Some Fixed Point Theorems with G-Iteration in Banach Space with the Help of Hemi Contractive Mapping

Akanksha Sharma[1] Kalpana Saxena[2] Ramakant Bhardwaj[3]
1.Department of Basic Science, Oriental college of Technology, Bhopal (M.P.)
2.Department of Mathematics, Govt. Motilal Vigyan Mahavidyalya, Bhopal (M.P.)
3.Department of Mathematics, Technocrats Institute of Technology, Bhopal (M.P.)

Abstract : the purpose of this paper is to obtain fixed point theorems with hemi contractive mapping in Banach space.
Keywords: common fixed point, hemi contractive mapping, G-iteration
2000 mathematics subject classification: 47J25 , 47H10, 54H25

1. Introduction: the class of pseudo contractive maps with fixed points is a subclass of the hemi contraction. By using G-iteration process which is introduced by Das and Debata [3], we are studying convergence of common fixed point for continuous hemi contractive mapping in Banach space.

2. Preliminaries:
Definition 2.1 [2] (i) A mapping T with domain D(T) and range R(T) in a Banach space is called pseudocontrative mapping, if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that
$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2$
(ii) [4] A mapping T with domain D(T) and range R(T) in E is called a hemicontrative mapping if F (T)$\neq \emptyset$ and for all x $\in D(T)$ $x^* \in$ F (T) such that,
$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|x - Tx^*\|^2$
Theorem 2.2: Dhage [1] has proved a fixed point theorem satisfying the inequality:
$\|Tx - Ty\| \leq a\|x - Tx\| + \|y - Ty\| + (1 - 2a)\max\{\|x-y\|, \|x - Ty\| \, \|y - Ty\|, \frac{1}{2}\|x - Tx\| + \|y - Ty\|, \frac{1}{2}\|x - Ty\| + \|y - Tx\|$
Definition 2.3: Let X be a normed space and T: $X \to X$ is a self mapping then T is said to satisfy a Lipschitz condition with constant q if $\|Tx - Ty\| \leq q\|x - y\|$ for all $x, y \in X$. if $q < 1$ then T is called a contraction mapping.
Our main theorem is related to the concept of quasi-contraction, initiated by Ljubomir ciric [5] in .We define hemi ω -contraction in following manner:
Let X be a normed space then a self mapping T of X is called hemi ω -contraction contractive mapping if $\|Tx - Ty\|^2 \leq \omega \max\{\|x-y\|^2, \|x - Tx\|^2, \|y - Ty\|^2, \|x - Ty\|^2, \|y - Tx\|^2\}$ for all x, $y \in X$, where $0 < \omega < 1$.
extened the definition of hemi ω -contraction for a pair of mapping. we defining hemi ω -contraction pair of mapping as follows:
Definition 2.4: Let X be a normed space then T_1 and T_2 be two self mappings of X are called hemi ω -contractive pair of mapping If :
$\|T_1x - T_2y\|^2 \leq \omega \max\{\|x-y\|^2, \|x - T_1x\|^2, \|y - Ty\|^2, \|x - T_2y\|^2, \|y - T_1x\|^2\}$ for all x, $y \in X$, where $0 < \omega < 1$.

3. MAIN RESULTS:
Theorem 3.1: Let X be a closed subset of normed linear space N and let T: $X \to X$ be a hemi ω - mapping and $\{x_n\}$ be the sequence of G-iterates associated with T then G-iteration process is defined in the following manner:
Let $x_0, x_1 \in$ X and
$$x_{n+2} = (\mu_n - \lambda_n - s_n - a_n)x_{n+1} + (\lambda_n + s_n + a_n)Tx_{n+1} + (1 - \mu_n - \lambda_n + k_n + b_n)Tx_n + (\lambda_n - k_n - b_n)x_n$$
Where $\{\mu_n\}$,$\{\lambda_n\}$,$\{k_n\}$,$\{s_n\}$,$\{a_n\}$,and $\{b_n\}$ satisfying
(i) $\mu_0 = \lambda_0 = k_0 = 1$

