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Effect of Gear Design Variables on the Dynamic Stress of Multistage Gears

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Abstract

This work presents a numerical model developed to simulate and optimize the dynamic stress of multistage spur gears. The model was developed by using the Lagrangian energy method and modified Heywood method, and applied to study the effect of three design variables on the dynamic stress on the gears. The first design variable considered was the module, and the results showed that increasing the module resulted to increased dynamic stress levels. The second design variable, pressure angle, had a strong effect on the stress levels on the pinion of a high reduction ratio gear pair. A pressure angle of 25^o resulted to lower stress levels for a pinion with 14 teeth than a pressure angle of 20^o. The third design variable, the contact ratio, had a very strong effect on bending stress levels. It was observed that increasing the contact ratio to 2.0 reduced dynamic stresses significantly. For the gear train design used in this study, a module of 2.5 and contact ratio of 2.0 for the various meshes was found to yield the lowest dynamic stress levels on the gears. The model can therefore be used as a tool for obtaining the optimum gear design parameters for optimal dynamic performance of a given multistage gear train.

Keywords: dynamic load, dynamic bending stress, gear design parameters, mesh stiffness, multistage gear train

1. Introduction

Gears are important machine elements in most power transmission applications, such as automobiles, industrial equipment, airplanes, helicopters and marine vessels. These power transmission elements are often operated under high speeds and/or high torques and hence their dynamic analysis becomes a relevant issue due to durability of the gears and controlling vibrations and noise (Tamminana et al., 2005).

The physical mechanism of gear meshing has a wide spectrum of dynamic characteristics including time varying mesh stiffness and damping changes during meshing cycle (Tamminana et al., 2005). Additionally, the instantaneous number of teeth in contact governs the load distribution and sliding resistance acting on the individual teeth. These complexities of the gear meshing mechanism have led prior researchers (Bonori et al., 2004; Faith & Milosav, 2004; Gelman et al., 2005; Kuang & Lin, 2001; Parker et al., 2000; Vaishya & R. Singh, 2001; Vaishya & Singh, 2003) to adopt analytical or numerical approaches to analyze the dynamic response of a single pair of gears in mesh. A large number of parameters are involved in the design of a gear system and for this reason; modeling becomes instrumental to understanding the complex behavior of the system. Provided all the key effects are included and the right assumptions made, a dynamic model will be able to simulate the experimental observations and hence the physical system considered. Thus a dynamic model can be used to reduce the need to perform expensive experiments involved in studying similar systems. The models can also be used as efficient design tools to arrive at an optimal configuration for the system in a cost effective manner. Mechanical power transmission systems are often subjected to static or periodic torsional loading that necessitates the analysis of torsional characteristics of the system (Timothy, 1998). For instance, the drive train of a typical tractor is subjected to periodically varying torque. This torque variation occurs due to, among other reasons, the fluctuating nature of the combustion engine that supplies power to the gearbox (Timothy, 1998). If the frequency of the engine torque variation matches one of the resonant frequencies of the drive train system, large torsional deflections and internal shear stresses occur. Continued operation of the gearbox under such a condition leads to early fatigue failure of the system components (Timothy, 1998). Dynamic analysis of gears is essential for the reduction of noise and vibrations in automobiles, helicopters, machines and other power transmission systems. Sensitivity of the natural frequencies and vibration modes to system parameters provide important information for tuning the natural frequencies away from operating speeds, minimizing response and optimizing structural design (J. Lin & R. G. Parker, 1999).

Few models for the dynamic analysis of a multistage gear train have been developed (Choy et al., 1989; Jia et al., 2003; Krantz & Rashidi, 1995; Jian Lin & Parker, 2002) and those that exist treat either the shafts of the gear system or the gear teeth as rigid bodies depending on the purpose of the analysis. Effect of varying gear design parameters on the dynamics of a multistage gearbox in order to obtain the optimum parameters for a given gear train has also not been explored. Herbert and Daniel (Sutherland & Burwinkle, 1995) showed that gearboxes must be evaluated for dynamic response on an individual basis. There is Therefore need to develop a general model for a multistage gear train vibrations and one that can be used to obtain the optimum gear design parameters (module, addendum and pressure angle) based on vibration levels, dynamic load and dynamic root stress.

