Some Results on Complete Metric Space

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Abstract
In the present paper we established some fixed point and common fixed point theorems in complete metric spaces for new rational expressions.Our results are genralizatyion of many known results

Kew words: Complete Metric Space, Fixed Point, and Common Fixed Point.

2. Introduction: Before starting the main results first we are giving some fundamental results.

THEOREM A: Banach [1] The well known Banach contraction principal states that "If X is complete metric space and T is a contraction mapping on X into itself, then T has unique fixed point in X".

THEOREM B: Kannan [16] proved that "If T is self mapping of a complete metric space X into itself satisfying:

$$d(Tx,Ty) \le \eta\,[d(Tx,x)+d(Ty,y)],\ for\ all\ x,y\ \varepsilon\ X,$$
$$where\ \eta\ \varepsilon\left[0,\frac{1}{2}\right], then\ T\ has\ unique\ fixed\ point\ in\ X$$

THEOREM C: Fisher [9] proved the result with

$$d(Tx,Ty) \le \mu[d(Tx,x)+d(Ty,y)]+\delta\, d(x,y)\ for\ all\ x,y\ \varepsilon\ X,$$
$$where\ \mu,\delta\ \varepsilon\left[0,\frac{1}{2}\right], then\ T\ has\ unique\ fixed\ point\ in\ X$$

THEOREM C: A similar conclusion was also obtained by Chaterjee [3].

$$d(Tx,Ty) \le \mu[d(Ty,x)+d(Tx,y)],\ for\ all\ x,y\ \varepsilon\ X,$$
$$where\ \mu\ \varepsilon\left[0,\frac{1}{2}\right], then\ T\ has\ unique\ fixed\ point\ in\ X$$

THEOREM D: Ciric [5] proved the result

$$d(Tx,Ty) \le \eta[d(x,T(x))+d(y,T(y))]+\mu[d(x,T(y))+d(y,T(x))]$$
$$+\delta\, d(x,y)\ Where\ \eta,\mu,\delta\ \varepsilon\ [0,1[,\ x,y\ \varepsilon\ X.$$
$$Then\ T\ has\ unique\ fixed\ point\ in\ X.$$

THEOREM E: Reich [22] proved the result

$$d(Tx,Ty) \le \mu[d(x,T(y))+d(y,T(x))]+\delta\, d(x,y),$$
$$where\ \mu,\delta\ \varepsilon\ [0,1[,\ x,y\ \varepsilon\ X. Then\ T\ has\ unique\ fixed\ point\ in\ X.$$

THEOREM F: In 1977, the mathematician Jaggi [14] introduced the rational expression first

$$d(Tx,Ty) \le \delta\, d(x,y) + \beta \frac{d(x,Tx)d(y,Ty)}{d(x,y)} \quad \text{for all } x,y \,\varepsilon\, X, x \ne y, 0 \le \delta + \beta \prec 1,$$

Then T has unique fixed point in X.

THEOREM G: In 1980 the mathematicians Jaggi and Das [15] obtained some fixed point theorems with the mapping satisfying:

$$d(Tx,Ty) \le \alpha d(x,y) + \beta \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \quad \text{for all } x,y \,\varepsilon\, X, x \ne y, \alpha + \beta \prec 1$$

In the present paper we shall establish some unique fixed point and common fixed point theorems, through new rational expressions in complete metric spaces. Our theorems include the fundamental result of Banach [1], Kannan [16], Fisher [9], Reich [22], Chatterjee [3] and Ciric [5].

3. Main result

Theorem3.1:-Let (X,d) be a complete metric space . Let T : X→ X be continous mapping satisfies the condition:

$$d(Tx,Ty) \le \propto d(x,y) + \beta \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \gamma\,[\,d(x,Tx) + d(y,Ty)]$$

$$+\delta\,[\,d(x.Ty) + d(y,Tx)] + \eta\,[\,d(x,Tx) + d(y,Ty)]$$

$$+\xi_1 [\frac{d(y,Tx)+d(y,Ty)}{1+d(y,Tx)\; d(y,Ty)}]$$

$+\xi_2 \max[\,d(y,Tx), d(y,Ty), d(y,Tx), d(x,y)\,]$ _ _ _ _ _ _ _ (3.1.1)

For all x , y $\in X$. $\propto, \beta, \gamma, \eta, \delta, \xi_1, \xi_2$ non negative with $0 \le \propto + \beta + 2\gamma + 2\delta + 2\eta + \xi_1 + \xi_2 < 1$ then T has unique fixed point .

