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A Common Fixed Point Theorems in Menger Space using Occationally Weakly Compatible Mappings

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Abstract
In this paper we have improved the result of Saurabh Manro [7] by using the concept of occasionally weakly compatible Maps and proved some results on fixed points in menger space.
Key words: Menger space, Common fixed point, occasionally weakly compatible mappings.

1. Introduction:

In 1942 Menger [4] introduced the notion of a probabilistic metric space (PM-space) which is in fact, a generalization of metric space. The idea in probabilistic metric space is to associate a distribution function with a point pair, say (x, y), denoted by F(x, y; t) where t > 0 and interpret this function as the probability that distance between x and y is less than t, whereas in the metric space the distance function is a single positive number. Sehgal [8] initiated the study of fixed points in probabilistic metric spaces. The study of these spaces was expanded rapidly with the pioneering works of Schweizer-Sklar [1]. A weakly compatible map in fuzzy metric space is generalized by A. Al. Thagafi and Nasser Shahzad [1] by introducing the concept of occasionally weakly compatible mappings. Our paper improves the result of Saurabh Manro [7] by using of occasionally weakly compatible Maps and proved some results on fixed points in menger space.

2. Preliminaries:

First, recall that a real valued function f defined on the set of real numbers is known as a distribution function if it is nondecreasing, continuous and inf f(x) = 0, sup f(x) = 1. We will denote by L, the set of all distribution functions.

Definition 2.1: A probabilistic metric space (PM-space) is a pair (X, F) where X is a set and F is a function defined on X X to L such that if x, y and z are points of X, then

(F-1) $F_{x,y}(t) = 1$ for every t > 0 iff x = y,

(F-2) $F_{x,y}(0) = 0$,

(F-3) $F_{x,y}(t) = F_{y,x}(t)$,

(F-4) if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(s + t) = 1$ for all x, y, z∈ X and s, t ≥ 0.

For each x, y∈ X and for each real number t > 0, $F_{x,y}(t)$ is to be thought of as the probability that the distance between and y is less than t.

It is interesting to note that, if (X, d) is a metric space, then the distribution function F(x, y; t) defined by the relation F(x, y; t) defined by the relation F(x, y; t) = H(t – d(x, y)) induces a PM-space where H(x) denotes the distribution function defined as follows:

$$H(x) = \begin{cases} 0 & if\ x \le 0 \\ 1 & if\ x > 0 \end{cases}$$

Definition 2.2: A t-norm is a 2-place function, t:[0,1]×[0,1]→[0,1] satisfying the following:

(i) t(0,0) = 0, (ii) t(0,1) = 1, (iii) t(a, b) = t(b, a), (iv) if a ≤ c, b ≤ d, then t(a, b) ≤ t(c, d),

(v) t(t(a, b),c) = t(a, t(b, c)) for all a, b, c∈ [0,1].

Definition 2.3: A Menger PM-space is a triplet (X, F, t) where (X, F) is a PM-space and t is a t-norm with the following condition:

(F-5) $F_{x,z}(s + p) \ge t(F_{x,y}(s), F_{y,z}(p))$,for all x, y, z ∈ X and s, p ≥ 0.

This inequality is known as Menger's triangle inequality.

In our theory, we consider (X, F, t) to be a Menger PM-space with the additional following postulate: (F-6) $lim_{t\to\infty} F_{x,y}(t) = 1$ ∀ x, y ∈ X.

Definition 2.4: A menger space (X, F, t) is said to be complete if and only if every Cauchy sequence in X is convergent.

In 1996, Jungck [2] introduced the notion of weakly compatible maps as follows:

Definitoin 2.5: A pair of self mappings (A, S) on set X is said to be weakly compatible if they commute at the coincidence points i.e. Au = Su for some u∈ X, then SAu = ASu.

We need the following Lemmas due to Schweizer and Skalr [1] and Singh and Pant [6], in the proof of the theorems:

Lemma 2.1: Let (X, F, t) be a menger space and if for a number k ∈ (0,1) such that $F_{x,y}(kt) \geq F_{x,y}(t)$. Then x = y.

Definition: Let X be a set, f and g selfmaps of X. A point x∈X is called a coincidence point of f and g iff fx = gx. We shall call w = fx = gx a point of coincidence of f and g.