With the advancement of Computer Aided Drafting and Design (CADD) softwares like Mechanical Desktop and Autodesk Inventor, the design of gear trains in terms of relative sizes has been made easy. With Autodesk inventor, it is possible to simulate the relative movement of various parts in the design and any interference can be corrected at this stage of the design without having to first fabricate the prototype. However, it is necessary to carry out vibration and dynamic analysis in order to predict the performance of the system before the various parts are fabricated. The effect of the various gear design parameters on the vibration and dynamic characteristics also need to be analyzed in order to optimize the design. The aim of this work is to develop a general model to analyze the vibrations of a multistage gear train taking into account time varying mesh stiffness, time varying frictional torque and shaft torsional stiffness. The model will then be used to analyze the effect of gear design parameters on the vibration levels and gear tooth root stress with the aim of identifying the optimum configurations of the gearbox.

2. Model Formulation

The model developed in this work is based on a four-stage reduction gearbox (Figure 1) with an overall reduction ratio of 54:1 on gear train I. The orthographic view is shown in Figure 2. The gearbox contains five pairs of gears in mesh, the input and output inertias, five shafts and bearings. The major assumptions made in developing the model include:

i. Gears are modeled as rigid disk with radius equal to the base circle radius and flexibility at the gear teeth.
ii. Each gear is supported by a pair of lateral springs to represent the lateral deflection of shafts and bearings. This implies the simplifying assumption that the gear may move laterally but do not tilt.
iii. Shaft torsion is represented by equivalent torsion spring constants.
iv. The casing is assumed to be rigid (deflections are much smaller than the deflections of the gear teeth, shafts and bearings and can be neglected.)
v. Static transmission error effects are much smaller than the dynamic transmission error effects and so they can be neglected (Jia et al., 2003).
vi. Gear teeth are assumed to be perfectly involute and manufacturing and assembly errors are ignored.
vii. Backlash is not considered in this model. This is because while running at steady state, the gears are loaded in a single direction only and thus tooth separation is not considered.

The resulting model is shown in Figure 3, while a detailed gear pair model is shown in Figure 4. The mathematical model shown in Figure 2 can be described by a total of 33 coordinates. The rotational position of the gears, input and output inertias require thirteen coordinates. The lateral positions of the gears due to the lateral deflection of the shafts and bearings require another twenty coordinates.

A set of governing equations of motion for the model was derived using the standard Lagrangian equation, which is given here without proof (James et al., 1994):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \quad (1)$$

Where,

- q_i Generalized coordinate
- T Total kinetic energy of the system
- U Change in potential energy of the system with respect to its potential energy in the static equilibrium position.
- Q_i Generalized non-potential forces or moments resulting from excitation forces or moments that add energy into the system, and damping forces and moments that remove energy from it.

The kinetic energy of the sytem is given by:

$$T = \frac{1}{2}\sum\left(J_i\dot{\theta}_i + m_i\dot{x}_i + m\dot{y}_i\right) \quad (2)$$

The potential energy is classified into three groups of stored energy caused by:

1) Distortion of the gear meshes, for example the potential energy stored in the gear mesh in Figure 4 is expressed as:

$$V_{m1} = \frac{1}{2}K_g(t)\left[R_2\theta_2 - R_3\theta_3 - (y_1 - y_2)\cos\gamma + (x_1 - x_2)\sin\gamma\right]^2 \quad (3)$$

2) Twisting of gear shafts, for example the potential energy stored in shaft 1 (Fig. 2) is expressed as:

$$V_{s1} = \frac{1}{2}K_{s1}\left[\theta_1 - \theta_2\right]^2 \quad (4)$$

3) Lateral deflection of the shafts and bearings, expressed as:

$$V_{sl} = \frac{1}{2}\sum\left(K_{xi}x_i^{\ 2} + K_{yi}y_i^{\ 2}\right) \quad (5)$$

1.2 Solution Method

A numerical computer program in FORTRAN code was developed to study the time domain behavior of the system(J. K. Kimotho, 2008). The time domain behavior of the system was obtained by integrating the set of governing differential equations using **4th order Runge-Kutta** method. The differential equations were linearized by dividing the mesh period of the output pair into many small intervals. The mesh period for any pair of teeth in mesh was taken as the time interval from the initial point of contact to the highest point of single tooth pair contact.

