Analytical Solution of Bending Stress Equation for Symmetric and Asymmetric Involute Gear Teeth Shapes with and without Profile Correction

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Abstract

In this work, a developed analytical method based on a previous trial graphical method has been introduced to find the solution of bending stress equation for symmetric and asymmetric involute gear teeth shapes with and without profile correction and for different gear design parameters. In order to achieve this analytical solution, certain geometrical properties for gear tooth shape of tooth loading angle, tooth critical section thickness, the load height from this section and Lewis form factor which are imperative to formulate the final form of the tooth bending stress equation must be determined analytically step-by-step. As a result of this work, the trial graphical method has been avoided by establishing a simple analytical expression which can be easily solved and it gives a higher accuracy and requires a shorter time.

Keywords: Gear, Involute Teeth, Symmetric, Asymmetric, Profile Correction, Critical Section Thickness, Lewis Form Factor, Bending Stress Equation, Analytical Solution.

1. Introduction

Since 1892, when Wilfred Lewis introduced an analytical equation for computing the bending stress in a gear tooth by approximating the tooth to parabolic beam of uniform strength, the analysis of gear stresses has been the subject of many investigations. Many analytical, experimental and numerical attempts were carried out to develop this Lewis equation, and the most of these attempts focused on the symmetric tooth. In this work, an analytical technique has been made to develop this equation to take into account the effect of asymmetric tooth profiles with and without profile correction (profile shift) for different gear design parameters such as module, number of teeth, pressure angles and root fillet radii for loaded and unloaded sides of tooth, as well as correction factor for each side of tooth. In order to formulate the final analytical solution of the tooth bending stress equation, the determination of certain geometrical properties of gear tooth shape such as (tooth loading angle, tooth critical section thickness, the load height from this section and tooth form factor) is imperative.

2. Involutometry

In considering the involute for a tooth profile form, it is necessary to be able to calculate many of geometrical characteristics of this involute profile before any tooth geometrical study. From (Figure 1) with the principle of involute construction, the following relations can be determined [1]:

$$\measuredangle\ AO_1L = \frac{arc\ (AL)}{O_1L} \tag{1}$$

and $$arc\ (AL) = LD \tag{2}$$

therefore; $\measuredangle AO_1L = \frac{LD}{O_1L} = tan\,\phi$ (3)

also, $\measuredangle AO_1D = \measuredangle AO_1L - \phi$ (4)

from equation (3) and equation (4), $\measuredangle AO_1D = tan\,\phi - \phi$ (5)

where ϕ: represents the tooth pressure angle, and the expression "$\tan\phi - \phi$" is called an involute function and its designed by "$inv\,\phi$".

i.e. $inv\,\phi = tan\,\phi - \phi$ (6)

and $\measuredangle AO_1D = inv\,\phi$ (7)

The involute function can be calculated when the angle ϕ is known (where ϕ is expressed in radians).

Also, in the same way, $inv\,\theta = tan\,\theta - \theta$ (8)

And $\measuredangle AO_1C = inv\,\theta$ (9)

then from (Figure 1) $\measuredangle AO_1E = \measuredangle AO_1D + \gamma$ (10)

$\measuredangle AO_1E = \measuredangle AO_1C + \alpha$ (11)

from equation (7) and equation (10), $\measuredangle AO_1E = inv\,\phi + \gamma$ (12)

also, from equation (9) and equation (11), it can be deduced that:

$\measuredangle AO_1E = inv\,\theta + \alpha$ (13)

by equating equation (12) with equation (13) yield:

$inv\,\theta + \alpha = inv\,\phi + \gamma$ (14)

also, from the same figure, $arc\,(DF) = R_p\,.\gamma$ (15)

and from the properties of involute, $arc\,(DF) = \frac{p_c}{4}$ (16)

$p_c = \pi\, m_o$ (17)

and $R_p = \frac{m_o.z}{2}$ (18)

where m_o: is module, p_c: is circular pitch, z: is number of teeth on gear, and R_p: is radius of pitch circle.

From equation (15), equation (16), equation (17) and equation (18), it can be concluded that:

$$\gamma = \frac{\pi}{2z} \quad (19)$$

By substituting equation (19) into equation (14) and re-arranged yield:

$$\alpha = \frac{\pi}{2z} + inv\,\phi + inv\,\theta \quad (20)$$

Referring to (Figure 1) again, the radius of base circle R_b can be found from the two following equations:

$$R_b = R\cos\theta \quad (21)$$

and $R_b = R_p \cos\phi$ (22)

Thus $\cos\theta = \frac{R_p}{R}\cos\phi$ (23)

i.e. $\theta = \cos^{-1}\left(\frac{R_p}{R}\cos\phi\right)$ (24)

Finally, by substituting equation (6), equation (8), and equation (24) into equation (20) yield:

$$\alpha = \frac{\pi}{2z} + (\tan\phi - \phi) - \tan\left[\cos^{-1}\left(\frac{R_p}{R}\cos\phi\right)\right] + \cos^{-1}\left(\frac{R_p}{R}\cos\phi\right) \quad (25)$$

Now, by using equation (25), the angle α for each point lies on the involute profile can be found, if its radius *R* is given.