Proof:- Let $x_0 \in X$ and define the sequence as followes.

T(x_0) = x_1, T(x_1)= x_2 _ _ _ _ _ _ T(x_n)= x_{n+1} _ _ _ _

Putting x = x_{n-1} and y = x_n in eq. (3.1.1) We have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\le \propto d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{d(x_{n-1},x_n)}$$

$$+\gamma\,[\,d(x_{n-1},Tx_{n-1}) + d(x_n,Tx_n)]$$

$$+\delta\,[\,d(x_{n-1},Tx_n) + d(x_n,Tx_{n-1})]$$

$$+\eta\,[\,d(x_{n-1},Tx_{n-1}) + d(x_{n-1},x_n)]$$

$$+\xi_1 [\frac{d(x_n,Tx_n)+d(x_n,Tx_n)}{1+d(x_n,Tx_{n-1})d(x_n,Tx_n)}]$$

$$+\xi_2 \max[\,d(x_n,Tx_{n-1}), d(x_n,Tx_n), d(x_n,T_{xn-1}), d)(x_{n-1},x_n)]$$

$$= \propto d(x_{n-1},x_n) + \beta \frac{d(x_{n-1},x_n)\,d(x_n,x_{n+1})}{d(x_{n-1},x_n)}$$

$+\gamma\ [\ d(x_{n-1}\,,x_n) +\ d(x_n\,,x_{n+1})]$

$+\ \delta\ [\ d(x_{n-1}\,,x_{n+1}) + d\ (x_n\,,x_n)]$

$+\ \eta\ [\ d(x_{n-1}\,,x_n) +\ d(x_{n-1}\,,x_n)] +\ \xi_1 \left[\frac{d(x_n\,,x_n)+ d(x_n\,,x_{n+1})}{1+d(x_n\,,x_n)d(x_n\,,x_{n+1})}\right]$

$+\xi_2$ max[$d(x_n\,,x_n), d(x_n\,,x_{n+1}), d(x_{n-1}\,,x_n), d(x_{n-1}\,,x_n)$]

$\leq\ \propto d(x_{n-1}\,,x_n) +\ \beta d(x_n\,,x_{n+1}) + \gamma d(x_{n-1}\,,x_n)$

$+\ \gamma\ d\ (x_n\,,x_{n+1}) + \delta\ d\ (x_{n-1}\,,x_n)$

$+\delta\ d(x_n\,,x_{n+1}) + 2\eta\ d\ (x_{n-1}\,,x_n)$

$+\ \xi_1\ d(x_n\,,x_{n+1}) + \xi_2$ max[$d(x_n\,,x_{n+1}), d(x_{n-1}\,,x_n)\,, d(x_{n-1}\,,x_n)$]

$\leq\propto d(x_{n-1}\,,x_n) +\ \beta d(x_n\,,x_{n+1})$

$+\ \gamma\ d\ (x_{n-1}\,,x_n) + \gamma\ d\ (x_n, x_{n+1})$

$+\ \ \delta\ d\ (x_{n-1}\,,x_n) + \delta\ d(x_n\,,x_{n+1})$

$+2\eta\ d\ (x_{n-1}\,,x_n) +\ \xi_1\ d(x_n\,,x_{n+1})$

$+\xi_2$ max[$d(x_n\,,x_{n+1}), d(x_{n-1}\,,x_n)$]

Case-I

Where max(a,b)=a

Where a=$d(x_n\,,x_{n+1})$

b=$d(x_{n-1}\,,x_n)$, then

$$d(x_n\,,x_{n+1}) \leq \left[\frac{\propto +\gamma +\ \delta + 2\eta}{1-(\beta+\gamma+\delta{+}\xi_1+\xi_2)}\right] d\ (x_{n-1}\,,x_n)$$

=λ(x_n, x_{n-1})

Where

λ $=\left[\frac{\propto+\gamma+\ \delta+2\eta}{1-(\beta+\gamma+\delta+\xi_1+\xi_2)}\right]$, 0≤λ<1

so,

$d(x_n\,,x_{n+1})$ ≤λ(x_n, x_{n-1})- - - - -(3.1.2)

