Extension of Transmuted Halfnormal Distribution Properties and Application

Folounso.O.P.[1] Akomolafe.A.A.[2] Ajiboye.A.S.[2]
1.Research student, Department of Statistics, Federal University of Technology, Akure, Nigeria
2.Lecturer, Department of Statistics, Federal University of Technology, Akure, Nigeria
[akomolafe01@yahoo.com][asajiboye@futa.edu.ng]

Abstract
This research extended the HalfNormal distribution to Transmuted HalfNormal Distribution (THND) and it generalizes the classical HalfNormal by expanding its scope to modeling high-class random process that cannot be easily modeled with existing probability distribution. The THND was compounded and its statistical properties were obtained such as moment, hazard function, reliability, probability density and cumulative distribution function. This hybrid model can be used to capture non-normal data with highly skewed, heavily tailed and leptokurtic distribution.
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Introduction

In many applied sciences such as medicine, engineering and finance, modeling and analyzing lifetime data is imperative. Many lifetime distributions have been used to model such type of phenomena from which the data was generated. In statistics, it is very crucial to understand the underlying probability distribution or phenomena which the available data followed before deciding the appropriate statistical test to be employed in analyzing the data. In this era of advancement in science and technology, many processes have become complex to the extents that it becomes difficult to accurately model the stochastic behavior of such processes using those classical probability distribution. Since the quality of the procedures used in a statistical analysis depends majorly on the assumed probability model, then there is need to construct new probability distribution that can capture the pattern of such processes and use such distribution to construct necessary statistical test, confidence interval and making prediction regarding the subsequent behavior of such phenomena. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models (existing distribution), hence, the need for mixing family of distribution,.

HalfNormal distribution was used to model brownian movement and can also be used in the modeling measurement data and lifetime data. Let $X \sim N(0,\sigma^2)$, then $Y = |X|$ follows half normal distribution. The Half-normal is a fold at the mean of an ordinary normal distribution with mean zero, where σ is the scale parameter. By obtaining the transmuted version of HalfNormal distribution, the resulting hybrid distribution is called transmuted HalfNormal distribution. This proposed mixture distribution has more number of parameters as compared to its respective parent distributions (HalfNormal) and it has wider applicability exceeding modeling particular size but in modeling many stochastic processes and stochastic phenomena which cannot be easily modeled by one parameter probability density (parent distribution) such as disease growth, epidemiological studies of disease, buying behavior of consumers towards certain economic product etc. It is in this view that this research is structured to propose new hybrid distributions (Transmuted HalfNormal) with a view to studying its properties and application to real life data to reflect the flexibility, stability and consistency of this hybrid model as compared to its parent distributions.

2. Literature Review

Many researchers have worked on aspect of compounding two or more probability distributions to obtain family of hybrid distributions which are more efficient than their parent distributions due to addition of more parameters which increase the flexibility of the mixture of distributions in tracking many random phenomena which cannot be easily modeled by their parent distributions. Many authors have also worked on compounding beta distribution with other distributions. The beta family of distribution became popular some years back, which include beta-normal (Eugene &Famoye, 2002); beta-Gumbel (Nadarajah & Kotz, 2004), beta-Weibull (Famoye, Lee& Olugbenga, 2005), Beta-exponential (Nadarajah &Kotz, 2006); beta-Rayleigh (Akinsete &Lowe, 2009); Beta Fre´chet by Nadarajah, and Gupta(2004), beta-halfnormal, Akomolafe AA and Maradesa A (2017), beta-Gamma, beta-f, beta-t, beta-beta, beta-modified weibull, beta-nakagami among others.

Cordeiro and de Castro discussed moment generating function of for generated beta distribution. When we consider σ as a random variable, the bayes convert our belief about the parameter σ of Halfnormal distribution (before seeing data) into posterior probability, P($\sigma|X|$) , by using the likelihood function P(X| σ) .The maximum a-posteriori (MAP) estimate is defined as:

$$\sigma = \ argmax\, P(\sigma|X) = \ argmax_{\sigma}\ \frac{P(\sigma).P(X|\sigma)}{P(X)}\ .$$

Having used the Maximum A Posteriori Estimation and Maximum Likelihood Estimation, the mathematical approach suggested that the Maximum A Posteriori Estimation is a better fit as compared to the maximum likelihood estimation.

Lifetime data can be modeled using several existing distributions. However, some of these lifetime data do not follow these existing distributions or are inappropriately described by them. Hence, the need to develop distributions that could better describes some of these phenomena and provide greater flexibility in the modeling of lifetime data than the baseline distributions.