To integrate initial value problems, an appropriate set of initial conditions is required. In this study, all generalized coordinates were set to zero. Starting with this initial estimates of $\theta_i(0)$ and $\dot{\theta}_i(0)$ at the initial contact point, the values of $\theta_i(t)$ and $\dot{\theta}_i(t)$ were calculated for one mesh period of the output pair of gears.

The calculated value of the relative displacement $\delta_i(\tau_j)$ and relative velocity $\dot{\delta}_i(\tau_j)$ after the end of one period τ_j of each pair of gears j in mesh were compared with the initial values $\delta_i(0)$ and $\dot{\delta}_i(0)$. Unless the difference between them was sufficiently small (≤ 0.002%), an iteration procedure was used to obtain the $(i+1)^{th}$ iteration values of $\theta_i(t)$ and $\dot{\theta}_i(t)$ by taking the ith iteration values of $\theta_i(t)$ and $\dot{\theta}_i(t)$ as the new initial trial conditions. Once the solution has converged, this state corresponds to the steady state rotational speed of the shafts.

1.2.1 Dynamic Bending Stress

Tooth bending failure at the root is a major concern in gear design. If the bending stress exceeds the fatigue strength, the gear tooth has a high probability of failure. In this study, a modified Heywood formula (P.-H. Lin et al., 1998) for tooth root stress was used for the dynamic stress calculation at the root of a gear tooth. This formula has been found to correlate well with experimental data and finite element analysis results (P.-H. Lin et al., 1998). This formula is expressed as:

$$\sigma_j = \frac{W_j \cos\beta_j}{F}\left[1+0.26\left(\frac{h_f}{2R_f}\right)^{0.7}\right]\left[\frac{6l_f}{h_f^{\,2}}+\sqrt{\frac{0.72}{h_f l_f}}\left(1-\frac{h_l}{h_f}\tan\beta_j\right)-\frac{\tan\beta_j}{h_f}\right] \tag{6}$$

where, σ_j is the root bending stress, h_f is the tooth thickness at the critical section, R_f is the fillet radius, l_f is the length of the tooth from the projected point of contact on the neutral axis to the critical section and γ is the angle between the form circle and the critical section. The tooth geometry is shown in Figure 5.

3. Results and Discussions

This section presents the results of the computed dynamic load and dynamic bending stress on the gears. Table I shows the operating conditions and gear parameters.

The relative dynamic displacement of gear *i* and *i* + 1 represents the deflection of the gear teeth from their mean position. If gear *i* is the driving gear, the following situations will occur (H.-H. Lin, 1985):

i. $\delta_i > 0$: This represents the normal operating case and the dynamic mesh force is given by:

$$W_{di} = K_{gi}(t)\delta_i + C_{gi}\dot{\delta}_i \tag{7}$$

ii. $\delta_i \le 0$ and $|\delta_i| \le bh$,

where *bh* is the backlash between meshing gears. In this case, gears will separate and contact between meshing teeth will be lost.

$$W_{di} = 0 \tag{8}$$

iii. $\delta_i < 0$ and $|\delta_i| < bh$,

In this case, gear i+1 will collide with gear i on the back side and the meshing force will be given by:

$$W_{di} = K_{gi}(t)[\delta_i - bh] + C_{gi}\dot{\delta}_i \tag{9}$$

Where, W_{di} is the dynamic load. In this study, one of the assumptions in the development of the model was that there was no backlash, therefore only the first case was considered.

The dynamic relationship between all the gear stages is coupled through the non-linear interactions in the gear mesh. The gear mesh forces and moments were evaluated as functions of relative motion and rotation between two meshing gears and the corresponding mesh stiffness as shown in Equation 7 (J. Kimotho & Kihiu, 2010).