3. Formulation of Tooth Loading Angle

The tooth loading angle β of the normal applied force F_n which is shown in (Figure 1), should be determined for each position of the force on the loaded side of the investigated tooth to achieve any stress analysis of a gear tooth shape. The general expression for the normal force position on the tooth profile may be established by locating point *G* on the tooth center line and determining the angle β between the force line of action and the horizontal line passing through point *G*.

Now, with referring to (Figure 1), let the normal force F_n is applied at a point *C* on the involute tooth profile with a radius *R* from the gear center O_1, therefore:

$$\beta = \theta - \alpha \quad (26)$$

By substituting equation (24) and equation (25) in equation (26) it can be found that:

$$\beta = \tan\left[\cos^{-1}\left(\frac{R_p}{R}\cos\phi\right)\right] - \frac{\pi}{2z} - (\tan\phi - \phi) \quad (27)$$

Equation (27) is used to find the loading angle β without profile correction for symmetric tooth and for asymmetric tooth by making ($\beta = \beta_l$), and ($\phi = \phi_l$), where β_l and ϕ_l represent the loading angle and pressure angle for the loaded side of tooth respectively.

As a result of tooth profile correction (which may be positive or negative depending on the value of correction factor *x*), new pressure angle, pitch radius and loading angle which are called corrected pressure angle ϕ_c, corrected pitch radius R_{pc} and corrected loading angle β_c respectively can be obtained for any correction factor value as shown below [2]:

By using equation (21) with $(R = R_{pc})$ where $(R_{pc} = R_p + x.m_o)$ and $(\theta = \phi_c)$, it can be determined that:

$$R_b = R_{pc} \cos \phi_c \quad (28)$$

Now, by solving equation (22) and equation (28), it can be deduced that:

$$\cos \phi_c = \left(\frac{R_p}{R_{pc}}\right) \cos \phi \quad (29)$$

Therefore; $$\phi_c = \cos^{-1}\left(\frac{R_p}{R_{pc}} \cos \phi\right) \quad (30)$$

From equation (27) with replacing R_p by R_{pc} and ϕ by ϕ_c, it can be found that:

$$\beta_c = \tan\left[\cos^{-1}\left(\frac{R_p + x.m_o}{R} \cos \phi_c\right)\right] - \frac{\pi}{2z} - (\tan \phi_c - \phi_c) \quad (31)$$

Equation (31) can be used to find the corrected loading angle β_c for symmetric tooth and for asymmetric tooth by making $(\beta_c = \beta_{cl})$, $(x = x_l)$ and $(\phi_c = \phi_{cl})$, where β_{cl}, x_l and ϕ_{cl}: represent the corrected loading angle, correction factor and corrected pressure angle for the loaded side of tooth respectively.

4. Determination of Tooth Critical Section Thickness

In gear design, the tooth critical section thickness t which is also termed as "weakest section thickness" is considered as a one of the most important geometrical properties of a gear tooth, because it gives insight to the tooth strength, also it used in the determination of the Lewis form factor and stress concentration factor of the tooth. In this work, two methods are adopted to determine the critical section thickness which can be specified by the section JV as shown in (Figure 2), the first method is a trial graphical method and the second method is a developed analytical method.

4.1 Trial Graphical Method

The trial graphical method which used to determine the tooth critical section thickness can be described as follows with all necessary steps [1]:

Firstly, there are two cases should be considered in this method:

a) Case (i): $(R_b - R_d) \geq r_f$ [i.e. when the tangency point *S* of the fillet circle lies on a radial line AO_1 of the tooth profile as shown in (Figure 3.a)].

By referring to (Figure 3.a), let point *Z* to be a reference point on the top land of the tooth and lies at the tooth center line and let *M* to be an intersection point of dedendum circle with tooth center line, therefore:

$$ZM = h_t = h_a + h_d \quad (32)$$

$$ZG = ZE + EG = (R_a - R) + EG \quad (33)$$

and $$EG = R\,[1 - \cos\alpha + \sin\alpha\,.\tan\beta] \quad (34)$$

where R_a & R_d: are the radii of addendum and dedendum circles respectively, h_a & h_d: are the addendum and dedendum heights respectively, h_t: is the total depth of tooth, and r_f: is the fillet radius of tooth.

By using a suitable scale in the construction of (Figure 2.b), the construction may be carried out according to the following steps:

1) Locate point *N* on the tooth center line at a vertical distance *ZN*, where from (Figure 3.a), it can be determined that:

$$ZN = ZM + MN \quad (35)$$

and $$MN = R_d\,[1 - \cos(\lambda + \psi)] \quad (36)$$

where, the angle λ can be found by putting $(R = R_b)$ in equation (25), therefore;

$$\lambda = \frac{\pi}{2z} + (\tan\phi - \phi) \quad (37)$$

also, from (Figure 3.a), $$\psi = \sin^{-1}\left(\frac{r_f}{r_f + R_d}\right) \quad (38)$$

2) Through a point *N*, draw a perpendicular line to the tooth center line and locate point *Q* on this line, then a horizontal distance *NQ* can be measured from point *N* as follows:

$$NQ = R_d\,\sin(\lambda + \psi) \quad (39)$$

3) Through a point *Q*, draw a line at an angle "$\lambda + \psi$" with the tooth center line as shown in (Figure 3.b), and set

($QO_2 = r_f$), where O_2 as a center and r_f as a radius, (i.e. by this step the fillet circle can be described).