Definition 2.6[3]: Two self mappings A and S of a non-empty set X are OWC iff there is a point x∈X which is a coincidence point of A and S at which A and S commute.

The notion of OWC is more general than weak compatibility (see [5]).

Lemma 2.2[3]: Let X be a non-empty set, A and B are occasionally weakly compatible self maps of X. If A and B have a unique point of coincidence, w = Ax = Bx, then w is the unique common fixed point of A and B.

3. Main Results:

In our result, we used the following implicit relation:

Definition (Implicit Relation): Let I= [0, 1] and Ω be the set of all real continuous functions ϕ : $I^6 \rightarrow R$ satisfying the condition:

(i) ϕ is non increasing or non decreasing in third and fourth argument and
(ii) If we have ϕ(u, v, 1, 1, v, v) ≥1, for all u, v ∈ (0, 1) ⇒ u ≥ v.

Example: We define ϕ : $I^6 \rightarrow R$ by $\phi(u_1, v_1, v_2, v_3, v_4, v_5) = u_1 - v_1 + v_2 - v_3 + v_4 - v_5$

Then clearly continuous function such that if we have ϕ(u, v, 1, 1, v, v) ≥1, for all u, v∈ (0, 1),

Then ϕ(u, v, 1, 1, v, v) = u - v+ 1- 1+ v- v = u – v ≥ 1 ⇒ u ≥ v.

Theorem 3.1: Let (X,F,t) be a Menger space. Let A, B, S and T be self maps of X satisfying the following conditions:

1. (A, S) and (B, T) are owc.
2. there exist k ∈ (0,1) and $\emptyset \in \Omega$ such that

$$\emptyset\left(\left(F_{Ax,By}(kt)\right),\left(F_{Sx,Ty}(t)\right),\left(F_{Ax,Sx}(t)\right),\left(F_{By,Ty}(t)\right),\left(F_{Ax,Ty}(t)\right),\left(F_{By,Sx}(t)\right)\right) \geq 1 \quad \text{(I)}$$

for all x, y ∈ X and t > 0.

Then there exists a unique point w ∈ X such that Aw = Sw = w and a unique point z ∈ X such that Bz = Tz = z. Moreover, z = w, so that there is a unique common fixed point A, B, S and T in X.

Proof: Since the pairs (A, S) and (B, T) are owc, there exist points x, y ∈ X such that Ax = Sx, ASx = SAx and By = Ty, BTy = TBy. Now we show that Ax = By.

Then we have by inequality (I),

$$\emptyset\left(\left(F_{Ax,By}(kt)\right),\left(F_{Sx,Ty}(t)\right),\left(F_{Ax,Sx}(t)\right),\left(F_{By,Ty}(t)\right),\left(F_{Ax,Ty}(t)\right),\left(F_{By,Sx}(t)\right)\right) \geq 1$$

$$\emptyset\left(\left(F_{Ax,By}(kt)\right),\left(F_{Sx,Ty}(t)\right),\left(F_{Ax,Sx}(t)\right),\left(F_{By,Ty}(t)\right),\left(F_{Ax,Ty}(t)\right),\left(F_{By,Sx}(t)\right)\right) \geq 1$$

$$\emptyset\left(\left(F_{Ax,By}(kt)\right),\left(F_{Ax,By}(t)\right),\left(F_{Ax,Ax}(t)\right),\left(F_{By,By}(t)\right),\left(F_{Ax,By}(t)\right),\left(F_{By,Ax}(t)\right)\right) \geq 1$$

$$\emptyset\left(\left(F_{Ax,By}(kt)\right),\left(F_{Ax,By}(t)\right),1,1,\left(F_{Ax,By}(t)\right),\left(F_{By,Ax}(t)\right)\right) \geq 1$$

$$\emptyset\left(\left(F_{Ax,By}(kt)\right)\right) \geq \left(F_{Ax,By}(t)\right)$$

Thus by lemma 2.1 Ax = By. Therefore Ax = Sx= By = Ty.

Moreover, if there is another point z such that Az = Sz. Then using inequality (I) it follows that Az = Sz = By = Ty, or Ax = Az.