3. Methodology

A random variable X is said to have a transmuted distribution if its distribution function is given by

$$f(x) = (1+\lambda)g(x) - 2\lambda g(x) \qquad |\lambda| \le 1$$

$$= g(x) + \lambda g(x) - 2\lambda g(x) \qquad (1)$$

And the density function is given by

$$F(x) = (1+\lambda)G(x) - \lambda(G(x))^2 \qquad (2)$$

$$F(x) = G(x)[(1+\lambda) - \lambda G(x)] \qquad (3)$$

Where $g(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$ and $G(x) = \text{erf}(\frac{x}{\sigma\sqrt{2}})$ are the cumulative distribution and density function of the parent distribution (halfnormal). The parent distribution can be any probability distribution from which we want to obtain its transmuted version.

From (1) and (3), the pdf and cdf of the Transmuted halfnormal can be obtained shown by (4) and (5) respectively

$$f_{THND}(\text{x}, \sigma, \lambda) = \left[(1+\lambda) - 2\lambda \text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}} \qquad (4)$$

We can expand (4) to get (5)

$$= \frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}} + \lambda\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}} - 2\lambda\, \text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right) \qquad (5)$$

$$F_{THND}(\text{x}, \sigma, \lambda) = (1+\lambda)\left(\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right) - \lambda(\text{erf}(\frac{x}{\sigma\sqrt{2}}))^2 \qquad (6)$$

3.1 Estimation of Parameter (Maximum Likelihood Method)

Using the maximum likelihood estimation technique, we estimate the parameter of the hybrid distribution. Given that $f_{THND}(x;\lambda,\sigma)$ is the pdf of THND, then the likelihood function is given by

$$\text{L}(f_{THND}(x;x;\lambda,\sigma) = \ \prod_{i=1}^{n}\left[(1+\lambda) - 2\lambda\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}} \qquad (7)$$

By taking the natural logarithm of (5) and obtaining the derivatives with respect to each of the parameters, we can obtain the estimate of those parameters when setting the derivative to zero and solve the equations.

3.2 Investigation of Some Properties of the Distribution

Certain descriptive properties of the proposed distribution will be verified using mathematical and graphical approach and other methods such as classical method of moment generating function and others. Among the properties to be investigated are:

Moment

$$\boldsymbol{Ex^r} = \ \int_{\mathbf{0}}^{\infty}\boldsymbol{x^r}f_{THND}(x;\lambda,\sigma)\,\text{dx} \qquad \textbf{(8)}$$

$$= (1+\lambda)\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\int_{\mathbf{0}}^{\infty}\boldsymbol{x^r}e^{-\frac{x^2}{2\sigma^2}} - \mathbf{2}\lambda\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\int_{0}^{\infty}\boldsymbol{x^r}\ \text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)e^{-\frac{x^2}{2\sigma^2}}dx \quad (9)$$

erf (x) can expressed in term of confluence hypergeometric function of the first kind

$$= \frac{2x}{\sqrt{\pi}}\, M\left(\frac{1}{2}, \frac{3}{2}, -x^2\right) \qquad = \frac{2x}{\sqrt{\pi}}\ e^{-x^2}M(\,1, \frac{3}{2}\,, x^2)$$

Therefore $\frac{d^n}{dx^n} \ = \ (-1)^n\ \frac{2}{\sqrt{\pi}}\ H_{n-1}\,(x)\, e^{-x^2}$; *where* $H_{n-1}\,(x)$ *is Hermite polynomial*

erf (x) $=\pi^{-1/2}\gamma\left(\frac{1}{2}, x^2\right)$, *where* $\gamma\left(\frac{1}{2}, x^2\right)$ *could be view as incomplete gamma function*, it can therefore be expressed by the maclaurin series in

$$= \frac{2}{\sqrt{\pi}}\ \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)} \qquad (10)$$

By substituting for the error function as a maclaurin series in (10), we obtain (11)

$$= (1+\lambda)\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\int_0^\infty x^r e^{-\frac{x^2}{2\sigma^2}}dx - 2\lambda\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\int_0^\infty x^r \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)} e^{-\frac{x^2}{2\sigma^2}}dx \quad (11)$$

$$= (1+\lambda)\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\int_0^\infty x^r e^{-\frac{x^2}{2\sigma^2}}dx - 2\lambda\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\int_0^\infty x^r x^{2n+1} e^{-\frac{x^2}{2\sigma^2}}dx \quad (12)$$

$$= (1+\lambda)\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\int_0^\infty x^r e^{-\frac{x^2}{2\sigma^2}}dx - 2\lambda\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\int_0^\infty x^{2nr+r} e^{-\frac{x^2}{2\sigma^2}}dx \quad (13)$$

