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Method Of Weighted Residual For Periodic Boundary Value Problem: Galerkin's Method Approach

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Abstract
Method of weighted residual has been one of the foremost approximation solution to partial differential equation problems. In this paper a time dependent and boundary-valued strain model obtained from a PPC/$CaCO_3$ composite experimental data was analyzed using the method of weighted residual. The Galerkin's approach of this approximate method was applied to the model at a constant stress of 17.33 MPa. The discretization of the composite material was therefore, designed to take the form of a linear and one dimensional problem setting.
Keywords: Method of Weighted Residual, PPC/$CaCO_3$ Composite, Galerkin's Method, Time Dependent, Boundary-Value.

1. Introduction

Modern structural designs increasingly incorporate man made composite materials in applications that require components with special material properties which are unavailable from conventional metals or alloys. From the structural mechanics point of view, composites are typically used to improve either the stiffness-to-weight ratio or the strength-to-weight ratio of a structural member. The study of composite materials and structures can be undertaken from two different approaches; micromechanical and macromechanical [1]. The micromechanical approach which is the main interest of this paper, the properties of the constituent fiber and matrix materials and their interactions through stress-strain constitutive relations are analyzed in order to predict the overall behavior of the composite structural member. The composite material under analysis of its stress-strain property is a PP/$CaCO_3$ composite, with the employment of Galerkin's method of weighted residual of finite element analysis.

Prior to development of the Finite Element Method, there existed an approximation technique for solving differential equations called the Method of Weighted Residual (MWR) [2]. The history of the criteria to various choices of weighting functions of the approximate methods could be found in Table 1.

TABLE I: History of Approximate Methods

Date	Investigator	Method
1915	Galerkin [9]	Galerkin method
1921	Pohlhausen	Integral method
1923	Biezeno and Koch	Subdomain method
1928	Picone	Method of least square
1932	Kravchuk [10]	Method of moments
1933	Kantorovich [11]	Method of reduction to ordinary differential equations
1937	Frazer, Jones and Skan [12]	Collocation method
1938	Poritsky [13]	Method of reduction to ordinary differential equations
1940	Repman [14]	Convergence of Galerkin's method
1941	Bickley [15]	Collocation, Galerkin, least squares for initial-value problems
1942	Keldysh [16]	Convergence of Galerkin's method, steady state
1947	Yamada [17]	Method of moments
1949	Faedo	Convergence of Galerkin's method unsteady state
1953	Green [18]	Convergence of Galerkin's method unsteady state
1956	Crandall [4]	Unification as method of weighted residual

The method of weighted residuals is an engineer's tool for finding approximate solutions to the equations of change of distributed systems. The method is applicable but not limited to nonlinear and nonself adjoint problems, one of its most attractive features [3]. MWR includes many approximation methods that are being used currently. It provides a vantage point from which it is easy to see the unity of these methods as well as the relationships between them. Some of the best early treatments of MWR are those by Crandall [4], who coined the name method of weighted residuals, Ames [5] and Collatz [6], who called these methods error-distribution principles.

Suppose we have a linear differential operator 'D' acting on a function 'u' to produce a product 'p'.

$$D\big(u(x)\big) = p(x) \qquad (1)$$

We wish to approximate u by a function ũ, which is a linear combination of basis functions chosen from a linearly dependent set, that is;

$$u \cong \tilde{u} = \sum_{i=1}^{n} a_i \varphi_i \qquad (2)$$

Now, when substituted into the differential operator, D, the result of the operations is not, in general, p(x). Hence an error or residual will exist:

$$E(x) = R(x) = D(\tilde{u}(x)) - p(x) \neq 0 \qquad (3)$$

The notion in the MWR is to force the residual to zero in some average sense over the domain, thus;

$$\int_x R_{(x)} W_i dx = 0 \qquad i = 1,2,3 \ldots\ldots n \qquad (4)$$

Where the number of weight functions W_i is exactly equal the number of unknown constants a_i in ũ. The result is a set of n algebraic equations for the unknown constants a_i. There are (at least) five MWR sub-methods according to the choices for the W_i's, these include; (1) Collocation method (2) Sub-domain method (3) Least square method (4) Method of moments and (5) Galerkin method. In this paper, we are going to apply the Galerkin's method in analyzing our obtained boundary valued data. Boundary value problems with periodic conditions appear in a number of applications, particularly in the homogenization of composite materials with a periodic microstructure [7, 8]. When such problems are soved numerically, the periodicity condition is often imposed strongly, i.e., the solution values on periodic edges are required to match exactly.

