An Investigation of Solving Multidimensional Multiphase Flow: Streamline front tracking method

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Abstract

In the present paper we have studied streamline front tracking method for two and three-phase flow contains capillary forces. Increased demand for assessment of uncertainties and history matching require fast and accurate flow simulations on multiple plausible geological models on a routinely basis. On the other hand, conventional reservoir simulators fail to fulfill this need, and there seems to be trend within the petroleum industry to simulate reduced sets of equations. Hence, typically streamline or front tracking methods are used to solve the hyperbolic Buckley Leverette equation for two-phase flow. In this paper we consider models of multi-phase flow which do capillary forces, allowing for three phase. In particular we shall investigate a streamline front tracking method and to account for capillary effects we use operator splitting. Thus we compare the streamline front tracking method (SFTM) with a fast marching method (FFM) and a modified method of characteristics (MMOC).

Keywords: Streamline front tracking method, three-phase immiscible flow, modified method of characteristics, Cartesian grid, Riemann solver, Fast Marching method.

1 Governing equations

In this paper we study three-phase immiscible flow in a porous medium using basic equations water (w), gas (g) and oil (o), are mass balance equations and Darcy's law. We assume that the flow is incompressible, and that gravity can be neglected, the equations can be written in a global pressure as well as total velocity formulation (G. Chavent and J. Jaffre., 1986; Z. Chen and R. Ewing, 1997), as

$$\nabla.\mathrm{v}=\mathrm{q}(\mathrm{x},\mathrm{t}) \tag{1}$$

$$\mathrm{v}=-\lambda_T(x, S_\alpha)\mathbf{K}(x).\nabla p \tag{2}$$

$$\phi\frac{\partial S_\alpha}{\partial t}+\nabla.[F_\alpha \mathrm{v}-\varepsilon_\alpha\nabla.\, D_\alpha(\mathbf{x}, S_\alpha)\nabla S_\alpha]=q_\alpha(\mathbf{x},t) \tag{3}$$

Where $\phi\, and\, \mathbf{K}(\mathbf{x})$ represents the porosity and absolute permeability of the porous medium; S_α,v_α, $k_{r\alpha}$, and q/q_α are respectively, the reduced phase saturation, Darcy velocity, relative permeability, viscosity of phase α and sink/ source terms. Moreover, a global pressure p is derived from the phase pressure and the capillary pressures [2].

2. Solution strategy

Now to decouple the pressure, velocity equations Eq.(1) and Eq.(2) from the saturation equations Eq.(3), we use sequential time stepping. Hence, for a given saturation field at time t^n we calculate the new velocity field. The saturation field is advanced to a new time - step t^{n+1} by solving Eq.(3), applying the most recent velocity field. Thus this is continued sequentially up to a predetermined time $t = T$.

Furthermore, to solve parabolic saturation equation Eq.(3) it is found that a given velocity field, after operator splitting. In our study we use modified-method of characteristics (MMOC) to solve the two-phase problem based on the ideas (H, Dahle, M, Espedal, and O, Saevaried, 1992) . Henceforth, this method works excellently when the wave structure of the solution is known a priori. Here we present two alternative methods to the MMOC method, which both – preserve the shape of self-sharpening fronts and are more flexible than the MMOC method in the sense that no priori knowledge of the wave structure is required.

2.1 *A streamline front tracking method*

In this section we assume that the computational domain is discretized by a regular Cartesian grid such that the velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}) \in RT_0$, where RT_0 denotes the lowest order Raviart-Thomas space. Furthermore, we assume that all variables are known at cell centres at time-level t^n. Again to obtain saturation values S^{n+1} at time-level t^{n+1} we split Eq.(3) into a hyperbolic equation

$$\frac{\partial S}{\partial t} + \mathbf{v}.\nabla F(S) = 0 \tag{4}$$

and a parabolic heat type equation

$$\frac{\partial S}{\partial t} = \varepsilon \nabla . (D \nabla S) \tag{5}$$

Where S and $F(S)$ are vectors of saturation and fluxes respectively. We solved the equations in a standard operator splitting method.