Case-ii

Max {a,b}=b, $then$

$d\ (x_{n-1}\,,x_n) \leq \frac{(\beta+\gamma+\delta+\xi_1\ \ )}{1-\propto+\gamma+\ \delta+2\eta+\xi_2}\ d\ (x_{n-2}\,,x_{n-1})$, if 0≤λ<1

$d\ (x_{n-1}\,,x_n)$ ≤λ$d\ (x_{n-2}\,,x_{n-1})$

Now we can

$d\ (x_n, x_{n+1}) \leq \lambda^2 d\ (x_{n-2}\ , x_{n-1})$

Continuting in this way ,we have

$d\ (x_n, x_{n+1}) \leq \lambda^n d\ (x_0\ , x_1)$

Since , $0 \leq \lambda < 1$, for $n \to \infty$,

We have $d\ (x_n, x_{n+1}) \to 0$

Similarly we have show that $d\ (x_{n+1} x_n) \to 0$

Hence $(x_n)_{n \in N}$ is a Cauchy sequence in complete metric space (X,d).so there exists $\mu \epsilon$X such that $(x_n)_{n \in N}$ *converges to* u

Since T is continuous, therefore

T(u)=T($\lim_{n \to \infty} x_n$)=$\lim_{n \to \infty} T(x_n)$ = $\log_{n \to \infty}(x_{n+1})$ =u

Thus, u is a fixed point of T.

Uniqueness: - Suppose u and v are two fixed point of T (u$\neq v\ , Tu = u\ , Tv = v$) .Let u be fixed point then by condition for u we have :

d(u,u)=d(Tu,Tu)

$\leq \propto d(u,u)\ + \beta d(u,u) + 2\gamma d(u,u) + 2\delta d(u,u) + 2\eta d(u,u) + \xi_1 d(u,u) + \xi_2 d(u,u)$

= $(\propto + \beta + 2\gamma + 2\delta + 2\eta + \xi_1 + \xi_2)$d(u,u)

Which implies that d(u,u)=0

Since $0 < \propto + \beta + 2\gamma + 2\delta + 2\eta + \xi_1 + \xi_2 < 1$

Thus d(u,u)=0 for fixed point u of T . Similary, we get d(v,v)=0 for v fixed point of T .

Now from (3.1.1) we have

d(u,v)=d(Tu,Tv)

$\leq \propto d(u,v) + \beta \frac{d(u,u)d(v,v)}{d(u,v)} + \gamma[d(u,u) + d(v,v)] + \delta[d(u,v) + d(v,u)]$

$+\eta[d(u,u) + d(u,v)] + \xi_1[\frac{d(v,u)d(v,v)}{1 - d(v,u)d(v,v)}] + \xi_2$max[d(v,u) ,d(v,v),d(v,u),d(,vu)]

$\leq \propto d(u,v) + 0 + 0 + \delta[d(u,v) + d(v,u)]$+ηd(u,v)+$\xi_1$d(v,u)

$+\xi_2$max[d(v,u), ,vd(u)]

$\leq \propto d(u,v) + \delta[d(u,v) + d(v,u)] + \eta d(u,v) + \xi_1 d(v,u) + \xi_2 d(v,u)$

=$(\propto + \delta + \eta)$ d(u,v) +$(\delta + \xi_1 + \xi_2)$ d(v,u)

Similarly

d(v,u)≤ $(\propto +\delta+\eta)$d(v,u)+$(\delta+\xi_1+\xi_2)$d(u,v)

Hence I d(u,v)-d(v,u) I ≤ $(\propto +\eta+\xi_1+\xi_2)$ I d(u,v) –d(v,u) I

Since $0<\propto +\eta < 1$, get

d(u,v) = d(v,u) - - - - - -(3.1.3)

Again replacing (3.1.3) in (3.1.1) .we have that d(v,u)≤ $(\propto +2\delta+\eta)$ d(u,v) , which gives d(u,v) = 0 .

Since 0≤ $(\propto +2\delta+\eta)$<1.

Furter ,d(u,v) =d(v,u)=0 . Which implies u=v . Hence fixed point is unique.

Theorem 3.2.: Let (X,d) be a complete dislocated metric space . Let S,T : X→X be continuous mapping satisfying the condition:

$$d(Sx,Ty)\leq h\max\{d(x,y), d(x,Sx), d(y,Ty), d(x,Ty), d(y,Sx)\frac{d(x,Sx)d(y,Ty)}{d(x,y)}\}$$
$$+k\{\frac{d(y,Sx)+d(x,Sx)+d(y,Ty)}{1+d(y,Sx)d(x,Sx)d(y,Ty)}\} \text{- - - - - - - - - (3.2.1)}$$

If h+k < ½, then S and T have common fixed point.