Let $y = \frac{x^2}{2\sigma^2}$; $x = \sigma\sqrt{2y}$; $\frac{dy}{dx} = \frac{2x}{2\sigma^2}$; $2\sigma^2 dy = 2x dx$; $\frac{\sigma^2 dy}{x} = dx$

$$= (1+\lambda)\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\int_0^\infty x^r e^{-y}\frac{\sigma^2 dy}{x} - 2\lambda\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\int_0^\infty x^{2nr+r} e^{-y}\frac{\sigma^2 dy}{x} \quad (14)$$

$$= (1+\lambda)\frac{\sigma^2\sqrt{2}}{\sigma\sqrt{\pi}}\int_0^\infty x^{r-1} e^{-y}dy - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\int_0^\infty x^{2nr+r-1} e^{-y}dy \quad (15)$$

$$= (1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\int_0^\infty \left(\sigma\sqrt{2y}\right)^{r-1} e^{-y}dy - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\int_0^\infty \left(\sigma\sqrt{2y}\right)^{2nr+r-1} e^{-y}dy \quad (16)$$

$$= (1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\left(\sigma\sqrt{2}\right)^{r-1}\int_0^\infty y^{\frac{r}{2}-\frac{1}{2}} e^{-y}dy - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2nr+r-1}\int_0^\infty y^{\frac{nr}{2}+\frac{r}{2}-\frac{1}{2}} e^{-y}dy \quad (17)$$

$$= (1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^r}{\sqrt{\pi}}\int_0^\infty y^{\frac{r}{2}-\frac{1}{2}} e^{-y}dy - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2nr+r-1}\int_0^\infty y^{\frac{nr}{2}+\frac{r}{2}-\frac{1}{2}} e^{-y}dy \quad (18)$$

$$\boldsymbol{Ex^r} = (1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^r}{\sqrt{\pi}}\Gamma\left(\frac{r+1}{2}\right) - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2nr+r-1}\Gamma\left(\frac{nr+r+1}{2}\right) \quad (19)$$

The (19) above represent the moment of Transmuted Halfnormal Distribution

$$E(x) = (1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\Gamma\left(\frac{1+1}{2}\right) - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n+1-1}\Gamma\left(\frac{n+1+1}{2}\right) \quad r = 1; \quad (20)$$

$$E(x) = (1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right) \quad r = 1$$

$$\boldsymbol{Ex^2} = (1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^2}{\sqrt{\pi}}\Gamma\left(\frac{2+1}{2}\right) - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+2-1}\Gamma\left(\frac{2n+2+1}{2}\right) \quad r = 2; \quad (21)$$

$$\boldsymbol{Ex^2} = (1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right); r = 2; \quad (22)$$

$$\boldsymbol{Ex^2} = (1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^2}{2} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right); r = 2 \quad (23)$$

$$\boldsymbol{Ex^3} = (1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^3}{\sqrt{\pi}}\Gamma\left(\frac{3+1}{2}\right) - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{6n+3-1}\Gamma\left(\frac{3n+3+1}{2}\right) \quad r = 3; \quad (24)$$

$$\boldsymbol{Ex^3} = (1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^3}{\sqrt{\pi}} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{6n+2}\Gamma\left(\frac{3n+4}{2}\right) \quad r = 3 \quad (25)$$

$$\boldsymbol{Ex^4} = (1+\lambda)\frac{3\left(\sigma\sqrt{2}\right)^4}{4} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{8n+3}\Gamma\left(\frac{nr+5}{2}\right) \quad r = 4 \quad (26)$$

Moment about the Mean

Variance $(\mu_2) = \boldsymbol{Ex^2} - \left(E(x)\right)^2$

$$= (1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right) - \left[(1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\right]^2 \quad (27)$$

$\mu_3 = E(x-\mu)^3$

By applying Binomial Expansion it gives

$$= (1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^3}{\sqrt{\pi}} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{6n+2}\Gamma\left(\frac{3n+4}{2}\right) - 3(1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\left((1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^2}{2} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right) + 2\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\Gamma\left(\frac{1+1}{2}\right) - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n+1-1}\Gamma\left(\frac{n+1+1}{2}\right)\right)^3\right) \quad (28)$$

$$= E(x-\mu)^4 = \binom{4}{0}.x^0.(-\mu)^{4-0} + \binom{4}{1}.x^1.(-\mu)^{4-1} + \binom{4}{2}x^2.(-\mu)^{4-2} + \binom{4}{3}.x^3.(-\mu)^{4-3} + \binom{4}{4}.x^4.(-\mu)^{4-4}$$