2. Analysis of Data: MWR the Galerkin's Method

The strain model obtained from a PPC/$CaCO_3$ composite experimental data at a constant stress of 17.33 MPa is given as follows;

$$\mathcal{E} = 0.0117 + 0.0081t + 0.0050t^2 - 0.0031t^3 \qquad (5)$$

Under the boundary conditions; $\mathcal{E}(t, \sigma) = \mathcal{E}(0,17.33) = 0.0117$

$$\mathcal{E}(t_{max}, \sigma) = \mathcal{E}(3.0,17.33) = 0.081$$

Where $\mathcal{E}$ = Strain, t = Time in hours, σ = Stress.

Our initial step is to take the first and the second derivatives of the given model, thus;

$$\frac{d\mathcal{E}}{dt} = 0.0081 + 0.0050t - 0.0031t^2 \qquad (6)$$

$$\frac{d^2\mathcal{E}}{dt^2} = 0.0050 - 0.0031t \qquad (7)$$

Expressing the last term on the right hand-side of equation (7) as equal to zero at t = 0, hence;

$$\frac{d^2\mathcal{E}}{dt^2} = 0.0050 \qquad (8)$$

We treat this composite material under constant stress and variable strain as a linear and one dimensional problem, at fixed boundary conditions, as illustrated in Fig (1) below;

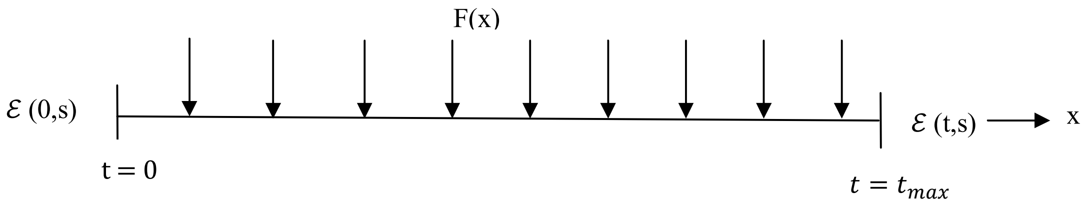

Fig (1)

Discretizing this composite into six equal elements and seven nodes, as in Fig (2) below, hence, at equal time intervals of 0.5hrs for each element, thus;

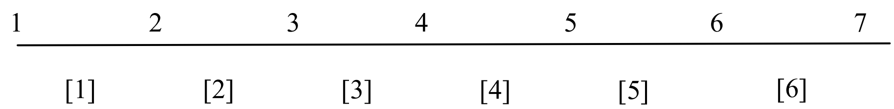

Fig (2)

Therefore this system equation (8) can be modeled with a one dimensional form of Poisson's equation, which is expressed as,

$$\frac{d^2\mathcal{E}}{dt^2} = -f(x) \qquad (9)$$

From equation (8) f(x) = 0.0050 and equation (9) can be expressed as;

$$\frac{d^2\mathcal{E}}{dt^2} + f(x) = 0 \qquad (10)$$

Therefore,

$$\frac{d^2\mathcal{E}}{dt^2} + 0.0050 = 0 \qquad (11)$$

Fig (3) shows an individual element and Fig (4) shows the approximation function (element equation) used in expressing the strain distribution along the element, and it's related as;

$$\bar{\mathcal{E}} = N_1\mathcal{E}_1 + N_2\mathcal{E}_2 \qquad (12)$$

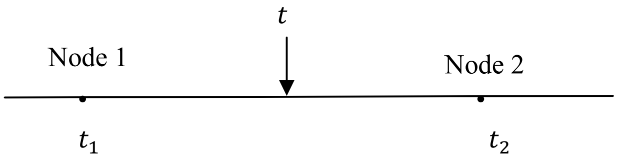

Fig (3)