(ii) $0 < \lambda_n < 1$, $0 < k_n < 1$, $0 < s_n < 1$, $0 < a_n < 1$, $0 < b_n < 1$ for $n > 0$
(iii) $\mu_n \geq \lambda_n$, $\mu_n \geq k_n$, $\mu_n \geq s_n$, $\mu_n \geq a_n$, $\mu_n \geq b_n$ for $n \geq 0$
(I v) $lim_{n\to\infty} \lambda_n = lim_{n\to\infty} s_n = lim_{n\to\infty} k_n = lim_{n\to\infty} a_n = \xi$ where $\xi > 0$
(v) $lim_{n\to\infty} \mu_n = 1$
(v I). $lim_{n\to\infty}. b_n = 0$
If $lim_{n\to\infty} x_n = z \in X$ then z is the fixed point of T

Proof: If $\{x_n\}$ converses on $z \in X$
i.e. $lim_{n\to\infty} x_n = z$.
We shall show that z is the fixed point of T.
Consider,

$$\|z - Tz\|^2 = \|z - x_{n+2} + x_{n+2} - Tz\|^2$$

$$\|z - Tz\|^2 \leq \|z - x_{n+2}\|^2 + \|x_{n+2} - Tz\|^2$$

$$\leq \|z - x_{n+2}\|^2 + \|(\mu_n - \lambda_n - s_n - a_n)x_{n+1} + (\lambda_n + s_n + a_n)Tx_{n+1} + (1 - \mu_n - \lambda_n + k_n + b_n)Tx_n + (\lambda_n - k_n - b_n)x_n - Tz\|^2$$

$$\leq \|z - x_{n+2}\|^2 + (\mu_n - \lambda_n - s_n - a_n)\|x_{n+1} - Tz\|^2 + (\lambda_n + s_n + a_n)\|Tx_{n+1} - Tz\|^2 + (1 - \mu_n - \lambda_n + k_n + b_n)\|Tx_n - Tz\|^2 + (\lambda_n - k_n - b_n)\|x_n - Tz\|^2$$

$$\leq \|z - x_{n+2}\|^2 + (\mu_n - \lambda_n - s_n - a_n)\|x_{n+1} - Tz\|^2 + (\lambda_n + s_n + a_n) \omega \max\{\|x_{n+1} - z\|^2, \|x_{n+1} - Tx_{n+1}\|^2, \|z - Tz\|^2, \|x_{n+1} - Tz\|^2, \|z - Tx_{n+1}\|^2\} + (1 - \mu_n - \lambda_n + k_n + b_n)\|Tx_n - Tz\|^2 + (\lambda_n - k_n - b_n)\|x_n - Tz\|^2 \quad (3.1.1)$$

We observed by the definition of G-iteration that

$$\|x_{n+1} - Tx_{n+1}\|^2 \leq \frac{1}{(\lambda_n + s_n + a_n)}\|x_{n+1} - x_{n+2}\|^2 + \frac{(\mu_n - 1)}{(\lambda_n + s_n + a_n)}\|x_{n+1} - Tx_n\|^2 + \frac{(\lambda_n - k_n - b_n)}{(\lambda_n + s_n + a_n)}\|x_n - Tx_n\|^2$$

And

$$\|z - Tx_{n+1}\|^2 \leq \|z - x_{n+1}\|^2 + \|x_{n+1} - Tx_{n+1}\|^2$$

$$\leq \|z - x_{n+1}\|^2 + \frac{1}{(\lambda_n + s_n + a_n)}\|x_{n+1} - x_{n+2}\|^2 + \frac{(\mu_n - 1)}{(\lambda_n + s_n + a_n)}\|x_{n+1} - Tx_n\|^2 + \frac{(\lambda_n - k_n - b_n)}{(\lambda_n + s_n + a_n)}\|x_n - Tx_n\|^2$$