$\mu_4 = Ex^4 - 4\mu Ex^3 + 6\mu^2 Ex^2 - 3(Ex)^4$

$$= (1+\lambda)\frac{3\left(\sigma\sqrt{2}\right)^4}{4} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{8n+3}\Gamma\left(\frac{nr+5}{2}\right) - 4\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - \right.$$

$$\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\Big)(1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^3}{\sqrt{\pi}}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{6n+2}\Gamma\left(\frac{3n+4}{2}\right)+6\Big((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\Big)^2\left((1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^2}{2}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right)-3\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\right)^4\right) \quad (29)$$

Skewness

This is obtained as

$$\boldsymbol{\gamma_1(x)} = \frac{\boldsymbol{(\mu_3)^2}}{\boldsymbol{(\mu_2)^3}} = \frac{\left((1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^3}{\sqrt{\pi}}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{6n+2}\Gamma\left(\frac{3n+4}{2}\right)-3(1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\left((1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^2}{2}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right)+2\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\Gamma\left(\frac{1+1}{2}\right)-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n+1-1}\Gamma\left(\frac{n+1+1}{2}\right)\right)^3\right)\right)^2}{\left((1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right)-\left[(1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\right]^2\right)^3} \quad \mathbf{(30)}$$

Kurtosis

This was derived as shown below

$$\boldsymbol{\gamma_2(x)}=\frac{\boldsymbol{\mu_4}}{\boldsymbol{(\mu_2)^2}}$$

$$= \frac{(1+\lambda)\frac{3\left(\sigma\sqrt{2}\right)^4}{4}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{8n+3}\Gamma\left(\frac{nr+5}{2}\right)-4\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\right)(1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^3}{\sqrt{\pi}}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{6n+2}\Gamma\left(\frac{3n+4}{2}\right)+6\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\right)^2\left((1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^2}{2}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right)-3\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\right)^4\right)}{\left((1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right)-\left[(1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}-\mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\right]^2\right)^2}$$

Hazard Rate Function

$$=\boldsymbol{h}(\boldsymbol{x};\lambda,\sigma) = \frac{[\mathbf{1}+\lambda-2\lambda\mathrm{G}(x)]\boldsymbol{g}(\boldsymbol{x})}{[\mathbf{1}-\boldsymbol{G}(\boldsymbol{x})][\mathbf{1}-\lambda\mathrm{G}(x)]} = \boldsymbol{h_G}(\boldsymbol{x};\sigma)\frac{[\mathbf{1}+\lambda-2\lambda\mathrm{G}(x)]}{\mathbf{1}-\lambda\mathrm{G}(x)}$$

where $\boldsymbol{h_G}(\boldsymbol{x};\sigma)$ is the baseline (parent) distribution.

$$=\boldsymbol{h}(\boldsymbol{x};\lambda,\sigma) = \frac{\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}}{\mathbf{1}-\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)} \cdot \frac{\mathbf{1}+\lambda-2\lambda\,.\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}{\mathbf{1}-\lambda\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)} = \frac{\left[\mathbf{1}+\lambda-2\lambda\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}}{\left[\mathbf{1}-\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\left[\mathbf{1}-\lambda\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]} \quad \mathbf{(31)}$$

$$=\frac{\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}+\lambda\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}-2\lambda\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}{\left[\mathbf{1}-\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\left[\mathbf{1}-\lambda\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]} \quad \mathbf{(32)}$$

Order Statistics

According to Marcelo Bourguignon et al (2016), the order statistics for transmuted family can be obtained by (30).

$= f_{r.n}(x;\lambda,\sigma) = \frac{1}{B(r,n-r+1)}F(x)^{r-1}[1-F(x)]^{n-r}$ (30) which can be view (30) when defining it in term of transmuted family using the required baseline (parent) distribution.

$$= \frac{1}{B(r,n-r+1)}\sum_{j=0}^{n-r}(-1)^j\binom{n-j}{j}\left[(1+\lambda)G(x;\sigma)-\lambda(\mathrm{G}(x;\sigma))^2\right]^{r-1+j}.\left[1+\lambda-2\lambda\mathrm{G}(x;\sigma)\right]g(x)$$

By substituting the pdf and cdf of parent distribution we obtain the order statistics of transmuted halfnormal distribution (THND) .