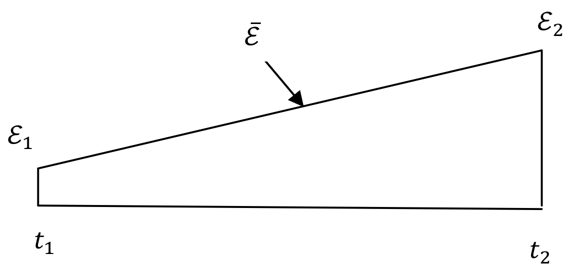

Fig (4)

Where N_1 and N_2 are linear interpolation functions expressed as, thus;

$$N_1 = \frac{t_2 - t}{t_2 - t_1} \qquad (13)$$

$$N_2 = \frac{t - t_1}{t_2 - t_1} \qquad (14)$$

Equation (12) can be substituted into equation (11) but, as equation (11) is not the exact solution, the hand-side of the resulting equation will not be equal to zero, but will be equal to a residual "R", thus;

$$R = \frac{d^2\mathcal{E}}{dt^2} + 0.0050 \qquad (15)$$

The method of weighted residual (MWR) consists of finding a minimum for the residual "R" according to the general formula, below;

$$\int RW_i \, dD = 0 \qquad i = 1,2,3 \ldots \ldots \ldots . m \qquad (16)$$

Where D = the solution domain

W_i = linear independent weighting functions

For the W_i the interpolation functions N_i can be employed to substitute it as the weighting functions. When these are substituted into equation (16), the result is referred to as the Galerkin's approach, thus;

$$\int_D RN_i \, dD = 0 \qquad i = 1,2,3 \ldots \ldots \ldots m \qquad (17)$$

Therefore,

$$\int_{t_1}^{t_2} \left[\frac{d^2 \bar{\mathcal{E}}}{dt^2} + f(x) \right] N_i dt \qquad i = 1,2,3 \ldots \ldots \ldots m \qquad (18)$$

Re-expressed as;

$$\int_{t_1}^{t_2} \frac{d^2 \bar{\mathcal{E}}}{dt^2} N_i(t) dt + \int_{t_1}^{t_2} f(x) N_i(t) dt \qquad i = 1,2,3 \ldots \ldots \ldots m \qquad (19)$$

Therefore,

$$\int_{t_1}^{t_2} \frac{d^2 \bar{\mathcal{E}}}{dt^2} N_i(t) dt = -\int_{t_1}^{t_2} 0.0050 N_i(t) dt \qquad i = 1,2,3 \ldots \ldots \ldots m \qquad (20)$$

Applying integration by parts, first on the left hand-side of the above equation (20), hence making our choices of terms;

$$U = N_i(t) \qquad \therefore du = dN_i$$

$$dv = \frac{d^2 \bar{\mathcal{E}}}{dt^2} \qquad \therefore V = \frac{d\bar{\mathcal{E}}}{dt}$$

Therefore,

$$\int_{t_1}^{t_2} \frac{d^2 \bar{\mathcal{E}}}{dt^2} N_i(t) dt - \left| N_i(t) \frac{d\bar{\mathcal{E}}}{dt} \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d\bar{\mathcal{E}}}{dt} \frac{dN_i}{dt} dt \qquad i - 1,2,3 \ldots . . m \qquad (21)$$

The first term on the right hand-side equation (21) can evaluated by taking $i = 1$ and it becomes;

$$\left| N_1(t)\frac{d\bar{\mathcal{E}}}{dt}\right|_{t_1}^{t_2} = N_1(t_2)\frac{d\bar{\mathcal{E}}(t_2)}{dt} - N_1(t_1)\frac{d\bar{\mathcal{E}}(t_1)}{dt} \qquad (22)$$

It has been established that $N_1(t_2) = 0$ *and* $N_1(t_1) = 1$, hence;

$$\left| N_1(t)\frac{d\bar{\mathcal{E}}}{dt}\right|_{t_1}^{t_2} = -1 \times \frac{d\bar{\mathcal{E}}(t_1)}{dt} = -\frac{d\bar{\mathcal{E}}(t_1)}{dt} \qquad (23)$$