Figures 6 and 7 compare the static and dynamic stress on a single tooth of all the gears in mesh. The root bending stress on the gear teeth depends on the magnitude of the dynamic force and the position of the force along the path of contact. For the driving gear, the point of contact moves from the lowest point of contact along the tooth profile to the highest point of contact and thus the cantilever beam length of the gear tooth increases along the path of contact. This explains why both the static and dynamic stresses increase with time for the driving gear. The converse is true for the driven gear.

3.1 Effect of Gear Design Variables

In order to optimize the design with respect to gear design parameters, the effect of varying the following gear design parameters was investigated:

I. module
II. pressure angle
III. contact ratio

3.1.1 Effect of module

The effect of the module of the gear dynamics was investigated by changing the module from 3.0 to 2.5 and 2.0, while holding the pressure angle and pitch radius constant. In order to maintain the pitch radius constant, the number of teeth was varied. However, since the number of teeth for any gear is an integer, the pitch radius of some gears varied slightly (by less than 0.5 mm).

Figure 8 shows sample dynamic bending stress curves for a pair or gears as function of the contact position. It can be observed that reducing the module of a pair of gears increases the dynamic bending stress significantly. This could be attributed to the smaller tooth thickness at the root for gears with a smaller module.

3.1.2 Effect of Pressure Angle

The pressure angle was increased from 20^o to 25^o while holding the module and number of teeth for the various meshes constant. From the sample bending stress levels (Figure 9) the peak bending stress on the pinion (14T) with a pressure angle of 20^o is higher than that with a pressure angle of 25^o which is attributed to the addendum modification of the teeth with a pressure angle of 20^o to reduce interference. This modification increases the length of the tooth and consequently the cantilever effects on the tooth.

3.1.3 Effect of Contact Ratio

The contact ratio of a pair of gears in mesh is given by Equation 9 and is affected by the following parameters:

- addendum
- center distance
- pressure angle
- module

$$C.R = \frac{\sqrt{R_{o1}^{2} - R_{b1}^{2}} + \sqrt{R_{o2}^{2} - R_{b2}^{2}} - (R_{p1} + R_{p2})\sin\phi}{p_c \cos\phi} \qquad (10)$$

The contact ratio of a gear pair can be increased by varying one of the above parameters or a combination of two or more of these parameters. Increasing the addendum is normally recommended for increasing the contact ratio since this can be achieved by simply adjusting the cutter depth (H.-H. Lin, 1985). The maximum permissible addendum modification coefficients are obtained by iteratively varying the addendum modification coefficient of the pinion and gear until the top land thickness is equal to the minimum allowable (usually 0.3*m*) (Dudley, 1962). In this research work, a code was developed to obtain the maximum possible contact ratio for a gear pair by varying the addendum and adjusting the center distance in order to avoid interference (Kuria & Kihiu, 2008).

A contact ratio close to 2.0 also results to a smooth root stress curve as shown on Figure 10. A contact ratio of 2.0 reduces the peak dynamic root stresses on the gear teeth by about 45% in both cases. In addition, the discontinuities in the stress curves that occur during the transition from double tooth contact to single tooth contact and vice versa are eliminated. This implies that the gears with a contact ratio of 2.0 would have a higher fatigue life than those with a contact ratio lower than 2.0.

The speed of the gearbox is varied by sliding the speed gears into mesh. This means that the rate of wear for these gears is very high. Thus, the wear rate should be taken into consideration when selecting the appropriate gear module for this application.

4. Conclusions

A mathematical model was developed to analyze the dynamic stress of multistage gears. The model consists of 33 equations of motion which were derived using the Lagrangian energy method and solved using Fourth Order Runge Kutta method. The main sources of excitation for the gear train were the time varying gear mesh stiffness and the time varying frictional torque on the gear teeth. The effect of torsional stiffness of the shafts and lateral stiffness of the shafts and bearing stiffness were considered in the model. Parametric studies were also conducted to examine the effects of three design variables, module, pressure angle and contact ratio on the dynamic stress of the gears. The following specific conclusions can be drawn from the study:

1) Reducing the module reduces the dynamic bending stress on the gear teeth and therefore increase gear life. Increased gear life means: Low maintenance costs, low operating costs, increased production of plant/ machine since there will be no downtime, fewer accidents in the plant.
2) Increasing the pressure angle of the gears from 20^o to 25^o results to reduced stress levels since the gears with a higher pressure angle requires less or no addendum modification depending on the number of teeth.
3) A high contact ratio also leads to lower bending stress levels. Particularly for a contact ratio of 2.0, the discontinuities on the stress curves observed due to variation in the number of teeth in contact is eliminated. Therefore, gears with a contact ratio of 2.0 would have a higher fatigue life.