Proof:- Let $x_0 \in X$ be arbitrary .Define the sequence $(x_n)_{n\in N}$ such that

x_1=S(x_0) , x_2 =T(x_1), - - - --x_{2n} = T(x_{2n-1}) , x_{2n+1} = S(x_{2n})

By the condition we have:

$$d(x_{2n+1}, x_{2n+1}) = d(Sx_{2n}, Tx_{2n+1})$$
$$\leq h\max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}), d(x_{2n}, Tx_{2n+1}),$$
$$d(x_{2n+1}, Sx_{2n})\frac{d(x_{2n},Sx_{2n})d(x_{2n+1},Tx_{2n+1})}{d(x_{2n},x_{2n+1})}\}$$
$$+k\{\frac{d(x_{2n+1},Sx_{2n})+d(x_{2n},Sx_{2n})+d(x_{2n+1},Tx_{2n+1})}{1+d(x_{2n+1},Sx_{2n})d(x_{2n},Sx_{2n})d(x_{2n+1},Tx_{2n+1})}\}$$

$$\leq h\max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}),$$
$$d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n=2})\frac{d(x_{2n+1},x_{2n+1})d(x_{2n+1},x_{2n+2})}{d(x_{2n},x_{2n+1})}\}$$
$$+k\frac{d(x_{2n+1},x_{2n+1})+d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})}{1+d(x_{2n+1},x_{2n+1})d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})}\}$$

$$\leq h\max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}),$$
$$d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}$$
$$+k\{d(x_{2n+1}, x_{2n+1})+d(x_{2n}, x_{2n+1})+d(x_{2n+1}, x_{2n+2})\}$$

$$\leq h\{d(x_{2n}, x_{2n+1})+d(x_{2n+1}, x_{2n+2})\}+k\{d(x_{2n}, x_{2n+1})+d(x_{2n+1}, x_{2n+2})\}$$

$$\leq (h+k)(d(x_{2n}, x_{2n+1})+d(x_{2n+1}, x_{2n+2}))$$

Therefore

d(x_{2n+1} , x_{2n+2}) ≤ $\frac{h+k}{1-(h+k)}$ d(x_{2n} , x_{2n+1})

Define r = $\frac{h+k}{1-(h+k)}$, 0< r <1 continuing in this way .

We get d(x_{2n+1} , x_{2n+2}) ≤ r^{2n}d(x_0, x_1),Since

0 < r< 1 , $r^{2n} \to \infty$.

Hence , $(x_n)_{n\in N}$ is a Cauchy sequence in complete metric space (X,d).

So there exists u∈ X , such that $(x_n)_{n\in N}$ converges to u .

Futher , the subsequence(Sx_{2n})→ u and (Tx_{2n+1})→u .

Since S ,T: X → X are continuous , will have Su= u and Tu = u .

Uniqueness:- Let u and v be fixed point of S,T .

Then d(u,v) = d(Su,Tv)

≤ h max { d(u,v) ,d(u,Su), d(v,Tv) ,d(u,Tv) ,d(v,Su) $\frac{d(u,Su),d(v,Tv)}{d(u,v)}$ }

+k {$\frac{d(v,Su)+d(u,Su)+d(v,Tv)}{1+d(v,Su)d(u,Su)d(v,Tv)}$ } - - - - -- (3.2.2)

≤ h max {d(u,v) ,d(u,u) ,d(v,v) , d(u,v) ,d(v,u) $\frac{d(u,u)d(v,v)}{d(u,v)}$ } +k {$\frac{d(v,u)+d(u,u)+d(v,v)}{1+d(v,u)d(u,u)d(v,v)}$ }

= h{ d(u,v) } + k { (u,v) }

= (h+k) d(u,v)

Replacing v by u in (3.2.3)

We get

d(u,u)≤ (h+k) d(u,u)

since 0< h < $\frac{1}{2}$.

Hance d(u,u) =0 - - - - - - - - - - (3.2.4)

Similary can show

d(v,u) = 0 - - - - - - - -- - - - - -(3.2.5)

Again from (3.2.3) d(u,v) < (h+k) d(u,v) , which implies that d(u,v) = 0.

Since X,d) is complete metric space we have u=v .

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