Now consider the solution of Eq.(4) and it is found that on streamlines $\mathbf{r} = \mathbf{r}(\xi)$ such that

$$\frac{d\mathbf{r}}{d\xi} = \mathbf{v} \tag{6}$$

Eq.(4) becomes one dimensional so that

$$\frac{\partial S}{\partial t} + F_{\xi}(S) = 0 \tag{7}$$

In addition, we exploit this to obtain new saturation-values at cell centres $\mathbf{x}_I$, in the following way, we have to first trace streamlines Eq.(6) analytically for $\mathbf{r}(0) = \mathbf{x}_I$, and $-\xi_{max} < \xi < \xi_{max}$. Furthermore, $\xi_{max} = |\lambda_{max}| (t^{n+1} - t^n)$ with λ_{max} being the estimate the of the maximum wave speed of the system, such that the streamline covers the domain of dependence for $(\mathbf{x}_I, t^{n+1})$. Moreover, the streamline is only traced in the upstream direction if all the wave speeds are positive. The piecewise constant cell values of the saturations are then projected onto these local streamlines, thus mentioning piecewise constant initial conditions for Eq.(7), and this conveniently arranges for Eq.(7) to be solved for the front tracking method.

It is observed that the main advantages of the front tracking method, is that the method is super fast (H, Hollden, L, Holden, and R, Hoegh-Krohn, 1998), and preserves the frontal structure of the solutions extremely well. On the other hand, the method depends on solving Riemann problems; it is not easy to extend the method to the three-phase flow. In section 2.2 we will demonstrate the solution strategy for so-called triangular systems which may be step towards an application solution procedure for three-phase flow problems.

Eq.(5) may be solved by finite element or finite difference based methods. In this work we have to convenience use a standard explicit central finite difference scheme. The overall significance is that a local time step, $\Delta t_{diff} \leq t^{n+1} - t^{n}$, is required to satisfy the stability constraint inherent in the explicit finite difference method.

2.2 *Triangular systems*

The extension of *Streamline Front Tracking Method* (SFTM) approach to three-phase flow, is to use an approximate Riemann solver to generate front speeds. However, to our knowledge, the construction of accurate and reliable Riemann solvers for fully coupled three-phase flow is not a trivial task, we have chosen a different approach in our studies. Moreover, the viscosity of gas phase is usually at least an order less than the viscosities of the liquid phases. Motivated by this fact that, it seems reasonable to assume that the fractional flow function of the gas phase can be approximated by a flux function which only depends on the saturation of the gas phase. We will demonstrate 2×2 triangular hyperbolic system, as an approximation to the hyperbolic part of the fully coupled system (H, Dahle M, Espedal and O, Saevaried, ,1992]

$$\frac{\partial S_g}{\partial t} + \frac{\partial}{\partial x} F_g\left(S_g\right) = 0 \tag{8}$$

$$\frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x} F_w\left(S_g, S_w\right) = 0 \tag{9}$$

systems in the above have been investigated in (T. Gimse, 1988; 1989, L. Holden, and R. Hoegh-Krohn. 1990).

2.3 *A Riemann solver for triangular systems*

The numerical construction of a solution for Riemann problem associated with equations (8)-(9) was developed by (T. Gimse, 1988). In the present work the idea is to solve the Riemann problem for Eq.(8) first. The approximate solution to Eq.(8) consists of a set of constant states as $S_g^L = S_g^1 < S_g^2 < S_g^3 \ldots\ldots\ldots . < S_g^{N+1} = S_g^R$, separated by jump discontinuities traveling with the Rankine-Hugoniot shock speed:

$$s_i = \frac{F_g\ (S_g^{i+1}) - F_g\ (S_g^i)}{S_g^{i+1} - S_g^i} \qquad i = 1,2,3\ldots\ldots.N \tag{10}$$

Within each wedge of the solution fan, the fractional flow of the water phase depends only on the water saturation and is follows as

$$F_w^i(S_w) \overset{def}{=} F_w\left(S_g^i, S_w\right), \qquad i = 1,2,3\ \ldots N+1$$

Hence we may easily solve for the water saturation within each wedge once we know the left and right-hand state of the water saturation within the wedge. Obviously, successive left and right-hand states over the discontinuities in the gas phase must also satisfy the jump condition

$$s_i = \frac{F_w^{i+1}(S_w^{i+1}) - F_w^i\,(S_w^i)}{S_w^{i+1} - S_w^i} \quad i = 1, 2, 3 \ldots \ldots N \tag{11}$$

There are infinitely many states S_w^i which satisfy conditions Eq.(11), leads H sets

$H_{1,in}$ The set of S_w - values in region $S_{g,}^L$ that can be reached from $S_{w,}^L$ with speed $\sigma \leq s_I$.

$H_{1,in}$ The set of S_w - values in region $S_{g,}^{i+1}$, i =1,2,3….N that can be reached from $H_{i,out}$ with speed σ such that $s_{i-1} \leq \sigma \leq s_i$.

$H_{i+1,out}$ The set of S_w - values in region $S_{g,}^{i+1}$, i =1,2,3….N+1 that can be reached from $H_{i,in}$ by shock with speed s_i.