4) By using a trial method, a line can be drawn tangent to the fillet circle such that $VW = WT$ (this condition is required to locate the critical section thickness t and the load height h from this section) [3].

b) Case (ii): $(R_b - R_d) < r_f$ [i.e. when the point *S* is the tangency point between the fillet circle and the tangent line of the tooth profile as shown in (Figure 4.a), this case may be occur when the number of teeth on a gear would be increased].

The same procedure of *case (i)* is used, but with replacing the angles λ and ψ by the angles $\bar{\lambda}$ and $\bar{\psi}$ respectively as shown in (Figure 4.b), where the angle $\bar{\lambda}$ can be found by substituting R_s instead of R in equation (25), with ($R_s = R_d + SP$) and ($SP = Q_v.r_f$), i.e. ($R_s = R_d + Q_v.r_f$), therefore:

$$\bar{\lambda} = \frac{\pi}{2z} + (\tan\phi - \phi) - \tan\left[\cos^{-1}\left(\frac{R_p}{R_d + Q_v.r_f}\cos\phi\right)\right] + \cos^{-1}\left(\frac{R_p}{R_d + Q_v.r_f}\cos\phi\right) \quad (40)$$

and the value of Q_v can be calculated from the following set of equations by using a trail graphical technique, [1]:

$$Q_v = \left[\frac{1-q}{1+q}\right]\left[\frac{cos(\bar{\lambda}+\bar{\psi}) + cos\,\bar{\theta} + sin\,\bar{\psi}}{1+ cos\,\bar{\psi}}\right] \quad (41)$$

where $q = \left(\frac{\bar{\theta}+\bar{\psi}}{2}\right)$ (42)

$$\bar{\theta} = cos^{-1}\left(\frac{R_b}{R_S}\right) \quad (43)$$

$$\bar{\psi} = 2\,tan^{-1}\sqrt{\frac{(r-r_3)(r-r_2)}{r(r-r_1)}} \quad (44)$$

$$r_1 = r_f \quad (45)$$

$$r_2 = R_S = R_d + Q_v.r_f \quad (46)$$

$$r_3 = R_d + r_f \quad (47)$$

and$r = \frac{1}{2}(r_1 + r_2 + r_3)$ (48)

4.2 Developed Analytical Method

According to the above trial graphical method, it's clear that, this method is difficult and required a long time to find the critical section thickness, besides that its result is approximate, because it depends on the manual measuring. Therefore; in this work, this trial graphical method has been avoided by establishing an analytical expression for the angle ξ as shown in (Figure 5), which can be solved easily and it gives a higher accuracy and requires shorter time comparing with the graphical method. The procedure of this developed method is shown below:

Referring to (Figure 5), it can be deduced that:

$$x_1 = r_f\,cos\,\xi + UW \quad (49)$$

and $UW = (y_1 + r_f\,cos\,\xi)\,tan\,\xi$ (50)

and from equation (49) and equation (50), it can be found that:

$$x_1 = r_f\,cos\,\xi + (y_1 + r_f\,cos\,\xi)\,tan\,\xi \quad (51)$$

From (Figure 5), it can be obtained that:

$$tan\,\xi = \frac{x_2 - r_f\,cos\,\xi}{y_2 + r_f\,sin\,\xi} \quad (52)$$

i.e. $$y_2 = \frac{x_2 - r_f \cos\xi}{\tan\xi} - r_f \sin\xi \quad (53)$$

and $$VT = \frac{y_2 + r_f \sin\xi}{\cos\xi} \quad (54)$$

According to the condition $VW = WT$, it can be concluded that [3]:

$$VT = 2WT \quad (55)$$

Therefore; from equation (54) and equation (55), it can be found that:

$$WT = \frac{y_2 + r_f \sin\xi}{2\cos\xi} \quad (56)$$

Also, from (Figure 5), $$WT = \frac{x_2 - x_1}{\sin\xi} \quad (57)$$

from equation (56) and equation (57), it can be obtained:

$$(y_2 + r_f \sin\xi)\tan\xi = 2(x_2 - x_1) \quad (58)$$

Now, by substituting for x_1 & y_2 from equation (51) and equation (53) respectively into equation (58), and simplifying, it can be obtained that:

$$x_2 \cos\xi - 2y_1 \sin\xi - r_f(1 + \sin^2\xi) = 0 \quad (59)$$

where from (Figure 5), $x_2 = NQ + r_f \sin(\lambda + \psi)$ (60)

$$NQ = R_d \sin(\lambda + \psi) \quad (61)$$

and from equation (60) and equation (61), it can be found that:

$$x_2 = (r_f + R_d)\sin(\lambda + \psi) \quad (62)$$

Also, from (Figure 5), $y_1 = R - EG - (r_f + R_d)\cos(\lambda + \psi)$ (63)

and by substituting for EG from equation (34) into equation (63), it can be determined that:

$$y_1 = R(\cos\alpha - \sin\alpha . \tan\beta) - (r_f + R_d)\cos(\lambda + \psi) \quad (64)$$

where the angles α and β are calculated from equation (25) and equation (27) respectively, and the angles λ and ψ are calculated from equation (37) and equation (38) respectively, for the case of $(R_b - R_d) \geq r_f$. While for the case of $(R_b - R_d) < r_f$ the angles λ and ψ are replaced by the angles $\bar{\lambda}$ and $\bar{\psi}$ respectively which are calculated from equation (40) and equation (44) respectively, but after finding the value of Q_v from equation (41).

