter to the option value c_{Ti} of a plain vanilla call option, we obtain a sample of N possible values of the option at its maturity c_{TiAF} (AF = After Filtering). We randomly draw without replacement N.P option values of c_{TiAF} from the sample.

3.4 Step 4

The Probabilistic Monte Carlo (PMC) model computes c_{uo} the value of a T-year European equity up and out barrier option using equation 2:

$$c_{uo} = \left[\frac{P}{N.P} \sum_{i=1}^{N.P} c_{TiAF} + K.(1-P) \right] .e^{-r.T} \tag{2}$$

That we simplify:

$$c_{uo} = \left[\frac{1}{N} \sum_{i=1}^{N.P} c_{TiAF} + K.(1-P) \right] .e^{-r.T} \tag{3}$$

With r the continuous risk free rate over time T, c_{TiAF} the option values after filtration at step 2 and the draws without replacement at step 3, P the probability that the option is not knocked out and K the rebate offered by the barrier option when the option is knocked out.

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3.5 Step 5

We extend our reasoning to down-and-out, up-and-in and down-and-in European call options. We obtain table 1:

Table 1: Methodology of the PMC model applied to single barrier call options

	Up-and-out call options	Down-and-out call options	Up-and-in call options	Down-and-in call options
	Spot stock price S_o starts below the barrier level H and has to move up for the option to be knocked out: $S_o < H$ and choose $X < H$	Spot stock price S_o starts above the barrier level H and has to move down for the option to be knocked out: $S_o > H$	Spot stock price S_o starts below the barrier level H and has to move up for the option to be knocked in (activated): $S_o < H$	Spot stock price S_o starts above the barrier level H and has to move down for the option to be knocked in (activated): $S_o > H$
Option value c_{Ti} at maturity of the option	$c_{Ti} = (S_{Ti} - X)^+$ Option value c_{Ti} of a PLAIN VANILLA CALL option at maturity T of the option for a given simulated trajectory i among N trajectories	*idem*	*idem*	*idem*
Option value c_{TiAF} after filtration	if $c_{Ti} \geq (H-X)$ *then* $c_{TiAF} = 0$ Else: $c_{TiAF} = (S_{Ti} - X)^+$	if H > X *then* $c_{TiAF} = (S_{Ti} - X)^+ > (H-X)$ *else* $c_{TiAF} = 0$ Else: $c_{TiAF} = (S_{Ti} - X)^+$	$c_{TiAF} = (S_{Ti} - X)^+ = c_{Ti}$ *(no filtration)*	$c_{TiAF} = (S_{Ti} - X)^+ = c_{Ti}$ *(no filtration)*
Compute Probability P	Probability P that the option is not knocked out during the life of the option	Probability P that the option is not knocked out during the life of the option	Probability P that the option is not knocked in during the life of the option	Probability P that the option is not knocked in during the life of the option
Draws without replacement	Draw *(N.P)* values without replacement from the sample of N c_{TiAF}	*idem*	*idem*	*idem*
Option price c	$c_{uo} = \left[\begin{array}{l} \frac{1}{N}\sum_{i=1}^{N.P} c_{TiAF} \\ + K.(1-P) \end{array}\right] .e^{-r.T}$	$c_{do} = \left[\begin{array}{l} \frac{1}{N}\sum_{i=1}^{N.P} c_{TiAF} \\ + K.(1-P) \end{array}\right] .e^{-r.T}$	$c_{ui} = \left[\left[\frac{(1-P)}{(N-N.P)}\sum_{i=1}^{N-N.P} c_{TiAF}.\right] + K.P\right] .e^{-r.T}$	$c_{di} = \left[\left[\frac{(1-P)}{(N-N.P)}\sum_{i=1}^{N-N.P} c_{TiAF}.\right] + K.P\right] .e^{-r.T}$

The same reasoning applies to put options. We obtain table 2:

Table 2: Methodology of the PMC model applied to single barrier put options

	Up-and-out put options	Down-and-out put options	Up-and-in put options	Down-and-in put options
	Spot stock price S_o starts below the barrier level H and has to move up for the option to be knocked out: $S_o < H$	Spot stock price S_o starts above the barrier level H and has to move down for the option to be knocked out: $S_o > H$ and choose $X > H$	Spot stock price S_o starts below the barrier level H and has to move up for the option to be knocked in (activated): $S_o < H$	Spot stock price S_o starts above the barrier level H and has to move down for the option to be knocked in (activated): $S_o > H$
Option value c_{Ti} at maturity of the option	$c_{Ti} = (X - S_{Ti})^+$ Option value c_{Ti} of a PLAIN VANILLA PUT option at maturity T of the option for a given simulated trajectory i among N trajectories	*idem*	*idem*	*idem*
Option value c_{TiAF} after filtration	if $X > H$ *then* $c_{TiAF} = (X - S_{Ti})^+ > (X - H)$ *else* $c_{TiAF} = 0$ Else: $c_{TiAF} = (X - S_{Ti})^+$	if $c_{TiAF} = (X - S_{Ti})^+ > (X - H)$ *then* $c_{TiAF} = 0$ Else: $c_{TiAF} = (X - S_{Ti})^+$	$c_{TiAF} = (X - S_{Ti})^+ = c_{Ti}$ *(no filtration)*	$c_{TiAF} = (X - S_{Ti})^+ = c_{Ti}$ *(no filtration)*
Compute Probability P	Probability P that the option is not knocked out during the life of the option	Probability P that the option is not knocked out during the life of the option	Probability P that the option is not knocked in during the life of the option	Probability P that the option is not knocked in during the life of the option
Draws without replacement	Draw $(N.P)$ values without replacement from the sample of N c_{TiAF}	*idem*	*idem*	*idem*
Option price p	$p_{uo} = \left[\frac{1}{N} \sum_{i=1}^{N.P} c_{TiAF} + K.(1-P) \right] .e^{-r.T}$	$p_{do} = \left[\frac{1}{N} \sum_{i=1}^{N.P} c_{TiAF} + K.(1-P) \right] .e^{-r.T}$	$p_{ui} = \left[\left[\frac{(1-P)}{(N-N.P)} \sum_{i=1}^{N-N.P} c_{TiAF^*} \right] + K.P \right] .e^{-r.T}$	$p_{di} = \left[\left[\frac{(1-P)}{(N-N.P)} \sum_{i=1}^{N-N.P} c_{TiAF^*} \right] + K.P \right] .e^{-r.T}$

The PMC model is benchmarked to two models: 1) A naïve numerical model using Monte Carlo (MC) simulation to price barrier options presented by Clewlow & Strickland (2000); 2) The analytical solution developed by Merton (1973) and emphasized by Reiner & Rubinstein (1991). The numerical model assumes that the crossing of the barrier is monitored daily by dividing the maturity of the option of 6 months in 125 steps (1 year = 250 steps). The analytical solution assumes that the crossing of the barrier is monitored continuously. As mentioned in the literature review, Broadie *et al.* (1997) developed an approximation for continuity correction to price discrete barrier options. The correction shifts the barrier away from the underlying asset price that reduces the probability of hitting the barrier and that mimics the effect of discrete monitoring that lowers the probability of hitting the barrier compared to continuous monitoring. The barrier HAC, After Correction is equal to:

$$H_{AC} = H.e^{\beta\sigma\sqrt{\Delta t}} \qquad (4)$$

When $H>S_0$,

And is equal to: $$H_{AC} = H.e^{-\beta\sigma\sqrt{\Delta t}} \qquad (5)$$

When $H<S0$

With H the initial barrier,

$\beta = -\dfrac{\zeta\left(\frac{1}{2}\right)}{\sqrt{2\pi}} = 0.5826$, where ζ is the Riemann zeta function, Δt = 0.5/125 = 0.004 (option maturity of 6 months divided in 125 days -steps), σ = volatility (either 0.25 or 0.30 in our example).