$$= \frac{1}{B(r,n-r+1)}\sum_{j=0}^{n-r}(-1)^j\binom{n-j}{j}\left[(1+\lambda)\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)-\left(\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right)^2\right]^{r-1+j}\left[1+\lambda-2\lambda\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}} \quad (33)$$

The k[th] order moment of $X_{r,n}$ is displayed below:

$$= \int_0^\infty x^k \left[\left(1 + \lambda G(x;\sigma) - \lambda\left(G(x)\right)^2\right)\right]^{r+j-1} [1 + \lambda - 2\lambda G(x;\sigma)] g(x;\sigma) \quad (34)$$

Equation (34) can be simplify by binomial expansion as (35)

$$= \frac{1}{B(r,n-r+1)} \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} J \quad (35)$$

$J = \int_0^1 \frac{1}{G(x;\sigma)} . (1 + \lambda - 2\lambda t) \ . \ [(1+\lambda)t - \lambda t^2]^{r+j-1}$, substitute for G(x; σ) in J to obtain the k[th] order moment of Transmuted Half-Normal.

$J = \int_0^1 \frac{1}{\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)} . (1 + \lambda - 2\lambda t) \ . \ [(1+\lambda)t - \lambda t^2]^{r+j-1}$. Put for J in equation(35).

$$= \frac{1}{B(r,n-r+1)} \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} \int_0^1 \frac{1}{\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)} . (1 + \lambda - 2\lambda t) \ . \ [(1+\lambda)t - \lambda t^2]^{r+j-1} \quad (36)$$

J can be evaluated using numerical integration.

3.3 Maximum Likelihood Method for THND

We consider the estimation of parameters of Transmuted family from samples by maximum likelihood. Let $x_1, x_2, \ldots x_n$ be observed values from this family with parameter$\theta = \lambda, \ \sigma$.

$$Lf_{THND}(x;\lambda,\sigma) = \frac{2^{\frac{n}{2}}}{\pi^{\frac{n}{2}}\sigma^n} e^{-\sum_{i=1}^{n} \frac{x^2}{2\sigma^2}} + \sum_{i=1}^{n} \left[1 + \lambda - 2\lambda erf\left(\frac{x}{\sigma\sqrt{2}}\right)\right] \quad (37)$$

$$lnLf_{THND}(x;\lambda,\sigma) = n\left(\frac{1}{2}ln2 - \frac{1}{2}ln\pi - ln\sigma\right) + \sum_{i=1}^{n} ln\left[1 + \lambda - 2\lambda \boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right] \quad (38)$$

$$\frac{\partial lnLf_{THND}(x;\lambda,\sigma)}{\partial\lambda} = \sum_{i=0}^{n} \left(\frac{1 - 2\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}{1 + \lambda - 2\lambda \boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}\right) \quad (39)$$

$$\frac{\partial lnLf_{THND}(x;\lambda,\sigma)}{\partial\sigma} = -\frac{n}{\sigma} + \sum_{i=0}^{n} \left(\frac{\frac{2\lambda\sqrt{2}(\sigma^2 - x^2)}{\sigma^4\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}}}{1 + \lambda - 2\lambda \boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}\right) \quad (40)$$

To obtain the information matrix, we follow this procedure and we obtain $u_{\lambda\lambda}$, $u_{\lambda\sigma}$ *and* $u_{\sigma\sigma}$, we can form information matrix called J (θ) and the likelihood of the two distributions can be used to test for the goodness of fit to determine if transmuted distribution is superior to the baseline distribution based on the available data. These equations can be solved numerically using Newton-raphson algorithm and the information matrix J (θ) is given by (36).