In this work, an iterative technique based on a Secant Method which is considered as the most common method of hand computation has been adopted to find this value of Q_v. The formula of this method can be formulated as shown below [4]:

$$Q_{v_{i+1}} = Q_{v_i} - \frac{f(Q_{v_i}).(Q_{v_i} - Q_{v_{i-1}})}{f(Q_{v_i}) - f(Q_{v_{i-1}})} \tag{65}$$

where from equation (41), the general expression of $f(Q_v)$ can be written as follows:

$$f(Q_v) = Q_v - \left[\frac{1-q}{1+q}\right]\left[\frac{cos(\bar{\lambda} + \bar{\psi}) + cos\,\bar{\theta} + sin\,\bar{\psi}}{1+ cos\,\bar{\psi}}\right] \tag{66}$$

Now, by starting with two initial guesses, $Q_{v_{-1}}$ and Q_{v_0}, and by using equation (66) the corresponding values of $f(Q_{v_{-1}})$ and $f(Q_{v_0})$ can be found respectively, then by repeatedly applying of equation (65) until the convergence criterion $|Q_{v_{i+1}} - Q_{v_i}| < \varepsilon$ is reached, where (ε) being the error tolerance. Only the latest value of Q_v has to be stored as a final result (where at this value of Q_v, the value of $f(Q_v)$ would be converged to zero). After finding the value of Q_v, the values of SP, $\bar{\psi}$, $\bar{\theta}$, and R_S can be easily determined.

Also, in order to find the root of an angle ξ of the non-linear equation (59), another iterative technique based on a Newton- Raphson algorithm, has been adopted for this purpose. Newton- Raphson algorithm is the best-known method of finding roots because it is simple and fast, but this method is usable only in the problems where the derivative of the function can be readily computed. The Newton–Raphson formula can be formulated as follows [4]:

$$\xi_{i+1} = \xi_i - \frac{f(\xi_i)}{f'(\xi_i)} \tag{67}$$

where $$f(\xi_i) = x_2\,cos\,\xi - 2\,y_1\,sin\,\xi - r_f\,(1 + sin^2\,\xi) \tag{68}$$

and $$f'(\xi_i) = -x_2\,sin\,\xi - 2\,y_1\,cos\,\xi - 2\,r_f\,cos\,\xi\,sin\,\xi \tag{69}$$

The algorithm for the Newton–Raphson method is simple: it repeatedly applies equation (67), starting with an initial value ξ_0, until the convergence criterion $|\xi_{i+1} - \xi_i| < \varepsilon$ is reached, where ε: is the error tolerance. The main advantage of this technique, its rapid convergence to the final root, often three iterations are sufficient, also it can be reduced these iterations by using a closer starting guess of the root ξ, and from the previous knowledge of involute tooth profile construction, the working range of angle (ξ) may be about ($\pi/18 \rightarrow \pi/6$). Therefore; with some mathematical configurations, it can be found a direct solution rather than iterative solution, as shown below:

Firstly, in order to simplicity the solution, the value ($\pi/6$) will be taken as a starting guess of the root ξ, therefore;

$$\xi_0 = \frac{\pi}{6} \tag{70}$$

From equation (67), equation (68) and equation (69) with equation (70), it can be found that:

$$\xi_1 = \xi_0 - \frac{x_2 \cos \xi_0 - 2\, y_1 \sin \xi_0 - r_f\,(1 + \sin^2 \xi_0)}{-x_2 \sin \xi_0 - 2\, y_1 \cos \xi_0 - 2\, r_f \cos \xi_0 \sin \xi_0} \tag{71}$$

i.e. $$\xi_1 = \frac{\pi}{6} + \frac{x_2 \cos\frac{\pi}{6} - 2\, y_1 \sin\frac{\pi}{6} - r_f\,(1 + \sin^2 \frac{\pi}{6})}{x_2 \sin\frac{\pi}{6} + 2\, y_1 \cos\frac{\pi}{6} + 2\, r_f \cos\frac{\pi}{6} \sin\frac{\pi}{6}} \tag{72}$$

Therefore; $$\xi_1 = \frac{\pi}{6} + \frac{\frac{\sqrt{3}}{2} x_2 - y_1 - \frac{5}{4} r_f}{\frac{1}{2} x_2 + \sqrt{3}\, y_1 + \frac{\sqrt{3}}{2} r_f} \tag{73}$$

Also, from equation (67), it can be found that:

$$\xi_2 = \xi_1 + \frac{x_2 \cos \xi_1 - 2\, y_1 \sin \xi_1 - r_f\,(1 + \sin^2 \xi_1)}{x_2 \sin \xi_1 + 2\, y_1 \cos \xi_1 + 2\, r_f \cos \xi_1 \sin \xi_1} \tag{74}$$

The expression of ξ_2 in equation (74) represents the final expression of root ξ, with the error tolerance, $\varepsilon = |\xi_2 - \xi_1| < 0.00001$.

