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Common Fixed Point Theorems in Generalized Cone Metric Space

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Abstract

The aim of this paper is to prove some common fixed point theorems for weakly compatible mappings in Generalized cone metric space.

Keywords: Generalized cone metric space, weakly compatible maps, fixed points.

1.Introduction:

The concept of D-metric space was introduced by Dhage (Dhage, 1992). It was shown that certain theorems involving Dhage's D -metric space are flawed and most of the results claimed by Dhage and others are invalid. These errors were pointed out by Mustafa and Sims (Mustafa and Sims, 2006), among others. They also introduced a valid generalized space structure, which they call G -metric spaces.

Huang and Xian (Huang and Xian, 2007) introduced the notion of cone metric space by replacing real numbers with an ordering Banach space and obtained some fixed point theorem. Afterwards some common fixed point theorems in cone metric spaces are studied by Abbas , Jungck and Rhoades (Abbas and Jungck, 2008, Abbas and Rhoades, 2008). Recently, Rezapour and Hamlbarani (Rezapour and Hamlbarani, 2008) omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space.

Jungck (Jungck, 1996) defined a pair self mappings to be weakly compatible if they commute at their coincidence point. In 2009, J.O.Olaleru (Olaleru, 2009) proved some common fixed point theorems in cone metric spaces for weakly compatible mappings. More recently Ismat Beg et. al (Beg et.al, 2010) introduced the concept of G-cone metric space by replacing the set of real numbers by an ordered Banach space, proved convergence properties of sequences and proved some common fixed point theorems in this space.

In this paper, we prove common fixed point theorems for two self mappings satisfying the concept of weak compatibility in G-cone metric spaces which generalizes the result of Ismat Beg et.al.

2.Preliminaries:

Let E be a real Banach space. A subset $P \ of \ E$ is called a *cone* if and only if:

(a) P is closed, non empty and $P \neq \{0\}$;

(b) $a, b \in R, a, b \geq 0, \ x, y \in P \Rightarrow ax + by \in P$; More generally if $a, b, c \in R$, $a, b, c \geq 0, x, y, z \in P \Rightarrow ax + by + cz \in P$;

(c) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \; implies \; \|x\| \leq K \|y\|.$$

The least positive number satisfying the above inequality is called the *normal constant* of P, while $x \ll y$ stands for $y - x \in intP$ (interior of P):

Rezapour and Hamlbarani (Rezapour and Hamlbarani, 2008) proved that there are no normal cones with normal constants $K < 1$ and for each $k > 1$ there are cones with normal constants $K > k$:

Definition 2.1(Beg et. al, 2010): Let X be a nonempty set. Suppose a mapping $G : X \times X \times X \to E$ satisfies:

(G_1) $G(x, y, z) = 0 \; if \; x = y = z,$

(G_2) $0 < G(x, x, y)$; $whenever \; x \neq y, for \; all \; x; \; y \in X,$

(G_3) $G(x, x, y) \leq G(x, y, z)$; $whenever \; y \neq z,$

(G_4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \cdots$ (Symmetric in all three variables),

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z) for \; all \; x, y, z, a \in X.$

Then G is called a generalized cone metric on X, and X is called a generalized cone metric space or more specifically a $G -$ cone metric space.

The concept of a $G -$ cone metric space is more general than that of a $G -$metric spaces and cone metric spaces.

Definition 2.2 (Beg et. al, 2010) : A $G -$ cone metric space X is symmetric if

$$G(x, y, y) = G(y, x, x) for \; all \; x, y \in X.$$

Following are examples of symmetric and non symmetric $G -$cone metric spaces respectively.

Example 2.3: Let (X, d) be a cone metric space. Define $G : X \times X \times X \to E$ by $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Example 2: Let$X = \{a, b\}, E = R^3, P = \{(x, y, z) \in E: x, y, z \geq 0$. Define $G : X \times X \times X \to E$ by

$$G(a, a, a) = (0,0,0) = G(b, b, b),$$

$$G(a, b, b) = (0,1,1) = G(b, a, b) = G(b, b, a),$$

$$G(b, a, a) = (0,1,0) = G(a, b, a) = G(a, a, b)$$

Note that X is nonsymmetric $G -$cone metric space as $G(a, a, b) \neq G(a, b, b)$.

Proposition 2.5(Beg et. al, 2010): Let X be a $G -$ cone metric space, define $d_G: X \times X \to E$ by

$$d_G(x, y) = G(x, y, y) + G(y, x, x)$$

Then (X, d_G) is a cone metric space.

