# An Explicit Positivity Preserving Scheme: Application to Biological Model

Mohammad Mehdizadeh Khalsaraei* Leila Davari Algoo, Reza Shokri Jahandizi
Department of Mathematics, Faculty of Science, University of Maragheh, Maragheh, Iran

**Abstract**
This paper deals with the construction of nonstandard finite difference method (NFSD) for nonlinear initial value problems modeled by a system of nonlinear ordinary differential equations. The proposed scheme preserves the positivity property as well as the requirement of conservation law and boundedness. In order to illustrate the accuracy of the new scheme, the numerical results compared with the standard ones.
**Keywords:** Positivity, Boundedness, Nonstandard finite difference.

## 1. Introduction

The modeling biological systems are commonly based on systems of nonlinear ordinary differential equations. Mathematical models and their simulations are important to understand qualitatively and quantitatively these systems. The aim of this paper is to investigate numerically the reliability and convenience of the nonstandard finite differences (NSFDs), applied to MSEIR models governed by the following five-dimensional system of nonlinear ordinary differential equations (Anguelov, et al., 2014):

$$
\begin{aligned}
\frac{dM}{dt} &= b(N-S)-(\delta+\mathrm{d})\mathrm{M},\\
\frac{dS}{dt} &= bS+\delta\mathrm{M}-\beta\mathrm{SI/\,N}-\mathrm{dS},\\
\frac{dE}{dt} &= \beta\mathrm{SI/\,N}-(\varepsilon+\mathrm{d})\mathrm{E},\\
\frac{dI}{dt} &= \varepsilon E-(\gamma+d)I,\\
\frac{dR}{dt} &= \gamma I-dR,\\
\frac{dN}{dt} &= (b-d)N,
\end{aligned}
\qquad (1)
$$

where $M$ , $S$ , $E$ , $I$ and $R$ are infants with passive immunity, susceptibles, exposed individuals, infectives and recovered individuals respectively. According to Anguelov, et al., (2014) the parameters $d$ , $q$ , $\beta$ , $\delta$ , $\varepsilon$ and $\gamma$ are positive numbers. In what follows, it is convenient to consider as dependent variables the fractions (divided by $N$ ) of the sub-populations in the compartments: $m=M/N$ , $s=S/N$ , $e=E/N$ , $i=I/N$ and $r=R/N$ . This reduces to the following equivalent system for the MSEIR:

$$
\begin{aligned}
\frac{dm}{dt} &= (d+q)(e+i+r)-\delta m,\\
\frac{ds}{dt} &= \delta m-\beta si,\\
\frac{de}{dt} &= \beta si-(\varepsilon+d+q)e,\\
\frac{di}{dt} &= \varepsilon e-(\gamma+d+q)i,\\
\frac{dr}{dt} &= \gamma i-(d+q)r.
\end{aligned}
\qquad (2)
$$

A major difficulty in the study of the ordinary differential equations (ODEs) systems is, in general, the lack of exact analytical solution or can not be solved by a straight forward formula. One way to proceed is to use numerical integration techniques to obtain useful information on the possible solution behaviors. A popular and important is one based on the use of finite difference to construct discrete models of the ODEs such as Euler, Runge- Kutta, or Adams methods (Hairer & Wanner, 1996, Lambert, 1991). But one of the shortcomings of the standard finite difference method is that the qualitative properties of the exact solution such as positivity usually are not transferred to the numerical solution. One way of avoiding this disadvantage is to employ finite difference schemes that are nonstandard in the sense of Mickens' definition (Mickens. 2007).

 

The present work which is motivated by many successful papers on the matter (Anguelov, et al., 2014, Herbert & Hethcote, 2000), introduces a generic family of finite-difference schemes to approximate dynamically consistent solution for (2) with respect to positivity.

The rest of the paper is organized as follows: In Section 2, we present preliminaries. We propose the new method and investigate the positivity and conservation law in Section 3. In Section 4, we apply the new method on (2) and compared the obtained results with standard explicit Euler scheme and second-order Runge - Kutta. Finally, we end the paper with some conclusions in Section 5.

## 2. Preliminaries

We now give a brief summary of the NSFDs for ODEs (Mickens (2000), Mehdizadeh Khalsaraei (2010), Mehdizadeh Khalsaraei & Khodadoosti (2014)). Let us consider the systems of ODEs, which can be written in the form

$$\frac{d}{dt}u(t) = F(u(t)), \quad (t \geq 0), \qquad u(0) = u_0 \tag{3}$$

where $u$ may be a single function or a vector of functions of length $k$ mapping $[t_0, T) \to \mathrm{C}^k$ and the corresponding $F$ a single function or a vector of functions of length $k$ mapping $([t_0, T) \to \mathrm{C}^k) \to ([t_0, T) \to \mathrm{C}^k)$. Discretization of the continuous differential equation, or beginning instead with a difference equation, we define $t_n = t_0 + n\Delta t$, where $\Delta t$ is a positive step size, and say that the discretized version of the function $u$ at time $t_n$ is

$$u_n \approx u(t_n). \tag{4}$$

Then the discretized version of Eq. (3) becomes

$$\mathrm{D}_{\Delta t} u_n = \mathrm{F}_n(F, u_n), \tag{5}$$

where $\mathrm{D}_{\Delta t} u_n$ represents the discretized version of $\frac{d}{dt}u(t)$ and $\mathrm{F}_n(F, u_n)$, approximates $F(u(t_n))$ at time $t_n$.

For the construction of the new schemes we will use the rules $1-2$, coming below of the non-standard modeling as in Mickens (Mickens, 2000, Mickens, 2003).