Now putting above values in (3.1.1) then we have

$$\|z - Tz\|^2 \leq \|z - x_{n+2}\|^2 + (\mu_n - \lambda_n - s_n - a_n)\|x_{n+1} - Tz\|^2 + (\lambda_n + s_n + a_n)\omega \max\left\{\frac{1}{(\lambda_n + s_n + a_n)}\|x_{n+1} - x_{n+2}\|^2 + \frac{(\mu_n - 1)}{(\lambda_n + s_n + a_n)}\|x_{n+1} - Tx_n\|^2\right\} + (1 - \mu_n - \lambda_n + k_n + b_n)\|Tx_n - Tz\|^2 + (\lambda_n - k_n - b_n)\|x_n - Tz\|^2$$

Letting $n \to \infty$ then we have

$$\|z - Tz\|^2 \leq (1 - 3\xi + 3\xi\omega\|z - Tz\|^2$$

$\Rightarrow \|z - Tz\|^2 = 0$ Since $0 < \omega < 1$ and $\xi > 0$
Hence $z = Tz$ is a fixed point of T

Theorem 3.2: Let X be a closed convex subset of normed linear space N and let T_1 and T_2 be hemi ω -

contractive pair of self mapping of X and $\{x_n\}$ be the sequence of G-iterates associated with T_1 and T_2 then G-iteration process is defined in the following manner:

Let $x_0, x_1 \in$ X and

$$x_{2n+2} = (\mu_n - \lambda_n - s_n - a_n)x_{2n+1} + (\lambda_n + s_n + a_n)T_1x_{2n+1} + (1 - \mu_n - \lambda_n + k_n + b_n)T_2x_{2n} + (\lambda_n - k_n - b_n)x_{2n}$$

And

$$x_{2n+3} = (\mu_n - \lambda_n - s_n - a_n)x_{2n+2} + (\lambda_n + s_n + a_n)T_2x_{2n+2} + (1 - \mu_n - \lambda_n + k_n + b_n)T_1x_{2n+1} + (\lambda_n - k_n - b_n)x_{2n+1}$$

Where $\{\mu_n\},\{\lambda_n\},\{k_n\},\{s_n\},\{a_n\}$,and $\{b_n\}$ satisfying

(i) $\mu_n = \lambda_n = k_n = 1$ if n=0

(ii) $0 < \lambda_n < 1$, $0 < k_n < 1$, $0 < s_n < 1$, $0 < a_n < 1$, $0 < b_n < 1$ for $n > 0$

(iii) $\mu_n \geq \lambda_n$, $\mu_n \geq k_n$, $\mu_n \geq s_n$, $\mu_n \geq a_n$, $\mu_n \geq b_n$ for $n \geq 0$

(iv) $lim_{n\to\infty} \lambda_n = lim_{n\to\infty} s_n = lim_{n\to\infty} k_n = lim_{n\to\infty} a_n = \xi$ where $\xi > 0$

(v) $lim_{n\to\infty} \mu_n = 1$

(vi) $lim_{n\to\infty} b_n = 0$

If $lim_{n\to\infty} x_n = z \in$ X then z is the common fixed point of T_1 and T_2

Proof: if $\{x_n\}$ converges on $z \in$ X

i.e. $lim_{n\to\infty} x_n = z$

We shall show that z is the fixed point of T.

Consider,

$$\|z - T_1z\|^2 \leq \|z - x_{2n+3}\|^2 + \|x_{2n+3} - Tz\|^2$$

$$\leq \|z - x_{2n+3}\|^2 + \left\| \begin{matrix} (\mu_n - \lambda_n - s_n - a_n)x_{2n+2} + (\lambda_n + s_n + a_n)T_2x_{2n+2} + \\ (1 - \mu_n - \lambda_n + k_n + b_n)T_1x_{2n+1} + (\lambda_n - k_n - b_n)x_{2n+1} - T_1z \end{matrix} \right\|^2$$

$$\leq \|z - x_{2n+3}\|^2 + (\mu_n - \lambda_n - s_n - a_n)\|x_{2n+2} - T_1z\|^2 + (\lambda_n + s_n + a_n)$$

$$\omega \max\left\{\|x_{2n+2} - z\|^2, \|x_{2n+2} - T_1x_{2n+2}\|^2, \|z - T_1z\|^2, \|z - T_{2x_{2n+2}}\|^2, \|x_{2n+2} - T_1z\|^2\right\}$$

$$+(1 - \mu_n - \lambda_n + k_n + b_n)\|T_1x_{2n+1}x_n - T_1z\|^2 + (\lambda_n - k_n - b_n)\|x_{2n+1} - T_1z\|^2$$