4.1 Fitting Transmuted HalfNormal to GLO-DATA Plan Data

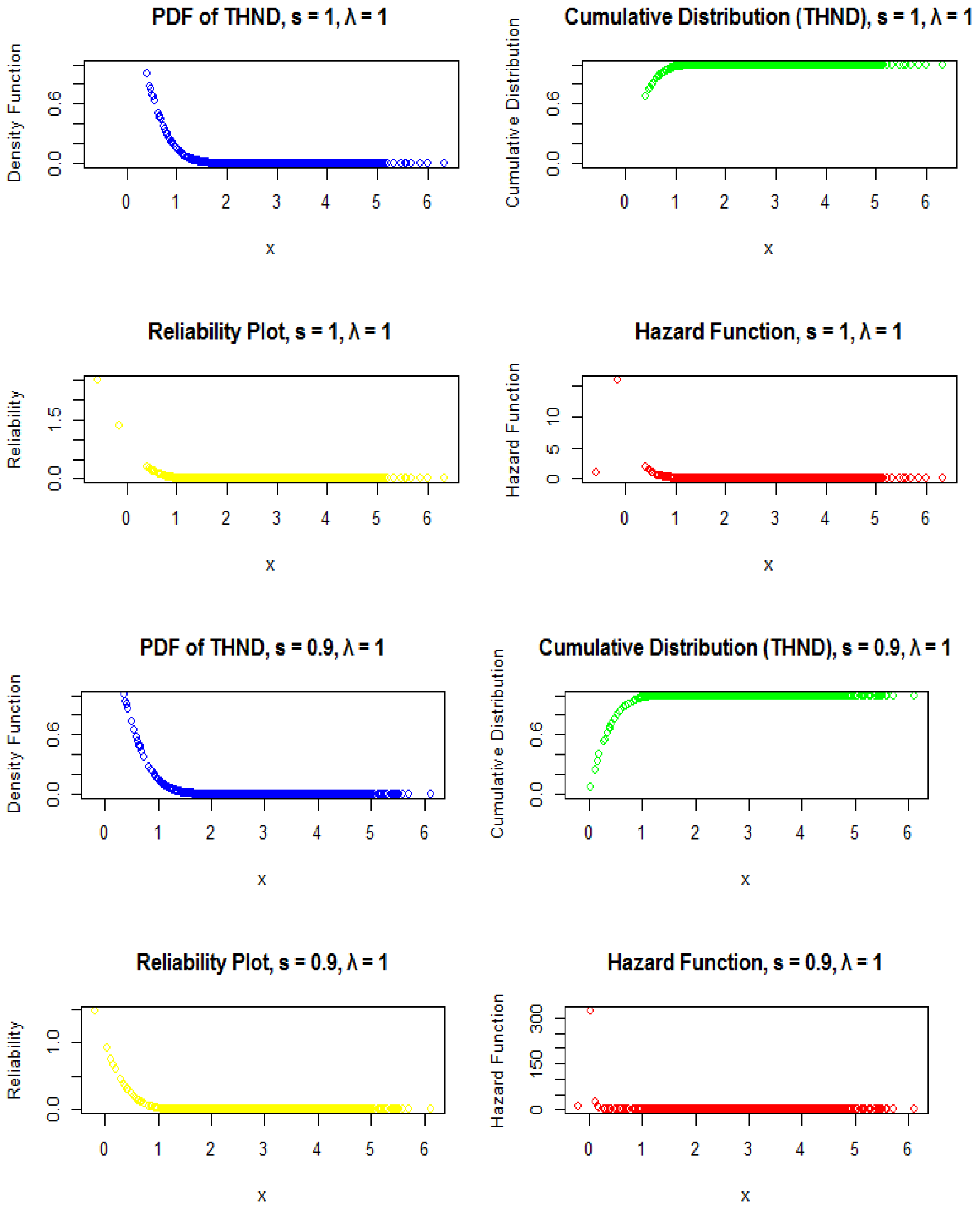

Fig 1: the Plot of THND at different value of parameter (λ, σ)
From fig 1 above, we can deduce that the distribution is heavily tailed