It can be noted that $G(x, y, y) \leq \frac{2}{3} d_G(x, y)$. If X is a symmetric $G -$ cone metric space, then

$$d_G(x, y) = 2G(x, y, y)$$

for all $x, y \in X$.

Definition 2.6 (Beg et. al, 2010): Let X be a $G -$ cone metric space and $\{x_n\}$ be a sequence in X.We say that $\{x_n\}$ is:

(a) Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is N such that for all n$, m, l > N$, $G(x_n, x_m, x_l) \ll c$.

(b) Convergent sequence if for every $c \; in \; E$ with $0 \ll c$, there is N such that for all $m, n > N$, $G(x_n, x_m, x) \ll c$ for some fixed x in X. Here x is called the limit of a sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

A $G -$ cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Proposition 2.7(Beg et. al, 2010): Let X be a $G-$ cone metric space then the following are equivalent.

(i) $\{x_n\}$ is converges to x.

(ii) $G(x_n, x_m, x) \to 0$ as $n \to \infty$.

(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$.

(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$

Lemma 2.8 (Beg et. al, 2010): Let $\{x_n\}$ be a sequence in a $G-$ cone metric space X and $x \in X$. If $\{x_n\}$ converges to $x \in X$, then $\{x_n\}$ is a Cauchy sequence.

Definition 2.9 (Jungck, 1996)**:** Let f and g be two self-maps defined on a set X then f and g are said to be weakly compatible if they commute at coincidence points.

3. Main Results:

Theorem3.1: Let X be a complete symmetric G-cone metric space and $T, S: X \to X$ be a mapping satisfying one of the following conditions

$$G(Tx,Ty,Tz) \le aG(Sx,Sy,Sz) + bG(Tx,Tx,Sx) + cG(Ty,Ty,Sy) + dG(Tz,Tz,Sz) + eG(Ty,Ty,Sx) + fG(Tx,Tx,Sy) \quad \ldots(1)$$

Or

$$G(Tx,Ty,Tz) \le aG(Sx,Sy,Sz) + bG(Tx,Sx,Sx) + cG(Ty,Sy,Sy) + dG(Tz,Sz,Sz) + eG(Ty,Sx,Sx) + fG(Tx,Sy,Sy) \quad \ldots(2)$$

For all $x, y, z \in X$ where $a, b, c, d, e, f \in [0,1)$ and $a + b + c + d + e + f < 1$. Suppose T and S are weakly compatible and $T(X) \subset S(X)$ s.t. $T(X)$ or $S(X)$ is a complete subspace of X, then the mappings T and S have a unique common fixed point. Moreover, for any $x_0 \in X$, the sequence $\{x_n\} \in X$, defined by $Sx_n = Tx_{n-1}$ for all n, converges to the fixed point.

Proof : Suppose that T satisfies condition (1) and (2), then for all $x, y \in X$

$$G(Tx,Ty,Ty) \le aG(Sx,Sy,Sy) + bG(Tx,Tx,Sx) + (c+d)G(Ty,Ty,Sy) + eG(Ty,Ty,Sx) + fG(Tx,Tx,Sy) \quad \ldots(3)$$

And

$$G(Ty,Tx,Tx) \le aG(Sy,Sx,Sx) + bG(Ty,Sy,Sy) + (c+d)G(Tx,Sx,Sx) + eG(Tx,Sy,Sy) + fG(Ty,Sx,Sx) \quad \ldots(4)$$

Since X is a symmetric G-cone metric space therefore by adding (3) and (4) we have ,

$$d_G(Tx,Ty) \le ad_G(Sx,Sy) + \frac{b+c+d}{2}d_G(Tx,Sx) + \frac{b+c+d}{2}d_G(Ty,Sy) + \frac{e+f}{2}d_G(Ty,Sx) + \frac{e+f}{2}d_G(Tx,Sy)$$

Let $a = \alpha$, $\frac{b+c+d}{2} = \beta = \gamma, \frac{e+f}{2} = \eta = \delta$

$\Rightarrow \alpha + \beta + \gamma + \eta + \delta < 1$

Or

$$d_G(Tx,Ty) \le \alpha d_G(Sx,Sy) + \beta d_G(Tx,Sx) + \gamma d_G(Ty,Sy) + \eta d_G(Ty,Sx) + \delta d_G(Tx,Sy) \quad \ldots(5)$$