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Table 1. Operating conditions and gear parameters for the initial design.

Input speed	1500 rpm
Nominal Torque	1300 Nm
Module	3.0 mm
Pressure angle	20^o
ζ_g	0.1
ζ_s	0.05

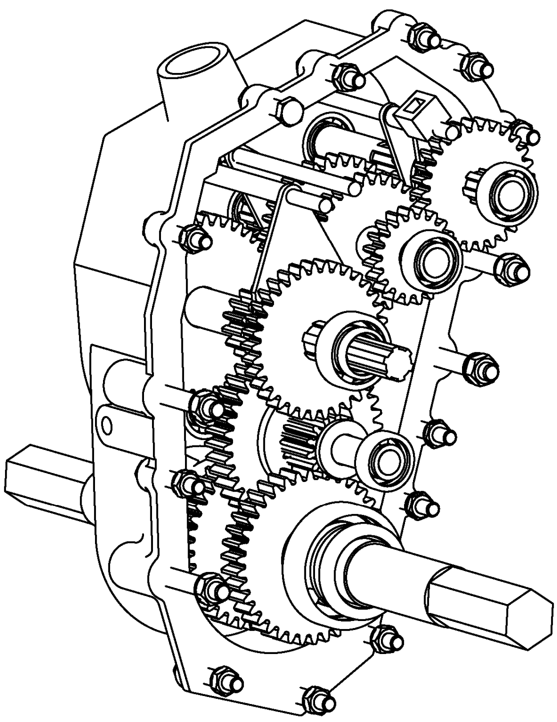

Figure 1. Multistage tractor gearbox.

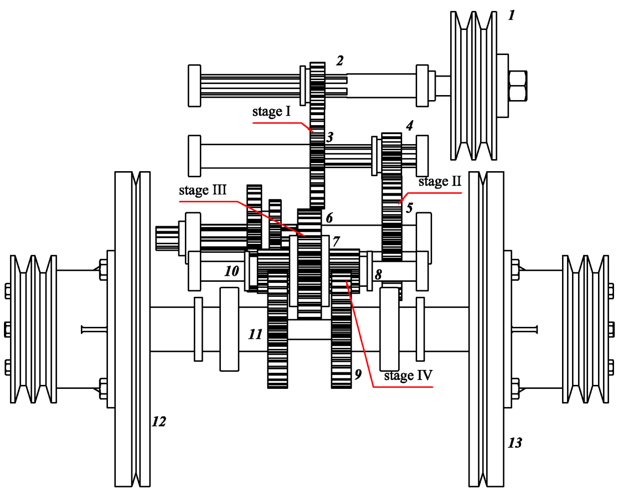

Figure 2. Gear Train for bottom gear ratio

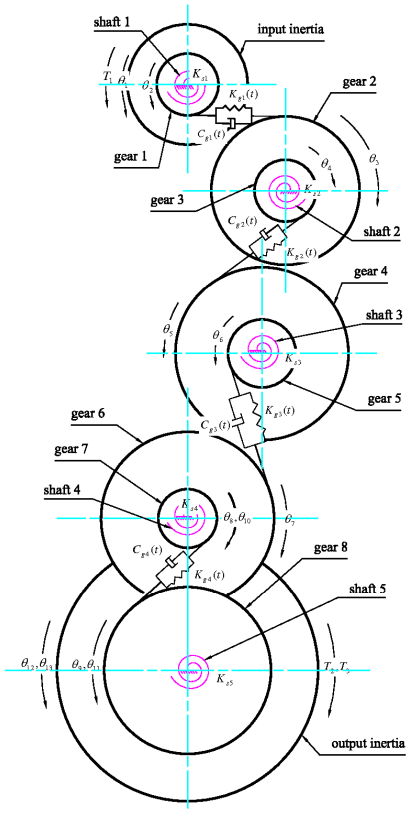

Figure 3. Gear train model.