Finally, after finding the angle ξ with referring to (Figure 5), it can be found that:

$$t = 2\,(x_2 - r_f \cos \xi) \tag{75}$$

and $$h = y_1 + r_f \sin \xi \tag{76}$$

Now, by using equation (75) and equation (76), the tooth critical section thickness t and the height of load from the critical section h can be determined at any radial position *R* of the normal load F_n on the involute tooth profile and for any pressure angle ϕ, fillet radius r_f and correction factor *x* for loaded and unloaded tooth sides.

5. Determination of Lewis Form Factor

In gear stress analysis, Lewis form factor Y represents one of the most important geometrical factors by which the tooth bending stress equation is governed. Therefore; the determination of this factor is more significant in the gear design technology because it has a direct effect on the tooth bending failure. In this section, two cases have been considered in the determination of this factor, the first is for symmetric tooth and the other is developed for asymmetric tooth.

5.1 Symmetric Tooth

Wilfred Lewis (1892) was the first person to give a formula for the bending stress at the root fillet of symmetric involute gear tooth shape [5]. He considered many assumptions to simulate this formula, and these assumptions can be summarized as follows:

1) The gear tooth is considered as a cantilever beam.
2) The normal force is applied on the tip of tooth that its mean the tooth is carrying the whole load alone and it represents the worst loading case.
3) The normal force F_n has a uniform distributed along the face width of tooth and it consists of tangential component F_t and radial component F_r.
4) This analysis considers only the component of the tangential force acting on the tooth, and does not consider the effect of the radial force, which will cause a compressive stress over the cross section of tooth root fillet, because this compressive stress has a small effect on the tooth bending strength comparing with that of bending stress for tangential force.
5) The involute profile of tooth is assumed as a parabolic curve (which has a head at the intersection point of the normal force line action and the tooth center line and has two ends tangent the root fillets at tension and compression sides of the actual tooth).
6) The critical section of the tooth is specified by the distance between the two tangent points of the two ends of parabola curve for the assumed tooth with the curves of the root fillets for the actual tooth, which can be specified by the section JV as shown in (Figure 2).

Therefore; the bending stress formula at the critical section can be formulated as follows:

$$\sigma = \frac{M.y}{I} \tag{77}$$

with $M = F_t.h$, $y = t/2$, and $I = b.t^3/12$, where *M:* is the bending moment at the critical section of symmetric tooth, *I:* is the area moment of inertia of this section, and *b:* is the tooth face width.

i.e. $$\sigma = \frac{(F_t.h).t/2}{b.t^3/12} = \frac{6\,F_t.h}{b.t^2} \tag{78}$$

From similar triangles in (Figure 2), it can be shown that:

$$\frac{k}{t/2} = \frac{t/2}{h} \tag{79}$$

Therefore; $$h = \frac{t^2}{4k} \tag{80}$$

By substituting equation (80) into equation (78), it can be determined that:

$$\sigma = \frac{6\,F_t\,\frac{t^2}{4k}}{b.t^2} = \frac{3F_t}{2bk} = \frac{3F_t}{2bk} * \frac{m_o}{m_o} \tag{81}$$

Therefore; $$\sigma = \frac{F_t}{b.m_o.Y} \tag{82}$$

where $$Y = \frac{2k}{3m_o} \tag{83}$$

Equations (82) & (83) represent traditional Lewis bending stress and form factor equations of symmetric tooth respectively. The Lewis form factor Y is a dimensionless factor also, it's a function of tooth geometry only, and it can be obtained from standard tables which are previously constructed for this purpose, and they did not take into account the effects of tooth fillet radius and tooth profile correction.

Also, from equations (80) & (83), it can be found that:

$$Y = \frac{1}{6m_o}\left(\frac{t^2}{h}\right) \tag{84}$$

By using equation (84), another technique can be carried out to determine the Lewis form factor for symmetric tooth at any value of tooth fillet radius with and without profile correction. This technique can be achieved by calculating the values of t and h from the developed analytical equations (75) & (76) respectively and then substituting these values in equation (84).