In addition the solution can be assembled by connecting the right state, $S_{w,}^L$, by admissible waves given by H sets. This has been explained in the in the following

$$S_w^R \rightarrow H_{n+1,out} \rightarrow H_{n,in} \rightarrow \cdots \ldots ..$$
$$H_{i+1,out} \rightarrow H_{n,in} \rightarrow \cdots$$
$$H_{2,out} \rightarrow H_{1,in} \rightarrow S_w^L \ldots .$$

In concrete it is found that a *jump* from one set to the next always happens at the first possible value in the current H set. Gimse has demonstrated (T. Gimse, 1988; 1989; L. Holden and R .Hoegh-Krohn., 1990) that the above construction gives a solution of the triangular system, if the following conditions hold:

(A) $F_g(0) = 0$, (B) $F_g(1) = 1$, (C) $F_w(S_g, 1- S_g) = 1- F_g(S_g)$

(D) $\frac{\partial F_w}{\partial S_w} \geq 0$, (E) $\frac{\partial F_w}{\partial S_g} < 0$, (F) $F_g'\left(S_g\right) \geq 0$.

3 Approximation of a full three – phase flow system by a triangular system

In the three-phase flow system the fractional flow function for the gas phase is nearly independent of the water saturation; it seems natural to decouple the gas phase from the other phases; and this can be done by plugging in a value for the water saturation, S_w^0, in gas fractional flow function:

$$F_g(S_w, S_g) \approx F_g(S_w^0, S_g) \stackrel{\text{def}}{=} F_g(S_g) \tag{12}$$

Moreover, we can choose S_w^0, so that the decoupled fractional flow function is as close as possible to the complete function in some norm over the admissible section of state space [H, Nordhaug, 1998)

In situation where the full solution for the gas phase consists of two rarefaction waves connected by an intermediate shock wave, the triangular approximation must fail because it cannot produce such a wave structure. In the above study, a possible to this problem is to S_w^0 be demonstrated locally for each Riemann problem which is solved. In addition, this will allow some feedback, and can be combined with the fact that the total mobility λ_T acts as an approximate invariant for the transport. Thus $\lambda_T\left(S_w, S_g\right) \approx \lambda_T\left(S_w^0,\ S_g\right)$ may be used to eliminate S_w from F_g in a more accurate way. However, this idea is not pursued any further in the paper. Another difficulty that arises from the approximation in Eq.(11) is that

$$F_w\left(1 - S_{g,}S_g\right) = 1 - F_g\left(1 - S_g, S_g\right) \neq 1 - F_g\left(S_w^0,\ S_g\right) \tag{13}$$

If the condition in Eq.(12) is not satisfied the construction of the solution may fail in two ways for values close to S_0 = 0: either the construction of appropriate H-sets will fail, or the tracking of admissible waves becomes impossible. Since $S_0 \approx 0 \Rightarrow S_w \approx 1–S_g$, we may circumvent the problem by replacing S_w^0 with $1–S_g$ in Eq.(12) when the oil phase is close to residual. Again, this requires a local representation of S_w^0 which has not been demonstrated in our simulator so far.

4.Results and Discussion

In the present paper we have studied a front taking streamline method (SFTM) for two-phase flow in porous media:

[i] it is found that the main advantage of the above method is to handle the advective part of the nonlinear transport using streamline information.

[ii] it is still be able to solve for diffusive / dispersive effects on a regular grid.

[iii] it is found that that the method has been compared with a modified method of characteristic (MMOC) and a Fast Marching (FMM) approach for two-phase flow problems.

[iv] it is also found that the solution obtained seem to be equally accurate.

[v] *front tracking streamline method* and the modified method of characteristics are comparable when it comes to computational efficiency ; whereas fast marching method gives a much faster advection solver.

[vi] however, it is compared to the modified method of characteristic and the fast marching method, the front tracking streamline method is more flexible and has lees restrictions with respect to the complexity of the problems that may be solved.

[vii] using the *H* set method, the front tracking streamline method approach has been extended to solve three-phase flow problems which are triangular.

[viii] since most three-phase flow problems are fully coupled in both saturations, triangular systems can only be approximate.

[ix] because the viscosity of the *gas* is much smaller than the viscosities of the liquid phases the gas phase is often nearly decoupled from the other phases.

[x] it is shown in this studies that the solution of a naïve triangular approximation of a fully coupled system, may and may not be a good approximation to the solution of the fully coupled system.

5 Acknowledgement

The present work has been financially supported by the University of Grants Commission under grant No. 40-236/2011 (SR). The authors would also like to acknowledge the financial support from the Council of Scientific and Industrial Research CSIR, (EMR-II) Division, New Delhi-110012, No. 25(0190)/10/EMR-II.

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