4. Results

Tables 3 and 4 gather the results respectively for European single barrier call and put options. We price a barrier option where S = 100, T = 0.5, r =0.08, q = 0.04, *rebate* = 3, σ = 0.25 or 0.30. We compute the option premiums for different levels of strike price X and barrier H as suggested in Haug (2006). For MC and PMC solutions, we simulate 1,000 trajectories for each option valuation.

			Merton et al. solution Continuous recording		Merton et al. solution + Broadie correction Daily recording		Naive MC solution Daily recording		Average simulation time in seconds			PMC solution Daily recording		Average simulation time in seconds		
Type	X	H	sigma = 0.25	sigma = 0.30	sigma = 0.25 (a) Merton + Broadie	sigma = 0.30 (b) Merton + Broadie	sigma = 0.25 (c) Naive MC	sigma = 0.30 (d) Naive MC		Squared error $[(a) - (c)]^2$	Squared error $[(b) - (d)]^2$	sigma = 0.25 (e) PMC	sigma = 0.30 (f) PMC		Squared error $[(a) - (e)]^2$	Squared error $[(b) - (f)]^2$
cdo	90	95	9.0246	8.8334	9.8132	9.7965	9.5641	9.6787	0.2105	0.0621	0.0139	5.8898	5.6865	0.3670	15.3931	16.8921
cdo	100	95	6.7924	7.0255	7.2081	7.6198	7.1549	6.9984	0.2030	0.0028	0.3861	4.6693	4.3796	0.3120	6.4455	10.4989
cdo	110	95	4.8759	5.4137	5.0209	5.7111	4.8899	5.4362	0.1950	0.0172	0.0756	3.3080	4.0481	0.3200	2.9340	2.7656
cdo	90	100	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	0.0930	0.0000	0.0000	3.0000	3.0000	0.2960	0.0000	0.0000
cdo	100	100	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	0.0930	0.0000	0.0000	3.0000	3.0000	0.2960	0.0000	0.0000
cdo	110	100	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	0.0930	0.0000	0.0000	3.0000	3.0000	0.2960	0.0000	0.0000
cuo	90	105	2.6789	2.6341	2.6872	2.6069	2.5976	2.5664	0.4880	0.0080	0.0016	2.7415	2.7000	0.3195	0.0029	0.0087
cuo	100	105	2.3580	2.4389	2.2512	2.3240	2.1510	2.2487	0.4595	0.0100	0.0057	2.1287	2.3165	0.3040	0.0150	0.0001
cuo	110	105	2.3453	2.4315	2.2464	2.3075	2.1877	2.2367	0.4525	0.0034	0.0050	2.2021	2.2252	0.3120	0.0020	0.0068
cdi	90	95	7.7627	9.0093	6.9676	8.0393	3.6459	3.5806	0.2035	11.0337	19.8800	10.2549	11.5281	0.3120	10.8063	12.1717
cdi	100	95	4.0109	5.1370	3.5889	4.5389	0.7898	0.6774	0.1950	7.8350	14.9112	5.9031	6.7304	0.3115	5.3555	4.8027
cdi	110	95	2.0576	2.8517	1.9061	2.5474	0.7898	0.6457	0.2030	1.2461	3.6165	3.7068	4.8809	0.3275	3.2425	5.4452
cdi	90	100	13.8333	14.8816	12.5268	13.4179	8.2121	8.0057	0.1170	18.6166	29.2919	12.6791	15.8472	0.2965	0.0232	5.9015
cdi	100	100	7.8494	9.2045	6.9542	8.1314	0.1268	0.1614	0.1020	46.6134	63.5209	7.4937	12.5064	0.2960	0.2911	19.1406
cdi	110	100	3.9795	5.3043	3.4707	4.6096	0.1585	0.0894	0.1090	10.9707	20.4322	4.5654	6.8170	0.3045	1.1984	4.8726
cui	90	105	14.1112	15.2098	14.0964	15.2299	12.1605	12.5005	0.2105	3.7477	7.4496	11.0642	11.6041	0.3120	9.1942	13.1464
cui	100	105	8.4482	9.7278	8.5485	9.8357	5.0094	5.3417	0.1950	12.5252	20.1960	6.6981	7.1493	0.3045	3.4240	7.2167
cui	110	105	4.5910	5.8350	4.7035	5.9221	0.7254	0.7005	0.2030	15.8253	27.2651	3.4569	3.9312	0.2970	1.5540	3.9637
									RMSE:	2.6720	3.3916			RMSE:	1.8239	2.4362
									AVG RMSE:	3.0318				AVG RMSE:	2.1301	