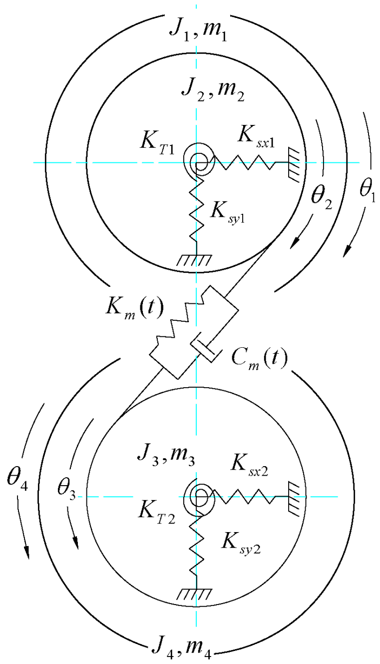

Figure 4. Detailed gear pair model.

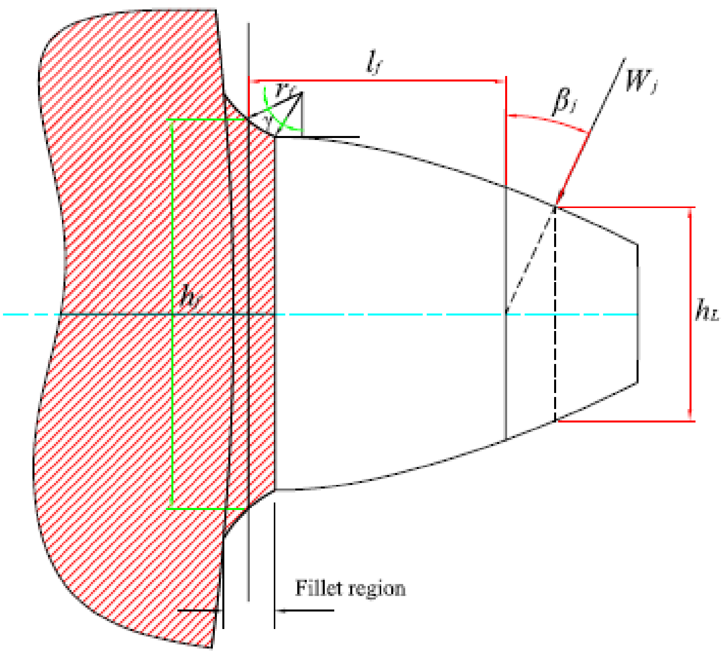

Figure 5. Tooth geometry nomenclature for root stress calculation (P.-H. Lin et al., 1998).

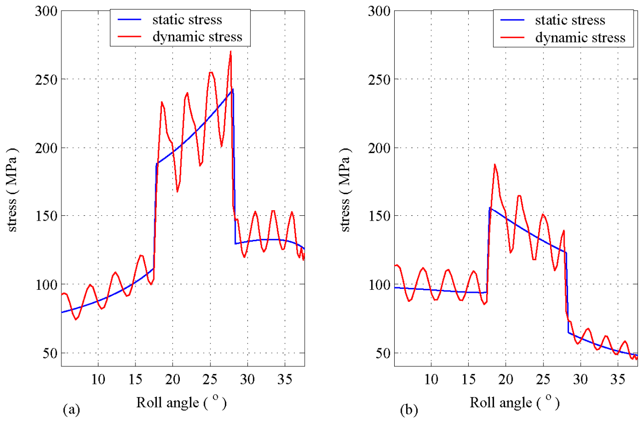

Figure 6. Tooth bending stress as a function of the contact position for gears in stage I, (a) pinion and (b) gear.