(3.2.1)

We observe by the definition of G-iteration that

$$\|x_{2n+2} - T_2x_{2n+2}\|^2 \leq \frac{1}{(\lambda_n + s_n + a_n)}\|x_{2n+2} - x_{2n+3}\|^2 + \frac{(\mu_n - 1)}{(\lambda_n + s_n + a_n)}\|x_{2n+2} - T_1x_{2n+1}\|^2$$
$$+\frac{(\lambda_n - k_n - b_n)}{(\lambda_n + s_n + a_n)}\|x_{2n+1} - T_1x_{2n+1}\|^2$$

Now putting the values in (3.2.1) then we have

$$\|z - T_1z\|^2 \leq \|z - x_{2n+3}\|^2 + (\mu_n - \lambda_n - s_n - a_n)\|x_{2n+2} - T_1z\|^2 + (\lambda_n + s_n + a_n)\omega \max$$

$$\left\{ \begin{matrix} \|x_{2n+2} - z\|^2, \frac{1}{(\lambda_n + s_n + a_n)}\|x_{2n+2} - x_{2n+3}\|^2 + \frac{(\mu_n - 1)}{(\lambda_n + s_n + a_n)}\|x_{2n+2} - T_1x_{2n+1}\|^2 \\ + \frac{(\lambda_n - k_n - b_n)}{(\lambda_n + s_n + a_n)}\|x_{2n+1} - T_1x_{2n+1}\|^2 \|z - T_1z\|^2, \|z - x_{2n+2}\|^2, \frac{1}{(\lambda_n + s_n + a_n)}\|x_{2n+2} - x_{2n+3}\|^2 \\ + \frac{(\mu_n - 1)}{(\lambda_n + s_n + a_n)}\|x_{2n+2} - T_1x_{2n+1}\|^2 + \frac{(\lambda_n - k_n - b_n)}{(\lambda_n + s_n + a_n)}\|x_{2n+1} - T_1x_{2n+1}\|^2, \|x_{2n+2} - T_1z\|^2 \end{matrix} \right\}$$

$$+(1 - \mu_n - \lambda_n + k_n + b_n)\|T_1x_{2n+1}x_n - T_1z\|^2 + (\lambda_n - k_n - b_n)\|x_{2n+1} - T_1z\|^2$$

Letting $n \to \infty$ then we have

$$\|z - T_1z\|^2 \leq (1 - 3\xi + 3\xi\omega\|z - T_1z\|^2$$

$\Rightarrow \|z - T_1z\|^2 = 0$ Since $0 < \omega < 1$ and $\xi > 0$

Hence $z = T_1z$ is a fixed point of T_1.

Similarly we can show that

$$\|z - T_2z\|^2 \leq (1 - 3\xi + 3\xi\omega\|z - T_2z\|^2$$

$\Rightarrow \|z - T_2z\|^2 = 0$ Since $0 < \omega < 1$ and $\xi > 0$

Hence $z = T_2 z$ is a fixed point of T_2.
Finally we can say that z is a common fixed point of T_1 and T_2.
This completes the proof.

References :

[1]. Dhage,B.C. "On common fixed point maps in Banach spaces" Acta Ciencia Indica vol XII no.1(1987),1-5.
[2]. F. E. Browder and W. V. Petryshyn "Construction of fixed points of nonlinear mappings in Hilbert space", J. Math.Anal. Appl. 20(1967), 197-228.
[3]. G,Das and J.P. Debata "Fixed Points of Quasinonexpencive mappings" Indian J. pure appl.Math.,17(11) (1986)1263-1269.
[4]. Liu Qihou "The convergence theorems of the sequence of Ishikawa iterates for hemicontractive mappings" journal of mathematical analysis and applications 148, (1990), 55-62
[5].Ljubomir ciric "Quasi-contrations in Banach Spases"Publications De L'Institut Mathematique Nouvelle serie, tome 21(35),1977,pp .41-48.