If $Tx_n = Tx_{n-1}$ for all $n \in N$ then $\{Tx_n\}$ is a Cauchy sequence. If $Tx_n \neq Tx_{n-1}$ for all $n \in N$ then put $x = x_{n+1}, y = x_n$ in (5), we get

$$d_G(Tx_{n+1}, Tx_n) \leq \alpha d_G(Sx_{n+1}, Sx_n) + \beta d_G(Tx_{n+1}, Sx_{n+1}) + \gamma d_G(Tx_n, Sx_n) + \eta d_G(Tx_n, Sx_{n+1}) + \delta d_G(Tx_{n+1}, Sx_n)$$

Using the fact that $Sx_n = Tx_{n-1}$ for all n, we have

$$d_G(Tx_{n+1}, Tx_n) \leq \alpha d_G(Tx_n, Tx_{n-1}) + \beta d_G(Tx_{n+1}, Tx_n) + \gamma d_G(Tx_n, Tx_{n-1}) + \eta d_G(Tx_n, Tx_n) + \delta d_G(Tx_{n+1}, Tx_{n-1})$$

$$d_G(Tx_{n+1}, Tx_n) \leq \alpha d_G(Tx_n, Tx_{n-1}) + \beta d_G(Tx_{n+1}, Tx_n) + \gamma d_G(Tx_n, Tx_{n-1}) + \eta d_G(Tx_n, Tx_n) + \delta(d_G(Tx_{n+1}, Tx_n) + d_G(Tx_n, Tx_{n-1}))$$

Or

$$\{1 - (\beta + \delta)\} d_G(Tx_{n+1}, Tx_n) \leq \{\alpha + \gamma + \delta\} d_G(Tx_n, Tx_{n-1})$$

It further implies that

$$d_G(Tx_{n+1}, Tx_n) \leq p d_G(Tx_n, Tx_{n-1})$$

where $p = \frac{\{\alpha+\gamma+\delta\}}{\{1-(\beta+\delta)\}} < 1$

Consequently

$$d_G(Tx_{n+1}, Tx_n) \leq p^n d_G(Tx_n, Tx_{n-1}) \quad (6)$$

Now for all $m, n \in N$ with $m > n$, we have

$$d_G(Tx_m, Tx_n) \leq d_G(Tx_m, Tx_{m-1}) + d_G(Tx_{m-1}, Tx_{m-2}) + \cdots + d_G(Tx_{n+1}, Tx_n)$$
$$= (p^{m-1} + p^{m-2} + \cdots + p^n) d_G(Tx_1, Tx_0)$$
$$\leq \frac{p^n}{1-p} d_G(Tx_1, Tx_0)$$

Let $0 \ll c$ be given. Following similar argument to those given in [9, theorem 2.3] , we conclude that $\frac{p^n}{1-p} d_G(Tx_1, Tx_0) \ll c$. So we have $d_G(Tx_m, Tx_n) \ll c$, for all $m > n$. Therefore $\{Tx_n\}$ is a Cauchy sequence. Since $T(X)$ or $S(X)$ is a complete subspace of , then there exist $x^* \in X$ such that $Tx_n \to x^*$ and $Sx_n \to x^*$. Let $z \in X$ such that $Sz = x^*$. We claim that $Sz = Tz$. From (5) we have,

$$d_G(Tx_n, Tz) \leq \alpha d_G(Sx_n, Sz) + \beta d_G(Tx_n, Sx_n) + \gamma d_G(Tz, Sz) + \eta d_G(Tz, Sx_n) + \delta d_G(Tx_n, Sz)$$

Letting $n \to \infty$

$$d_G(x^*, Tz) \leq \alpha d_G(x^*, Sz) + \beta d_G(x^*, x^*) + \gamma d_G(Tz, x^*) + \eta d_G(Tz, x^*) + \delta d_G(x^*, Sz)$$
$$d_G(x^*, Tz) \leq (\gamma + \eta) d_G(Tz, x^*)$$

Hence $x^* = Sz = Tz$.

Since $Sz = Tz$ and T & S are weakly compatible then $Tx^* = T(Sz) = S(Tz) = Sx^*$.

Next we show that $x^* = Tx^* = Sx^*$. Suppose $x^* \neq Tx^*$, then we have

$$d_G(Tx^*, Tz) \leq \alpha d_G(Sx^*, Sz) + \beta d_G(T x^*, Sx^*) + \gamma d_G(Tz, Sz) + \eta d_G(Tz, Sx^*) + \delta d_G(Tx^*, Sz)$$
$$\leq \alpha d_G(Tx^*, Tz) + \eta d_G(Tz, Tx^*) + \delta d_G(Tx^*, Tz)$$
$$= (\alpha + \eta + \delta) d_G(Tx^*, Tz) < d_G(Tx^*, Tz)$$

This is a contradiction and hence $x^* = Tx^* = Sx^*$. Thus x^* is a common fixed point of $T \& S$. The uniqueness follows from (1).