Table 3: Premiums of European Barrier Call Options (S =100, T=0.5, r=0.08, q=0.04, rebate = 3) computed with 4 models (for MC and PMC solutions: 1,000 simulations)

			Merton et al. solution Continuous recording		Merton et al. solution + Broadie correction Daily recording		Naive MC solution Daily recording		Average simulation time in seconds			PMC solution Daily recording		Average simulation time in seconds		
Type	X	H	sigma = 0.25	sigma = 0.30	sigma = 0.25 (a) Merton + Broadie	sigma = 0.30 (b) Merton + Broadie	sigma = 0.25 (c) Naive MC	sigma = 0.30 (d) Naive MC		Squared error $[(a) - (c)]^2$	Squared error $[(b) - (d)]^2$	sigma = 0.25 (e) PMC	sigma = 0.30 (f) PMC		Squared error $[(a) - (e)]^2$	Squared error $[(b) - (f)]^2$
pdo	90	95	2.2798	2.4170	2.1575	2.2956	2.0350	2.2425	0.4840	0.0150	0.0028	2.0407	2.2713	0.3115	0.0136	0.0006
pdo	100	95	2.2947	2.4258	2.1857	2.3144	2.1574	2.2646	0.4760	0.0008	0.0025	2.2260	2.3146	0.3120	0.0016	0.0000
pdo	110	95	2.6252	2.6246	2.6319	2.6013	2.5498	2.4458	0.4835	0.0067	0.0242	2.8313	2.7852	0.3200	0.0398	0.0338
pdo	90	100	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	0.0930	0.0000	0.0000	3.0000	3.0000	0.3120	0.0000	0.0000
pdo	100	100	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	0.0930	0.0000	0.0000	3.0000	3.0000	0.3120	0.0000	0.0000
pdo	110	100	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	0.0930	0.0000	0.0000	3.0000	3.0000	0.3120	0.0000	0.0000
puo	90	105	3.7760	4.2293	3.8193	4.3792	3.5835	4.4614	0.4600	0.0556	0.0068	2.7533	2.8946	0.3200	1.1364	2.2040
puo	100	105	5.4932	5.8032	5.7966	6.2554	5.4966	5.8779	0.4520	0.0900	0.1425	3.6586	3.8920	0.3200	4.5710	5.5857
puo	110	105	7.5187	7.5649	8.1849	8.3981	8.4449	8.2022	0.4605	0.0676	0.0384	5.1286	5.4700	0.3120	9.3410	8.5738
pdi	90	95	2.9586	3.8769	3.0745	3.9915	0.7725	0.6607	1.1540	5.2992	11.0942	2.1701	3.3584	0.3195	0.8179	0.4008
pdi	100	95	6.5677	7.7989	6.6703	7.9034	4.8786	5.1644	0.4680	3.2102	7.5021	4.4645	5.9924	0.3200	4.8656	3.6519
pdi	110	95	11.9752	13.3078	11.9621	13.3242	11.7010	12.6445	0.4835	0.0682	0.4620	9.2133	9.5404	0.3200	7.5559	14.3171
pdi	90	100	2.2845	3.3328	2.4078	3.4509	0.1499	0.1066	0.2190	5.0981	11.1843	2.5901	3.9574	0.3120	0.0332	0.2565
pdi	100	100	5.9085	7.2636	6.0319	7.3816	1.1663	1.3578	0.2105	23.6741	36.2862	5.3388	6.7993	0.3040	0.4804	0.3391
pdi	110	100	11.6465	12.9713	11.7450	13.0711	10.2182	10.5818	0.2180	2.3311	6.1966	9.2165	11.1055	0.2960	6.3933	3.8636
pui	90	105	1.4653	2.0658	1.4154	1.9089	0.7350	0.6244	0.4680	0.4629	1.6499	2.5959	2.8700	0.3435	1.3936	0.9237
pui	100	105	3.3721	4.4226	3.0622	3.9634	0.7062	0.7177	0.4680	5.5507	10.5346	5.9681	6.1137	0.3275	8.4443	4.6238
pui	110	105	7.0846	8.3686	6.4119	7.5284	3.6323	3.5316	0.4760	7.7262	15.9744	9.4208	9.8662	0.3275	9.0535	5.4653
									RMSE:	1.7265	2.3700			RMSE:	1.7343	1.6707
									AVG RMSE:	2.0483				AVG RMSE:	1.7025	