Hence w = Ax = Sx is the unique point of coincidence of A and S. By lemma 2.2, w is the unique common fixed point of A and S. Similarly, there is a unique point z∈ X such that z = Bz = Tz. Suppose that w ≠z and using inequality (I), we get

$$\emptyset\left(\left(F_{w,z}(kt)\right),\left(F_{w,z}(t)\right),\left(F_{w,w}(t)\right),\left(F_{z,z}(t)\right),\left(F_{w,z}(t)\right),\left(F_{z,w}(t)\right)\right) \geq 1$$

$$\emptyset\left(\left(F_{w,z}(kt)\right),\left(F_{w,z}(t)\right),1,1,\left(F_{w,z}(t)\right),\left(F_{z,w}(t)\right)\right) \geq 1$$

$$\emptyset\left(\left(F_{w,z}(kt)\right)\right) \geq \left(F_{w,z}(t)\right)$$

Thus by lemma 2.1 w = z.

Therefore z = Sz = Tz = Az = Bz.

To prove uniqueness, let u and v are two common fixed points of A, B, S and T in X. Therefore, by definition, Au = Bu = Tu = Su = u and Av = Bv = Tv = Sv = v.

Then by (I), take x = u and y = v, we get

$$\emptyset\left(\left(F_{u,v}(kt)\right),\left(F_{u,v}(t)\right),\left(F_{u,u}(t)\right),\left(F_{v,v}(t)\right),\left(F_{u,v}(t)\right),\left(F_{v,u}(t)\right)\right) \geq 1$$

$$\emptyset\left(\left(F_{u,v}(kt)\right),\left(F_{u,v}(t)\right),1,1,\left(F_{u,v}(t)\right),\left(F_{v,u}(t)\right)\right) \geq 1$$

$\emptyset\left(\left(F_{u,v}(kt)\right)\right) \geq \left(F_{u,v}(t)\right)$

Therefore, by lemma 2.1, u = v.

Hence the self maps A, B, S and T have a unique common fixed point in X.

Theorem 3.2: Let (X, M, t) be a menger space and let A, B, S, T, P and Q be self maps of X satisfying the following conditions:

(3.2.1) the pairs (A, SP) and (B, TQ) are owc;

(3.2.2) there exists $k\in(0,1)$ and $\phi\in\Omega$ such that $\phi\{F_{Ax,By}(kt), F_{SPx,TQy}(t), F_{Ax,SPx}(t), F_{By,TQy}(t), F_{Ax,TQy}(t), F_{By,SPx}(t)\} \geq 1$, $\forall$ x, y$\in$X and t > 0,

(3.1.3) the pairs (A, P), (S, P), (B, Q) and (T, Q) are commuting;

then A, B, S, T, P and Q have a unique common fixed point in X.

Proof: Since the pairs (A, SP) and (B, TQ) are owc, so there are points x, y$\in$X such that Ax = SPx implies A(SP)x = (SP)Ax and By = TQy implies B(TQ)y = (TQ)By.

We claim that Ax = By. Now by inequality (3.2.2) $\phi\{F_{Ax,By}(kt), F_{SPx,TQy}(t), F_{Ax,SPx}(t), F_{By,TQy}(t), F_{Ax,TQy}(t), F_{By,SPx}(t)\} \geq 1$,

We have, $\phi\{F_{Ax,By}(kt), F_{Ax,By}(t), F_{Ax,Ax}(t), F_{By,By}(t), F_{Ax,By}(t), F_{By,Ax}(t)\} \geq 1$, $\phi\{F_{Ax,By}(kt), F_{Ax,By}(t), 1, 1, F_{Ax,By}(t), F_{By,Ax}(t)\} \geq 1$,

$\Rightarrow F_{Ax,By}(kt) \geq F_{Ax,By}(t)$, thus by lemma 2.1 Ax=By.

Therefore Ax = SPx = By = TQy = z (say), then Az = SPz and Bz = TQz.

We claim that Az = Bz. Now by inequality (3.2.2) $\phi\{F_{Ax,By}(kt), F_{SPx,TQy}(t), F_{Ax,SPx}(t), F_{By,TQy}(t), F_{Ax,TQy}(t), F_{By,SPx}(t)\} \geq 1$,

We have, $\phi\{F_{Az,Bz}(kt), F_{SPz,TQz}(t), F_{Az,SPz}(t), F_{Bz,TQz}(t), F_{Az,TQz}(t), F_{Bz,SPz}(t)\} \geq 1$, $\phi\{F_{Az,Bz}(kt), F_{Az,Bz}(t), 1, 1, F_{Az,Bz}(t), F_{Bz,Az}(t)\} \geq 1$,

$\Rightarrow F_{Az,Bz}(kt) \geq F_{Az,Bz}(t)$, thus by lemma 2.1 Az = Bz. Therefore Az = SPz = Bz = TQz.