Likewise taking $i = 2$ *as* $N_2(t_2) = 1$ *and* $N_2(t_1) = 0$

Therefore,

$$\left| N_1(t)\frac{d\bar{\mathcal{E}}}{dt}\right|_{t_1}^{t_2} = 1 \times \frac{d\bar{\mathcal{E}}(t_2)}{dt} \qquad (24)$$

Substituting equations (21) through (24) back into equation (20), we obtain that;

At $i = 1$

$$-\frac{d\bar{\mathcal{E}}(t_1)}{dt} - \int_{t_1}^{t_2}\frac{d\bar{\mathcal{E}}}{dt}\frac{dN_1}{dt}dt = -\int_{t_1}^{t_2} 0.0050 N_1(t)dt \qquad (25)$$

And,

$$\int_{t_1}^{t_2}\frac{d\bar{\mathcal{E}}}{dt}\frac{dN_1}{dt}dt = -\frac{d\bar{\mathcal{E}}(t_1)}{dt} + \int_{t_1}^{t_2} 0.0050 N_1(t)dt \qquad (26)$$

At $i = 2$

$$\frac{d\bar{\mathcal{E}}(t_2)}{dt} - \int_{t_1}^{t_2}\frac{d\bar{\mathcal{E}}}{dt}\frac{dN_2}{dt}dt = -\int_{t_1}^{t_2} 0.0050 N_2(t)dt \qquad (27)$$

And,

$$\int_{t_1}^{t_2}\frac{d\bar{\mathcal{E}}}{dt}\frac{dN_2}{dt}dt = \frac{d\bar{\mathcal{E}}(t_2)}{dt} + \int_{t_1}^{t_2} 0.0050 N_2(t)dt \qquad (28)$$

The left hand-side of equations (26) and (28) governs the element's strain distribution, and in terms of the finite element method, it is the element property matrix.
Recalling from equations (13) and (14),

$$\frac{dN_1}{dt} = -\frac{1}{t_2 - t_1} \quad and \quad \frac{dN_2}{dt} = \frac{1}{t_2 - t_1} \tag{29}$$

Also from equation (12)

$$\frac{d\bar{\mathcal{E}}}{dt} = \frac{1}{t_2 - t_1}(\ \mathcal{E}_1 + \mathcal{E}_2) \tag{30}$$

Substituting equations (29) and (30) into the left hand-side terms of equations (26) and (28);

At $i = 1$

$$\int_{t_1}^{t_2} \frac{d\bar{\mathcal{E}}}{dt}\frac{dN_1}{dt}dt = \int_{t_1}^{t_2} \frac{\mathcal{E}_1 - \mathcal{E}_2}{(t_2 - t_1)^2} = \frac{1}{t_2 - t_1}(\mathcal{E}_1 - \mathcal{E}_2) \tag{31}$$

At $i = 2$

$$\int_{t_1}^{t_2} \frac{d\bar{\mathcal{E}}}{dt}\frac{dN_2}{dt}dt = \int_{t_1}^{t_2} \frac{-\mathcal{E}_1 + \mathcal{E}_2}{(t_2 - t_1)^2} = \frac{1}{t_2 - t_1}(-\mathcal{E}_1 + \mathcal{E}_2) \tag{32}$$

Combining equations (31) and (32) and expressing the out come in matrix form, thus;

$$\frac{1}{t_2 - t_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{bmatrix}$$

We can now substitute the above result into equations (26) and (28) and expressing the result in matrix form to give us the final version of element equations starting with element 1, thus;

$$\frac{1}{t_2 - t_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{bmatrix} = \begin{bmatrix} \frac{-d\mathcal{E}(t_1)}{dt} \\ \frac{d\mathcal{E}(t_2)}{dt} \end{bmatrix} + \begin{bmatrix} \int_{t_1}^{t_2} 0.0050N_1(t)dt \\ \int_{t_1}^{t_2} 0.0050N_2(t)dt \end{bmatrix} \tag{33}$$

Assigning values to equation (33), as $t_1 = 0$ and $t_2 = 30mins$, creating our system topology,

TABLE 2: The System Topology for the Finite Element Segmentation Scheme from Fig (2)