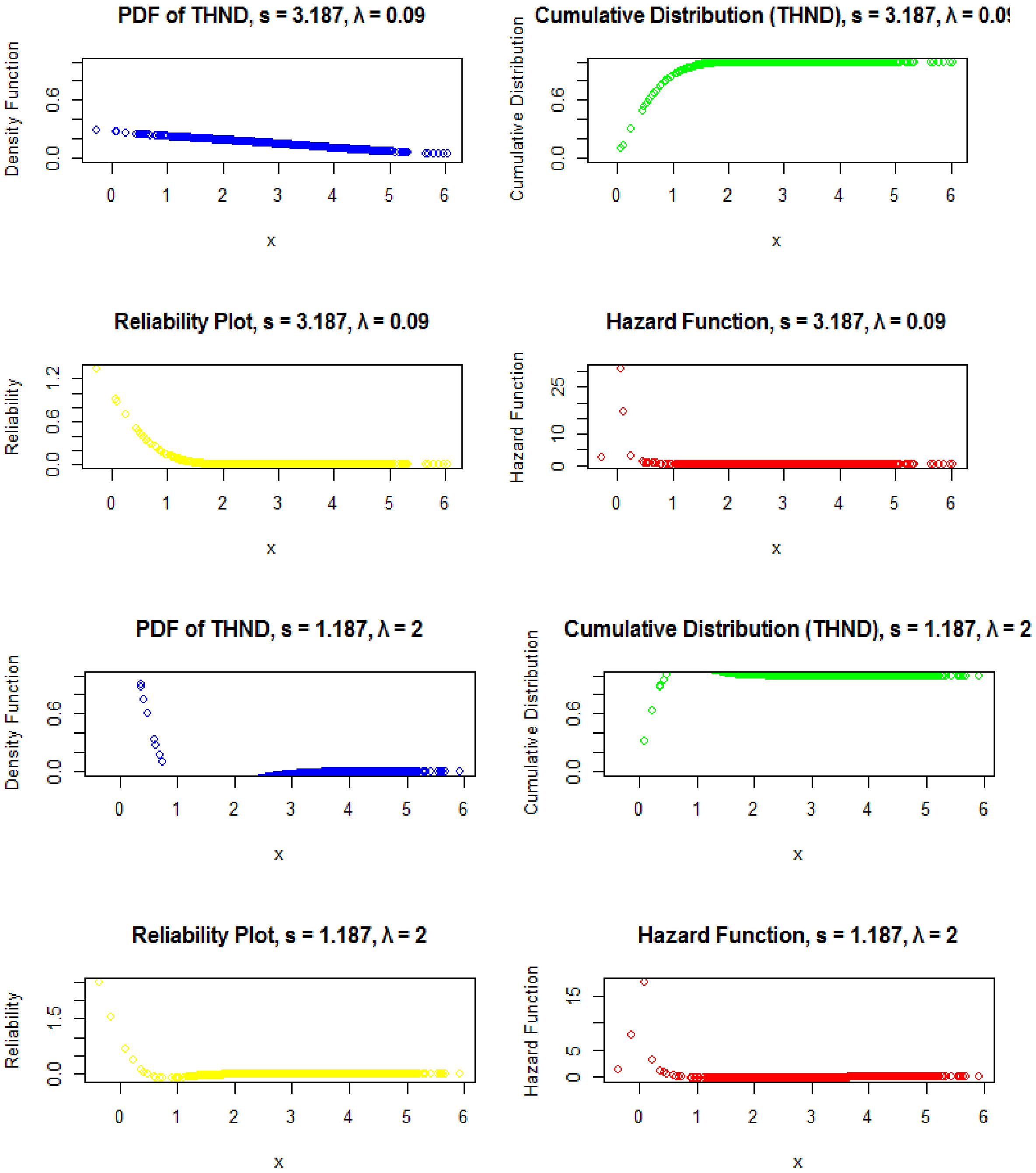

Fig 2: the Plot of THND at different value of parameter (λ, σ)

From fig 2 above, we can deduce that the distribution is heavily tailed and highly skewed at different parameter of the model, the transmuted parameter λ give the transmuted HalfNormal this flexibility and hence making it accurate when it come to modeling data whose distribution is
heavily tailed and highly skewed

4. Analysis and Results.

Accessing Normality

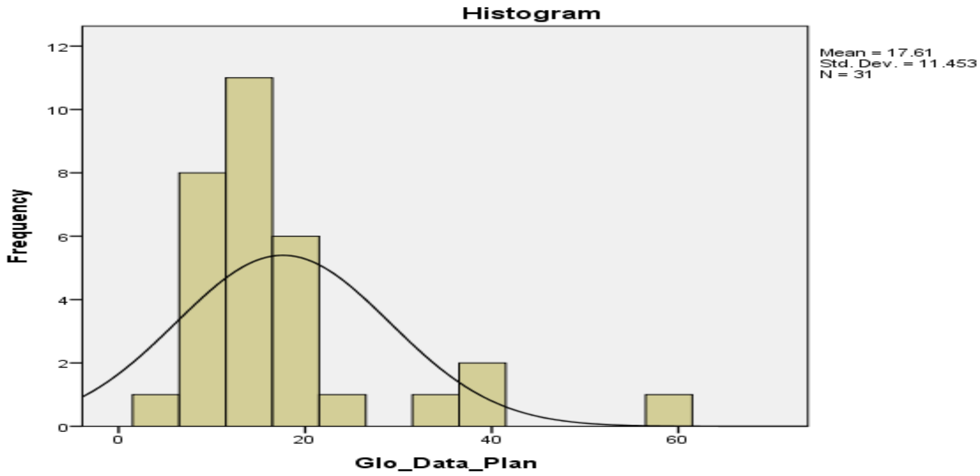

Fig 3: the plot showing the distribution of the data generated from the Glo Data Plan

From the fig 3, we can deduce that the data is not normally distributed as it is heavily tailed, the THND can therefore be fitted to the data. Since this data shows the impression of the THND, then we can fit the hybrid model to the data.