5.2 Asymmetric Tooth

By referring to (Figure 6), it can be found that the bending stress formula for asymmetric tooth at the critical section is:

$$\sigma' = \frac{M'.y'}{I'} \tag{85}$$

Where $M' = F_t.h' - F_r.e$, $F_r = F_t.\tan\beta_l$, $e = (t_u - t_l)/2$, $y' = t'/2$, $I' = b.t'^3/12$ and $t' = (t_u + t_l)$,

M': is the bending moment at the critical section of asymmetric tooth and I'& t': are the area moment of inertia and the total thickness of this section respectively.

t_l & t_u: are the critical section thicknesses for the two tooth parts of loaded and unloaded sides respectively. It's calculated by applying equation (75) with a radial load position R and with tooth design parameters for loaded side ϕ_l, x_l, and r_{fl} and for unloaded side ϕ_u, x_u, and r_{fu} respectively, with taking into account that the resulted value of t from this equation must be divided by two.

h': is the load height from the tooth critical section with a radial distance R and its calculated by applying equation (76) with design parameters of loaded tooth side ϕ_l, x_l, and r_{fl}.

β_l: is calculated from equation (27) without profile correction (i.e. $x_l = 0$), and it's calculated from equation (31) with tooth profile correction and this correction may be positive or negative depending on the design value of x_l.

Therefore; $$\sigma' = \frac{[F_t.h' - F_t.\tan\beta_l.(t_u - t_l)/2].(t_u + t_l)/2}{\frac{b.(t_u + t_l)^3}{12}} \tag{86}$$

$$\sigma' = \frac{3F_t}{b}\left[\frac{2h' - \tan\beta_l.(t_u - t_l)}{(t_u + t_l)^2}\right] * \frac{m_o}{m_o} \tag{87}$$

Finally, $$\sigma' = \frac{F_t}{b.m_o.Y'} \tag{88}$$

where $$Y' = \frac{1}{3\,m_o}\left[\frac{(t_u + t_l)^2}{2h' - \tan\beta_l.(t_u - t_l)}\right] \quad (89)$$

Equation (88) represents the developed Lewis bending stress equation of the asymmetric tooth with and without profile correction and this equation is governed by the developed Lewis factor Y' of asymmetric tooth which is calculated from equation (89).

6. Verification of Results

In order to verify the results of present analytical method, two strategies have been adopted for this purpose. The first strategy is by comparing the results of this method with those of a trail graphical method for the two cases of $(R_b - R_d) \geq r_f$ and $(R_b - R_d) < r_f$ as shown in (Table 1) and (Table 2) respectively. The second strategy is by comparing the values of Lewis form factors for symmetric and asymmetric teeth calculated using the present analytical method with those of previous work of Ref.[6], for the loading case when the load acts at the tip of tooth with ($R = R_a$) as shown in (Table 3). According to these results, it's clear that the verification of this analytical method has been very successful.

7. Conclusions

In this paper, a developed analytical method based on a previous trial graphical method is achieved to find the solution of bending stress equation for symmetric and asymmetric gear teeth shapes with and without profile correction and for different design parameters of loaded and unloaded tooth sides. Also, by this work, the trial graphical method has been avoided by establishing a simple analytical expression which can be easily solved and it gives a higher accuracy and requires a shorter time.

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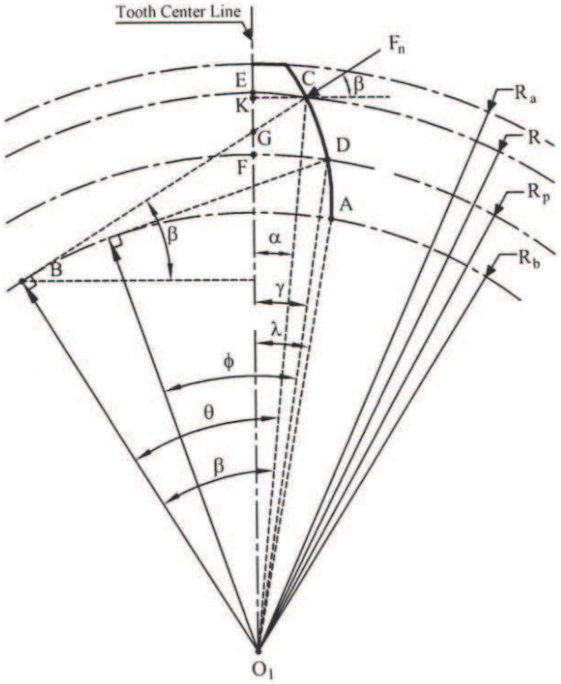

Figure 1. Involutmetry

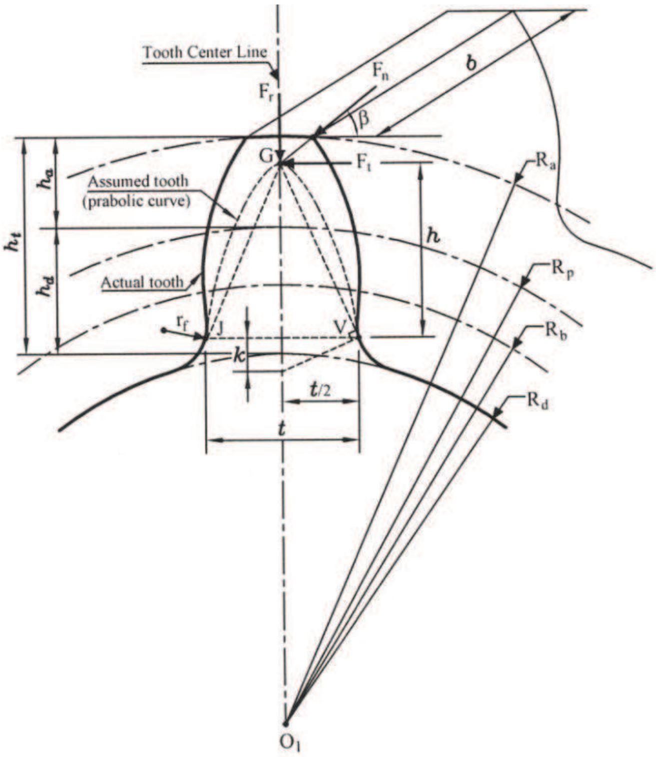

Figure 2. Critical section of symmetric tooth.