Table 4: Premiums of European Barrier Put Options (S =100, T=0.5, r=0.08, q=0.04, rebate = 3) computed with 4 models (for MC and PMC solutions: 1,000 simulations)

Based on the criteria of RMSE (Root Mean Square Error), using as a benchmark the analytical solution developed by Merton (1973) and emphasized by Reiner & Rubinstein (1991), adjusted for discretization by Broadie *et al* (1997), the PMC model beats the naïve numerical model using MC simulation, whatever call or put options. However, the naïve MC model applies to the pricing of barrier put options with a volatility $\sigma = 0.25$ has a slightly lower RMSE (1.7265 versus 1.7343) than the PMC model.

Figures 1 and 2 illustrate call and put options prices when $\sigma = 0.30$. Overall, we observe the superior accuracy of the PMC model compare to the naïve MC model when pricing single barrier options.

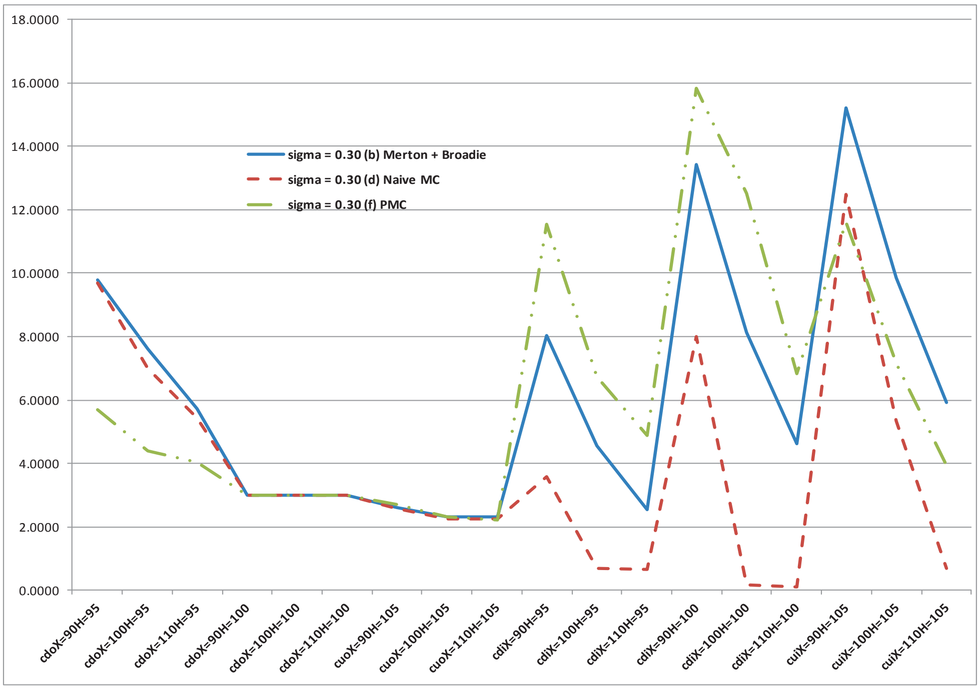

Figure 1. Premiums of European discrete barrier call options (S =100, T=0.5, r=0.08, q=0.04, σ =0.30) computed with PMC and MC models versus the benchmark Merton *et al*. (1973, 1991) + Broadie *et al*. (1997); for MC and PMC solutions: 1,000 simulations