Element	**Node Numbers**	
	Local	**Global**
1	1 2	1 2
2	1 2	2 3
3	1 2	3 4
4	1 2	4 5
5	1 2	5 6
6	1 2	6 7

And starting our solving with the second term of the right hand-side of equation (33), thus;

At $i = 1$

$$\int_{t_1}^{t_2} 0.0050 N_1(t) dt = \int_0^{0.5} 0.0050 \frac{0.5 - t}{0.5} dt$$

$$\xrightarrow{yields} 0.0050 \int_0^{0.5} \frac{0.5}{0.5} dt - \int_0^{0.5} \frac{t}{0.5} dt = 0.0050 \left[t - \frac{t^2}{1.0} \right]_0^{05}$$

$$= 0.0050\left[0.5 - \frac{0.5^2}{1.0}\right] - 0.0050\left[0 - \frac{0^2}{1.0}\right]$$

$$= \mathbf{0.00125}$$

And at $i = 2$

$$\int_{t_1}^{t_2} 0.0050 N_2(t)dt = \int_0^{0.5} 0.0050\frac{t-0}{0.5}dt$$

$$\xrightarrow{yields} 0.0050\int_0^{0.5}\frac{t}{0.5}dt - \int_0^{0.5} 0\,dt = 0.0050\left[\frac{t^2}{1.0}\right]_0^{0.5}$$

$$= 0.0050\left[\frac{0.5^2}{1.0}\right] - 0.0050[0] = \mathbf{0.00125}$$

Then solving the left hand-side term of equation (33);

$$\frac{1}{0.5-0}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{bmatrix} - \begin{bmatrix} 2\mathcal{E}_1 - 2\mathcal{E}_2 \\ -2\mathcal{E}_1 + 2\mathcal{E}_2 \end{bmatrix}$$

Combining the above results and substituting into the element equation (33), gives the following;

$$2\mathcal{E}_1 - 2\mathcal{E}_2 = -\frac{d\mathcal{E}}{dt}(t_1) + 0.00125 \qquad (34)$$

And,

$$-2\mathcal{E}_1 + 2\mathcal{E}_2 = \frac{d\mathcal{E}}{dt}(t_2) + 0.00125 \qquad (35)$$

Combining equations (34) and (35) gives us element 1 stiffness matrix which forms the element matrix for other elements, and since we have 6 elements and 7 nodes, we will have a system of 7 equations like the one below;

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{bmatrix} = \begin{bmatrix} -\frac{d\mathcal{E}}{dt}(t_1) + 0.00125 \\ \frac{d\mathcal{E}}{dt}(t_2) + 0.00125 \end{bmatrix}$$

For the assembly of all elements equations, we form a Zero 7 x 7 initialized stiffness matrix, thus;

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Assembly of the elements equations based on global coordinate points, hence Element 1 assembly;

$$\begin{bmatrix} 2.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -2 & 2.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{d\varepsilon}{dt}(t_1) + 0.00125 \\ \frac{d\varepsilon}{dt}(t_2) + 0.00125 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

After assembly of Element 2, we have;

$$\begin{bmatrix} 2.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -2 & 2+2 & -2 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -2 & 2.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{d\varepsilon}{dt}(t_1) + 0.00125 \\ 0.00125 + 0.00125 \\ \frac{d\varepsilon}{dt}(t_3) + 0.00125 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

After assembly of Element 3, we have;

$$\begin{bmatrix} 2.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -2 & 4.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -2 & 2+2 & -2 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -2 & 2.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{d\varepsilon}{dt}(t_1) + 0.00125 \\ 0.0025 \\ 0.00125 + 0.00125 \\ \frac{d\varepsilon}{dt}(t_4) + 0.00125 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

After assembly of Element 4, we have;

$$\begin{bmatrix} 2.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -2 & 4.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -2 & 4.0 & -2 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -2 & 2+2 & -2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -2 & 2.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \\ \mathcal{E}_4 \\ \mathcal{E}_5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{d\mathcal{E}}{dt}(t_1) + 0.00125 \\ 0.0025 \\ 0.0025 \\ 0.00125 + 0.00125 \\ \frac{d\mathcal{E}}{dt}(t_5) + 0.00125 \\ 0 \\ 0 \end{bmatrix}$$