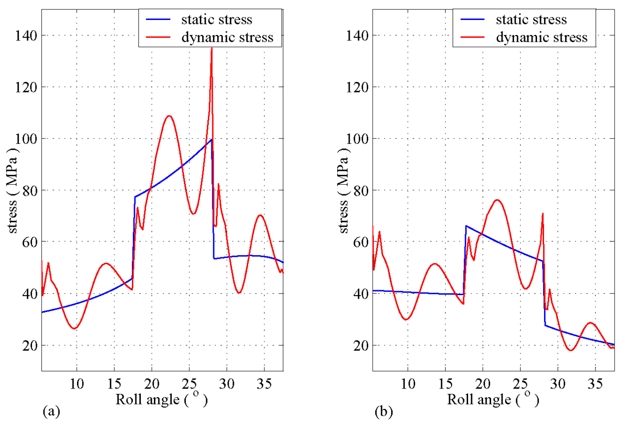

Figure 7. Tooth bending stress as a function of the contact position for gears in stage II, (a) pinion and (b) gear.

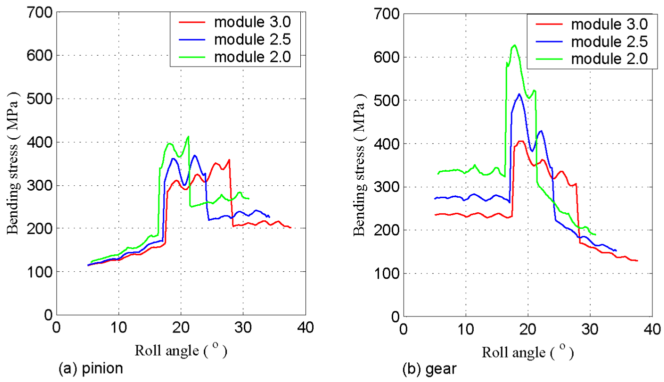

Figure 8. Root stress on stage IV of gear train 1 for different modules.

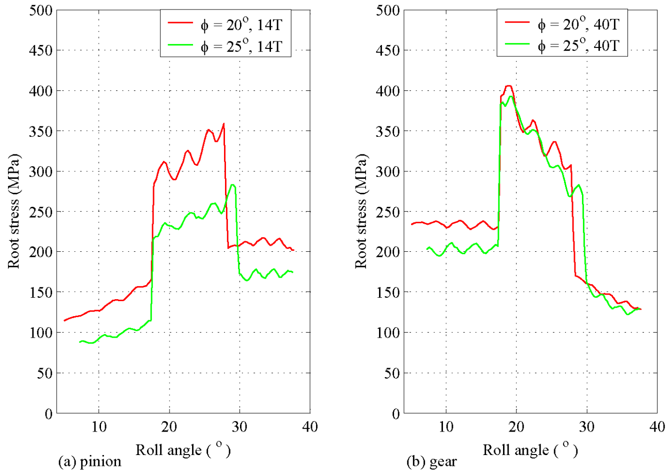

Figure 9. Sample root stress for gears with different pressure angles.

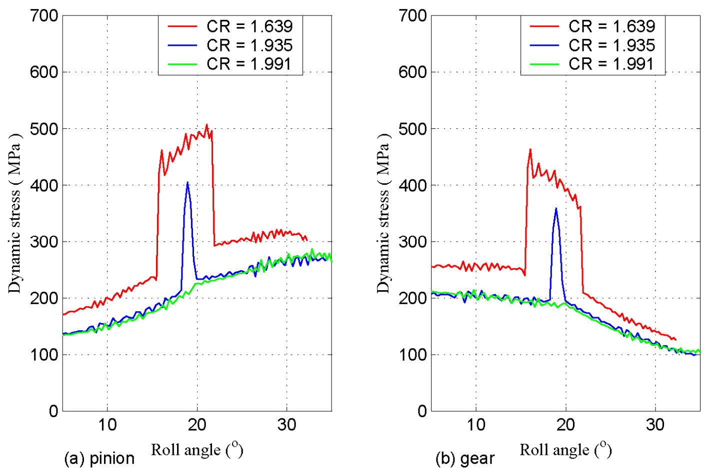

Figure 10. Root stress on stage IV gears of gear train 1 for different contact ratios using a module of 2.5.