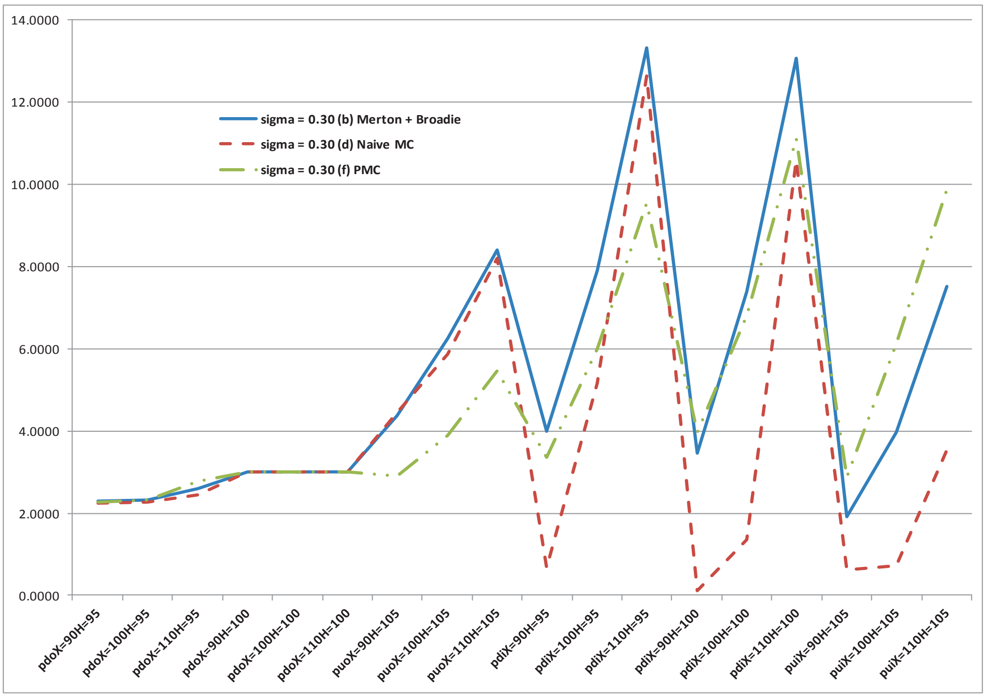

Figure 2. Premiums of European discrete barrier put options (S =100, T=0.5, r=0.08, q=0.04, σ = 0.30) computed with PMC and MC models versus the benchmark Merton *et al*. (1973, 1991) + Broadie *et al*. (1997); for MC and PMC solutions: 1,000 simulations

However, looking in details at Figures 1 and 2 and Tables 3 and 4, we observe that the naïve MC model outperforms the PMC model for down-and-out and up-and-out calls and puts (lower sum of squared errors). It is the opposite for down-and-in and up-and-in calls and puts where the PMC model performs best.

Investigating the power of convergence of the naïve MC model versus the PMC model, we choose an up-and-out call with H = 105, X = 100 and σ = 0.30. Based on Figure 3, we observe that the PMC model converges faster than the naïve MC model.