After assembly of Element 5, we have;

$$\begin{bmatrix} 2.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -2 & 4.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -2 & 4.0 & -2 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -2 & 4.0 & -2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -2 & 2+2 & -2 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -2 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \\ \mathcal{E}_4 \\ \mathcal{E}_5 \\ \mathcal{E}_6 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{d\mathcal{E}}{dt}(t_1) + 0.00125 \\ 0.0025 \\ 0.0025 \\ 0.0025 \\ 0.00125 + 0.00125 \\ \frac{d\mathcal{E}}{dt}(t_6) + 0.00125 \\ 0 \end{bmatrix}$$

After assembly of Element 6, we have;

$$\begin{bmatrix} 2.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -2 & 4.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -2 & 4.0 & -2 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -2 & 4.0 & -2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -2 & 4.0 & -2 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -2 & 2+2 & -2 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -2 & 2.0 \end{bmatrix} \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \\ \mathcal{E}_4 \\ \mathcal{E}_5 \\ \mathcal{E}_6 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{d\mathcal{E}}{dt}(t_1) + 0.00125 \\ 0.0025 \\ 0.0025 \\ 0.0025 \\ 0.00125 + 0.00125 \\ \frac{d\mathcal{E}}{dt}(t_6) + 0.00125 \\ 0 \end{bmatrix}$$

Element 6 stiffness matrix can be re-written as below, i.e completed assembly at node 7;

$$\begin{bmatrix} 2.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -2 & 4.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -2 & 4.0 & -2 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -2 & 4.0 & -2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -2 & 4.0 & -2 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -2 & 4.0 & -2 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -2 & 2.0 \end{bmatrix} \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \\ \mathcal{E}_4 \\ \mathcal{E}_5 \\ \mathcal{E}_6 \\ \mathcal{E}_7 \end{bmatrix} = \begin{bmatrix} -\frac{d\mathcal{E}}{dt}(t_1) + 0.00125 \\ 0.0025 \\ 0.0025 \\ 0.0025 \\ 0.0025 \\ 0.0025 \\ \frac{d\mathcal{E}}{dt}(t_7) + 0.00125 \end{bmatrix}$$

The equation depicts the assembly matrix of the 6 elements at 7 nodes. Applying the initial boundary conditions, where $\mathcal{E}_1$ *and* $\mathcal{E}_7$ were given as; $\mathcal{E}_1 = 0.0117$ *and* $\mathcal{E}_7 = 0.081$and computing into the assembly matrix, in order to obtain the values of $\mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, \mathcal{E}_6$.The matrix below is obtained;

$$\begin{bmatrix} 1.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -2 & 4.0 & -2 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -2 & 4.0 & -2 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -2 & 4.0 & -2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -2 & 4.0 & -2 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -2 & 4.0 & -2 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -2 & -1 \end{bmatrix} \begin{bmatrix} \frac{d\varepsilon}{dt}(t_1) \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \\ \frac{d\varepsilon}{dt}(t_7) \end{bmatrix} = \begin{bmatrix} -0.0222 \\ 0.0259 \\ 0.0025 \\ 0.0025 \\ 0.0025 \\ 0.1645 \\ -0.1608 \end{bmatrix}$$

Therefore using Mat lab software and algorithm, the above 7×7 matrix was solved and the following result was obtained;

$$\frac{d\varepsilon}{dt}(t_1) = \mathbf{0.0325}, \quad \varepsilon_2 = \mathbf{0.0274}, \quad \varepsilon_3 = \mathbf{0.0418}, \quad \varepsilon_4 = \mathbf{0.0549}, \quad \varepsilon_5 = \mathbf{0.0668}, \quad \varepsilon_6 = \mathbf{0.0775},$$

$$\frac{d\varepsilon}{dt}(t_7) = \mathbf{0.0059}$$

Note: Above matrix could also be solved traditionally (numerically) using LU decomposition or Gaussian elimination methods.

3. Concluding Remarks

At the assumption of a linear, one dimensional periodic boundary problem, the composite was discretized into six(6) elements of 7 nodes. Through the application of the Galerkin's method of weighted residual, the six elements' stiffness matrices were assembled and the approximate numerical solution found.

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