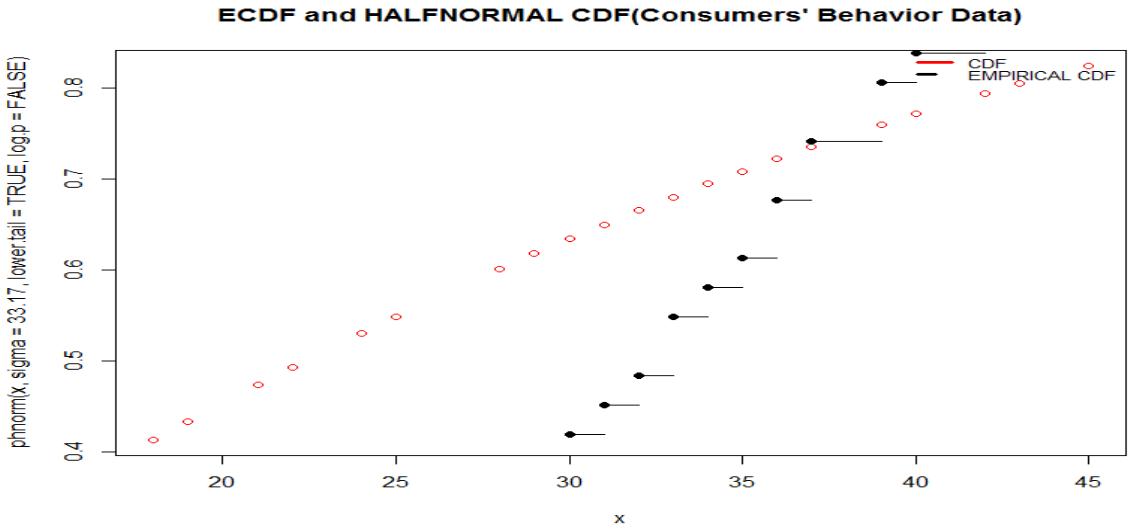

Fig 4: CDF and ECDF of HAlfNormal Distribution when fitted to Buying Behavior Data

From the fig 4 above, we can say that the HalfNormal Distribution does not fit the data reasonably well.

Fitting THND to the available Data

Table 1: Parameter Estimate

Model	**THND[Transmuted HalfNormal]**		**HND[HalfNormal]**	
Parameter	**EStimate**	**Std.Error**	**EStimate**	**Std.Error**
σ	7.178e-08	9.406e-17	33.317	3.425
Λ	3.614e-01	4.007e-11	-	-
Comparison Criterion	**AIC** =-1.857456e+20		**AIC** =264.3871	
LogLikelihood	9.287281e-19		131.1936	
	H= $\begin{pmatrix} 9.215123e+31 & -3.228008e+26 \\ -3.228008e+26 & -5.078385e+20 \end{pmatrix}$ w = 5.006941e+19			

5. Conclusion

The plot of the hybrid distribution that is the Transmuted HalfNormal distribution (THND) shows that the additional parameter (transmuted parameter) control the tail of the model by making it heavily tailed, from the monthly Glo data plan, since p value (0.5383) > α (0.05), then there is great statistical evidence that the data tested are not from a normally distributed population (the data are not normal). Then the THND can be fitted to the data because the distribution shows heavy tail due to addition of the transmuted parameter $(\boldsymbol{\lambda})$. Therefore THND can be used to capture non-normal data. The wald test carried out indicate that since the w(5.006941e+19) > $\chi_q{}^2(0.103)$, then we say that THND captures the data reasonably well as compared to parent distribution because of the additional parameter that controls the flexibility of the distribution. The likelihood ratio test shows that $T\ (262.3872) > \boldsymbol{\chi}^2{}_{(0.9\ ,\ 31)}$ (**20.599**), indicating that the transmuted Halfnormal Distribution (THND) fits the data reasonably well and is better than HND, this was established even when comparing the fitted Glo monthly data plan using the Akaike' information criterion. By obtaining the transmuted version of probability distributions, we get the corresponding hybrid distribution with increased number of parameters which gives the newly compounded distribution more flexibility, consistency and stability. After careful application of the newly compounded distribution to customer buying behavior of Monthy data consumption of the Glo subscribers, we therefore conclude that the additional parameter (Transmuted Parameter) gives THND more flexibility over HND in modeling highly skewed and heavily tailed data.

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