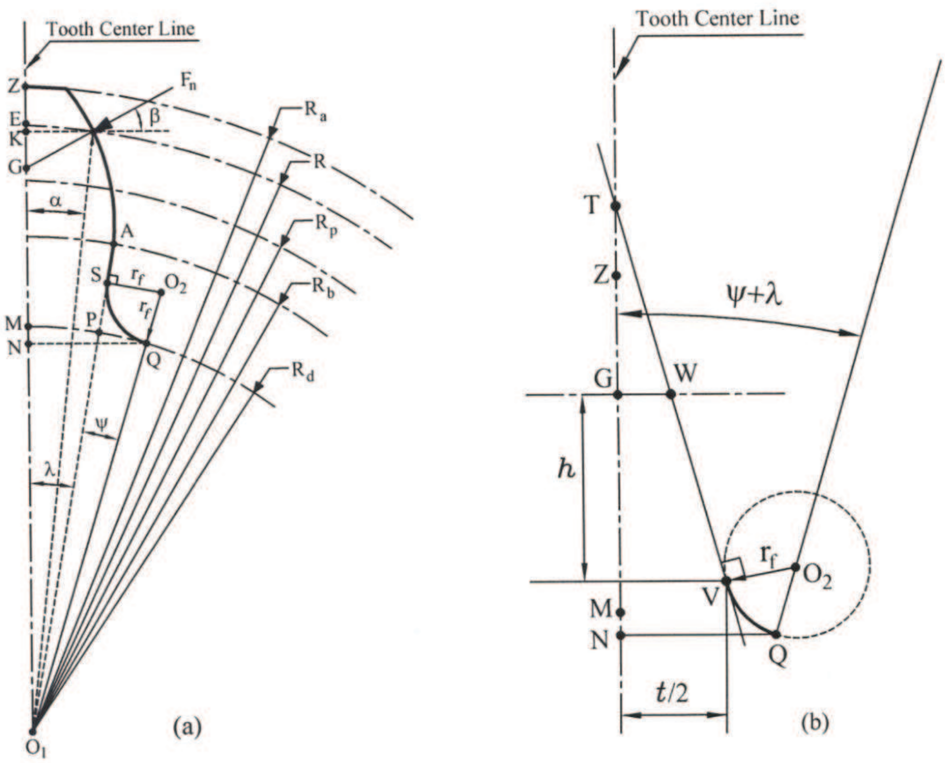

Figure 3. Describtion of graphical method to determine critical section thickness for the case $(R_b - R_d) \geqslant r_f$ [1].

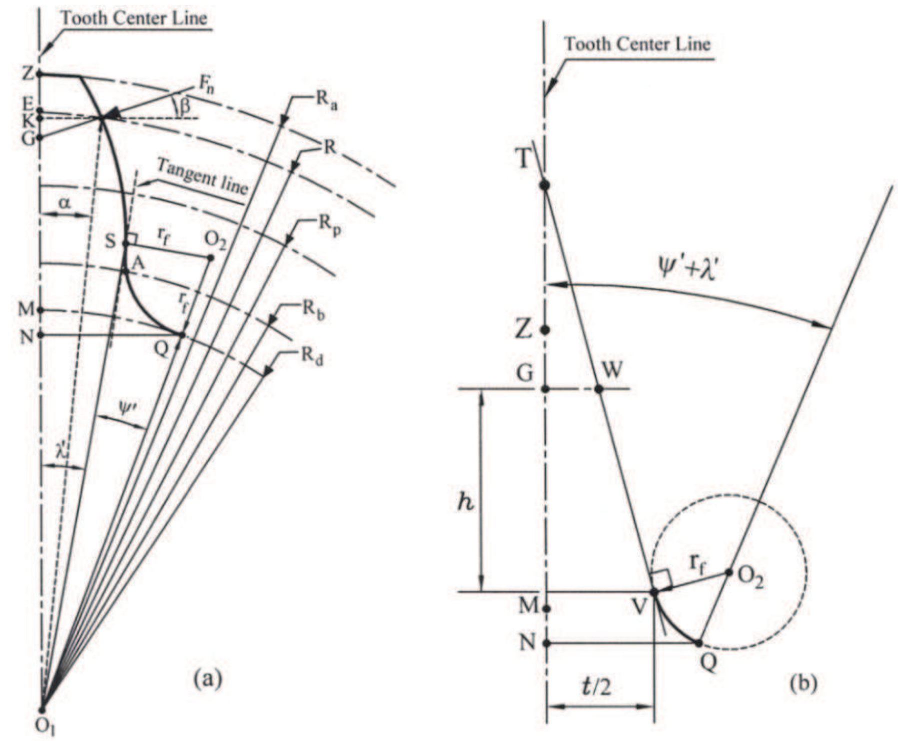

Figure 4. Describtion of graphical method to determine critical section thickness for the case $(R_b - R_d) < r_f$ [1].