Theorem3.2: Let X be a complete symmetric G-cone metric space and $T, S: X \to X$ be a mapping satisfying one of the following conditions:

$$G(Tx, Ty, Tz) + p \max\{G(Tx, Tx, Sx), G(Ty, Ty, Sx), G(Ty, Tz, Sx)\} \leq$$
$$p \max\{G(Sx, Sy, Sz), G(Ty, Ty, Sy), G(Ty, Tz, Sy)\} +$$
$$q \min\{G(Ty, Ty, Sx), G(Tx, Ty, Tz), G(Tx, Tx, Sy)\} \quad \ldots(7)$$

For all $x, y \in X$ where $p > 0, q > 0$ s.t. $p + q < 1$. Suppose T and S are weakly compatible and $T(X) \subset S(X)$ s.t $T(X)$ or $S(X)$ is a complete subspace of X, then the mappings T and S have a unique common fixed point. Moreover ,for any $x_0 \in X$, the sequence $\{x_n\} \in X$, defined by $Sx_n = Tx_{n-1}$ for all n, converges to the fixed point.

Proof: Putting $x = x_{n+1}, y = z = x_n$ in (7), we get

$$G(Tx_{n+1}, Tx_n, Tx_n) + p \max\{G(Tx_{n+1}, Tx_{n+1}, Sx_{n+1}), G(Tx_n, Tx_n, Sx_{n+1}), G(Tx_n, Tx_n, Sx_{n+1})\} \leq$$
$$p \max\{G(Sx_{n+1}, Sx_n, Sx_n), G(Tx_n, Tx_n, Sx_n), G(Tx_n, Tx_n, Sx_n)\} +$$
$$q \min\{G(Tx_n, Tx_n, Sx_{n+1}), G(Tx_{n+1}, Tx_n, Tx_n), G(Tx_{n+1}, Tx_{n+1}, Sx_n)\}$$

$$G(Tx_{n+1}, Tx_n, Tx_n) + p \max\{G(Tx_{n+1}, Tx_{n+1}, Tx_n), G(Tx_n, Tx_n, Tx_n), G(Tx_n, Tx_n, Tx_n)\}$$
$$\leq p \max\{G(Tx_n, Tx_{n-1}, Tx_{n-1}), G(Tx_n, Tx_n, Tx_{n-1}), G(Tx_n, Tx_n, Tx_{n-1})\}$$
$$+ q \min\{G(Tx_n, Tx_n, Tx_n), G(Tx_{n+1}, Tx_n, Tx_n), G(Tx_{n+1}, Tx_{n+1}, Tx_{n-1})$$

Or

$$\frac{1}{2} d_G(Tx_{n+1}, Tx_n) + p \max\left\{\frac{1}{2} d_G(Tx_{n+1}, Tx_n), 0,0\right\} \leq$$
$$p \max\left\{\frac{1}{2} d_G(Tx_n, Tx_{n-1}) \frac{1}{2} d_G(Tx_n, Tx_{n-1}) \frac{1}{2} d_G(Tx_n, Tx_{n-1})\right\} + q \min\{0, \frac{1}{2} d_G(Tx_n, Tx_{n+1}), \frac{1}{2} d_G(Tx_{n+1}, Tx_{n-1})\}$$

…(8)

$$\frac{1}{2}\{d_G(Tx_{n+1}, Tx_n) + p d_G(Tx_{n+1}, Tx_n)\} \leq \frac{p}{2} d_G(Tx_n, Tx_{n-1}) + 0$$

$$d_G(Tx_{n+1}, Tx_n) \leq \frac{p}{p+1} d_G(Tx_n, Tx_{n-1})$$

Which implies that

$$d_G(Tx_{n+1}, Tx_n) \leq k d_G(Tx_n, Tx_{n-1})$$

Where $k = \frac{p}{p+1} < 1$

Consequently

$$d_G(Tx_{n+1}, Tx_n) \leq k^m d_G(Tx_n, Tx_{n-1})$$

Now for all $m, n \in N$ with $m > n$, we have

$$d_G(Tx_m, Tx_n) \leq \frac{k^n}{1-k} d_G(Tx_1, Tx_0)$$

Let $0 \ll c$ be given. Following similar argument to those given in (Rezapour and Hamlbarani, 2008, theorem 2.3) , we conclude that $\frac{k^n}{1-k} d_G(Tx_1, Tx_0) \ll c$. So we have $d_G(Tx_m, Tx_n) \ll c$, for all $m > n$. Therefore $\{Tx_n\}$ is a Cauchy sequence. Since $T(X)$ or $S(X)$ is a complete subspace of , then there exist $x^* \in X$ such that $Tx_n \to x^*$ and $Sx_n \to x^*$. Let $z \in X$ such that $Sz = x^*$. We claim that $Sz = Tz$.