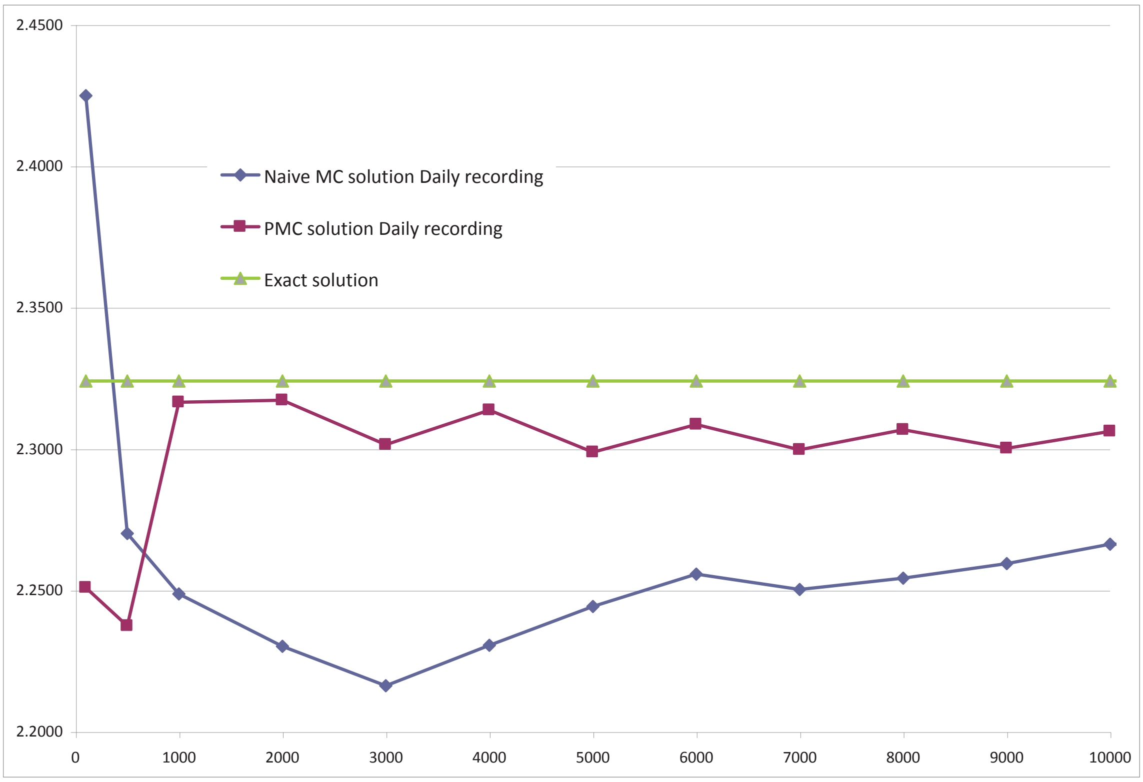

Figure 3: Illustration of the convergence for an up-and-out barrier call option (S=100, T=0.5, r=0.08, q=0.04, H=105, X=100, σ = 0.30) computed with PMC and MC models versus the benchmark Merton *et al*. (1973, 1991) + Broadie *et al*. (1997) that we call exact solution. We increase the number of simulations from 100 to 10,000.

5. Conclusion

We present an original Probabilistic Monte Carlo (PMC) model based on Monte Carlo (MC) simulation that prices discrete barrier options: we call it probabilistic since it computes the probability of not crossing the barrier for knock-out options and crossing the barrier for knock-in options. This probability is then multiplied by an average sample discounted payoff of a plain vanilla option that has the same inputs as the barrier option but without barrier and to which we have applied a filter. We test the consistency of our model with an analytical solution (Merton 1973 and Reiner & Rubinstein 1991) adjusted for discretization by Broadie *et al.* (1997) and a naïve numerical model using MC simulation. Overall, based on call and put options and all types of single-barrier European options (up-and-out, down-and-out, up-and-in and down-and-in) and based on the criteria of RMSE, the PMC model is superior to the naïve MC simulation. We show through an example that it also converges faster towards the exact solution. However, the naïve MC simulation offers better results for down-and-out and up-and-out calls and puts (lower sum of squared errors). It is the opposite for down-and-in and up-and-in calls and puts where the PMC model performs best.

Further works will focus on discrete double-barrier options pricing with the PMC model and will investigate variance reduction techniques applied to the PMC model.

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