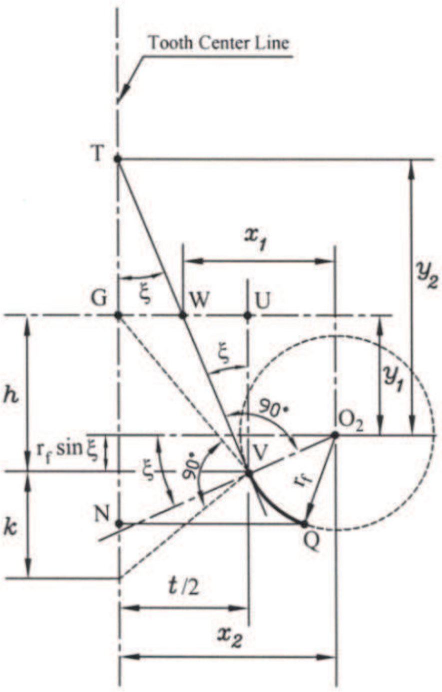

Figure 5. Describtion of developed method.

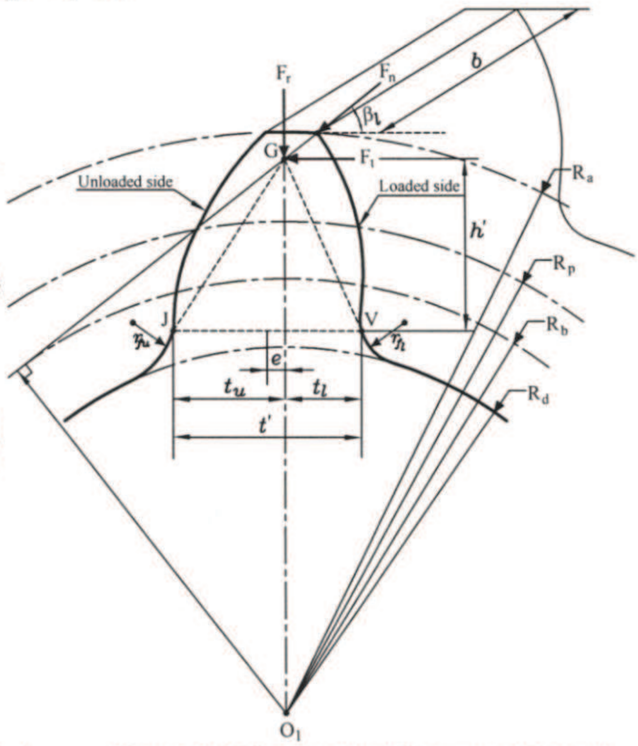

Figure 6. Critical section for asymmetric tooth.

Table 1. Comparison between the developed analytical method and the trial graphical method for the case $(R_b - R_d) \geq r_f$ with $\phi = 20^{\circ}, m_o =$ 5 mm, $z =$ 20 teeth, $r_f =$ 1.42 mm, and $R =$ 52 mm.

	t	h
Developed Analytical Method	8.68 mm	5.83 mm
Trial Graphical Method, Ref.[1]	8.56 mm	5.94 mm
Percentage Error	1.38 %	1.89 %

Table 2. Comparison between the developed analytical method and the trial graphical method for the case $(R_b - R_d) < r_f$ with $\phi = 20^{\circ}, m_o =$ 5 mm, $z =$ 40 teeth, $r_f =$1.22 mm, and $R = R_a$ = 105 mm.

	t	h	Y
Developed Analytical Method	10.11 mm	9.17 mm	0.372
Trial Graphical Method, Ref.[1]	10.27 mm	9.13 mm	0.371
Percentage Error	1.58 %	0.44 %	0.27 %

Table 3. Comparison of Lewis factor values for symmetric & asymmetric teeth calculated using the present work with those of previous work of Ref.[6], for the loading case when the load acts at the tip of tooth with $R = R_a$.

Number of Teeth on Gear	$z =$ 30			$z =$ 80		
Pressure Angles of Tooth Profiles, ϕ_u/ϕ_l	$20^{\circ}/20^{\circ}$	$20^{\circ}/25^{\circ}$	$20^{\circ}/30^{\circ}$	$20^{\circ}/20^{\circ}$	$20^{\circ}/25^{\circ}$	$20^{\circ}/30^{\circ}$
Lewis Factor of Present Work	0.359	0.407	0.464	0.430	0.478	0.539
Lewis Factor of Previous Work, Ref.[6]	0.358	0.421	0.463	0.429	0.469	0.515
Percentage Error	0.28 %	3.44 %	0.22 %	0.23 %	1.85 %	4.5 %