Putting $x = x_n, y = z$ in (7), we get

$$\tfrac{1}{2} d_G(Tx_n, Tz) + p \max\left\{\tfrac{1}{2} d_G(Tx_n, Tz), \tfrac{1}{2} d_G(Tz, Sx_n), \tfrac{1}{2} d_G(Tz, Sx_n)\right\} \leq$$
$$p \max\left\{\tfrac{1}{2} d_G(Sx_n, Sz) \tfrac{1}{2} d_G(Tz, Sz), \tfrac{1}{2} d_G(Tz, Sz)\right\} + q \min\{\tfrac{1}{2} d_G(Tz, Sx_n), \tfrac{1}{2} d_G(Tx_n, Tz), \tfrac{1}{2} d_G(Tx_n, Sz)\}$$

When $n \to \infty$

$$\frac{1}{2} d_G(x^*, Tz) + p \max\left\{\frac{1}{2} d_G(x^*, Tz), \frac{1}{2} d_G(Tz, x^*), \frac{1}{2} d_G(Tz, x^*)\right\}$$
$$\leq p \max\left\{\frac{1}{2} d_G(x^*, x^*), \frac{1}{2} d_G(Tz, x^*), \frac{1}{2} d_G(Tz, x^*)\right\}$$
$$+ q \min\{\frac{1}{2} d_G(Tz, x^*), \frac{1}{2} d_G(x^*, Tz), \frac{1}{2} d_G(x^*, x^*)$$

$$\frac{1}{2}(1+p) d_G(x^*, Tz) \leq \frac{p}{2} d_G(Tz, x^*)$$

or

$$d_G(x^*, Tz) \leq \frac{p}{1+p} d_G(Tz, x^*) < d_G(x^*, Tz)$$

which is a contradiction and so $x^* = Tz = Sz$.

Since $Tz = Sz$ and $T\&S$ are weakly compatible then $Tx^* = T(Sz) = S(Tz) = Sx^*$.

Next we show that $x^* = Tx^* = Sx^*$. Suppose $x^* \neq Tx^*$.

Let $x = x^*, y = z = w$ in (7) ,we have

$$\frac{1}{2} d_G(Tx^*, Tw) + p \max\left\{\frac{1}{2} d_G(Tx^* Sx^*), \frac{1}{2} d_G(Tw, Sx^*), \frac{1}{2} d_G(Tw, Sx^*)\right\}$$
$$\leq p \max\left\{\frac{1}{2} d_G(Sx^*, Sw), \frac{1}{2} d_G(Tw, Sw), \frac{1}{2} d_G(Tw, Sw)\right\}$$
$$+ q \min\{\frac{1}{2} d_G(Tw, Sx^*), \frac{1}{2} d_G(Tx^*, Tw), \frac{1}{2} d_G(Tx^*, Sw)$$

$$\frac{1}{2}d_G(Tx^*,Tw)+p\max\left\{\frac{1}{2}d_G(Tx^*\,Tx^*),\frac{1}{2}d_G(Tw,Tx^*),\frac{1}{2}d_G(Tw,Tx^*)\right\}$$
$$\le p\max\left\{\frac{1}{2}d_G(Tx^*,Tw),\frac{1}{2}d_G(Tw,Tw),\frac{1}{2}d_G(Tw,Tw)\right\}$$
$$+q\min\{\frac{1}{2}d_G(Tw,Tx^*),\frac{1}{2}d_G(Tx^*,Tw),\frac{1}{2}d_G(Tx^*,Tw)$$

i.e. $\frac{1}{2}(1+p)d_G(Tx^*,Tw)\le\frac{1}{2}(p+q)d_G(Tw,Tx^*)$

Or $d_G(Tx^*,Tw)\le\frac{p+q}{1+p}d_G(Tw,Tx^*)<d_G(Tw,Tx^*)$

which is a contradiction and hence $x^*=Tx^*=Sx^*$. Thus x^* is a common fixed point of $T\,\&\,S$. The uniqueness follows from (1).

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