Now we prove Az = z, Now by inequality (3.2.2), we have (by taking x = z and By = z)

$\phi\{F_{Az,z}(kt), F_{Az,z}(t), F_{Az,Az}(t), F_{z,z}(t), F_{Az,z}(t), F_{z,Az}(t)\} \geq 1$, $\phi\{F_{Az,z}(kt), F_{Az,z}(t), 1, 1, F_{Az,z}(t), F_{z,Az}(t)\} \geq 1$,

$\Rightarrow F_{Az,z}(kt) \geq F_{Az,z}(t)$, thus by lemma 2.1 Az = z. Therefore z = Az = Bz = SPz = TQz.

Now we put x = Pz and y = z in inequality (3.2.2) we get $\phi\{F_{APz,Bz}(kt), F_{SPPz,TQz}(t), F_{APz,SPPz}(t), F_{Bz,TQz}(t), F_{APz,TQz}(t), F_{Bz,SPPz}(t)\} \geq 1$,

Since (A, P) and (S, P) are commuting; $\phi\{F_{Pz,z}(kt), F_{Pz,z}(t), F_{Pz,Pz}(t), F_{z,z}(t), F_{Pz,z}(t), F_{z,Pz}(t)\} \geq 1$, $\phi\{F_{Pz,z}(kt), F_{Pz,z}(t), 1, 1, F_{Pz,z}(t), F_{z,Pz}(t)\} \geq 1$,

$\Rightarrow F_{Pz,z}(kt) \geq F_{Pz,z}(t)$, thus by lemma 2.1 Pz = z. Since z = SPz $\Rightarrow$ Sz = z.

To show Qz = z, we put x = z and y = Qz in inequality (3.2.2) we get $\phi\{F_{Az,BQz}(kt), F_{SPz,TQQz}(t), F_{Az,SPz}(t), F_{BQz,TQQz}(t), F_{Az,TQQz}(t), F_{BQz,SPz}(t)\} \geq 1$,

Since (B, Q) and (T, Q) are commuting; $\phi\{F_{z,Qz}(kt), F_{z,Qz}(t), F_{z,z}(t), F_{Qz,Qz}(t), F_{z,Qz}(t), F_{Qz,z}(t)\} \geq 1$, $\phi\{F_{z,Qz}(kt), F_{z,Qz}(t), 1, 1, F_{z,Qz}(t), F_{Qz,z}(t)\} \geq 1$,

$\Rightarrow F_{Qz,z}(kt) \geq F_{Qz,z}(t)$, thus by lemma 2.1 Qz = z. Since z = TQz $\Rightarrow$ Tz = z.

Therefore Az = Bz = Sz = Tz = Pz = Qz = z. i.e. z is the common fixed point of A, B, S, T, P and Q.

To prove uniqueness: let r and s be two distinct common fixed points of A, B, S, T, P and Q.

Then Ar = Br = Sr = Tr = Pr = Qr = r and As = Bs = Ss = Ts = Ps = Qs = s,

Now by inequality (3.1.2), we have (at x = r and y = s) $\phi\{F_{r,s}(kt), F_{r,s}(t), F_{r,r}(t), F_{s,s}(t), F_{r,s}(t), F_{s,r}(t)\} \geq 1$,

$\phi\{F_{r,s}(kt), F_{r,s}(t), 1, 1, F_{r,s}(t), F_{s,r}(t)\} \geq 1$,

$\Rightarrow F_{r,s}(kt) \geq F_{r,s}(t)$, thus by lemma 2.1 r = s.

This completes the proof of the theorem.

Conclusion: Our theorem is an improvement of theorem 3.1 of saurabh manro [7]. In our theorem we do not require the completeness & continuity of the space and also condition (1) of [7, theorem 3.1]. Our theorem is true for any continuous t-norm. In our result we do not require to define many implicit relations.

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