Rule 1:

The denominator function for the discrete derivatives must be expressed in terms of more complicated function $\varphi$ of the step-sizes than those conventionally used. This rule allows the introduction of complex analytic function of $\Delta t$ in the denominator with the condition that

$$\varphi(\Delta \mathrm{t}) = \Delta t + O(\Delta \mathrm{t}^2) \quad \text{as} \quad 0 < \Delta t \to 0. \tag{6}$$

Rule 2:

The non-linear terms must in general be modeled (approximated) non-locally on the computational grid or lattice in many different ways, for instance, the non-linear terms $u$, $u^2$ and $u^3$ can be modeled as follows as in Anguelov and Lubuma, (Anguelov & Lubuma, 2001)

$$\begin{aligned} u &\approx \frac{2u_{n+1} + u_n}{3}, \\ \mathrm{u} &\approx 3\mathrm{u}_{n+1} - 2u_n, \\ u^2 &\approx au_n^2 + (1-a)u_n u_{n+1}, \qquad a \in R, \\ u^3 &\approx \frac{1}{2}(u_n^3 + u_{n+1}^3). \end{aligned} \tag{7}$$

As mentioned above the preservation of the qualitative properties of the considered differential equation is of great interest in finite difference methods for solving the differential equations. The major consequence of this result is that such schemes do not allow numerical instabilities to occur.

## 3. Scheme Construction

Making use of the Rules 2 for three first equations in (2) and by the renormalization of denominator of the discrete derivatives it is now desired to find an accurate explicit nonstandard scheme which is positivity preserving which can be written as:

$$\frac{m^{k+1}-m^k}{\phi(h)} = (d+q)(e^k + i^k + r^k) - \frac{\delta}{2}(m^{k+1}+m^k), \tag{8}$$
$$\frac{s^{k+1}-s^k}{\phi(h)} = \frac{\delta}{2}(m^{k+1}+m^k) - \beta \mathrm{s}^{k+1} i^k,$$
$$\frac{e^{k+1}-e^k}{\phi(h)} = \beta \mathrm{s}^{k+1} i^k - (\varepsilon+\mathrm{d}+\mathrm{q})\,\mathrm{e}^k,$$
$$\frac{i^{k+1}-i^k}{\phi(h)} = \varepsilon e^k - (\gamma+\mathrm{d}+\mathrm{q})\,\mathrm{i}^k,$$
$$\frac{r^{k+1}-r^k}{\phi(h)} = \gamma \mathrm{i}^k - (\mathrm{d}+\mathrm{q})\,\mathrm{r}^k,$$

after simplifying the equations we have:

$$m^{k+1} = ((d+q)(e^k + i^k + r^k)\phi(\mathrm{h}) + \mathrm{m}^k(1-\frac{\delta\phi(h)}{2}))/(1+\frac{\delta\phi(h)}{2}), \tag{9}$$
$$\mathrm{s}^{k+1} = (\mathrm{s}^k + \frac{\delta\phi(h)}{2}(\mathrm{m}^{k+1}+\mathrm{m}^k))/(1+\beta\phi(\mathrm{h})\,\mathrm{i}^k),$$
$$e^{k+1} = e^k(1-(\varepsilon+\mathrm{d}+\mathrm{q})\phi(\mathrm{h})) + \beta\phi(h)\,\mathrm{s}^{k+1} i^k,$$
$$i^{k+1} = i^k(1-(\gamma+\mathrm{d}+\mathrm{q})\phi(\mathrm{h})) + \phi(h)\varepsilon e^k,$$
$$r^{k+1} = r^k(1-(\mathrm{d}+\mathrm{q})\phi(\mathrm{h})) + \phi(h)\gamma i^k.$$

**Theorem 1.** Sufficient condition for scheme (8) to be positive is,

$$0 \le \phi(h) \le \frac{1}{(\gamma+d+q)}.$$

**Proof.** Assume that $m^k$, $s^k$, $e^k$, $i^k$ and $r^k$ are nonnegative real numbers so for positivity of scheme it is enough to show that

$$1-\frac{\delta\phi(h)}{2} \ge 0,$$
$$1-(\mathrm{d}+\mathrm{q})\phi(\mathrm{h}) \ge 0,$$
$$1-(\varepsilon+\mathrm{d}+\mathrm{q})\phi(\mathrm{h}) \ge 0,$$
$$1-(\gamma+\mathrm{d}+\mathrm{q})\phi(\mathrm{h}) \ge 0,$$

which leads to

$$0 \le \phi(h) \le \frac{1}{(\gamma+\mathrm{d}+\mathrm{q})},$$

and this completes the proof. □

**Proposition 1.** The new scheme preserves the conservation law.

**Proof.** With collectors the right-hand side of (8) we have:

$$\frac{d}{dt}(m+s+e+i+r)$$
$$= (d+q)(e^k+i^k+r^k) - \frac{\delta}{2}(m^{k+1}+m^k) + \frac{\delta}{2}(m^{k+1}+m^k) - \beta\mathrm{s}^{k+1} i^k$$
$$+\beta\mathrm{s}^{k+1} i^k - (\varepsilon+\mathrm{d}+\mathrm{q})\,\mathrm{e}^k + \varepsilon e^k - (\gamma+\mathrm{d}+\mathrm{q})\,\mathrm{i}^k + \gamma\mathrm{i}^k - (\mathrm{d}+\mathrm{q})\,\mathrm{r}^k$$
$$= 0$$

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and also

$$m^{k+1} - m^k + s^{k+1} - s^k + e^{k+1} - e^k + i^{k+1} - i^k + r^{k+1} - r^k = 0$$

since

$$m^k + s^k + e^k + i^k + r^k = 1,$$

we have

$$m^{k+1} + s^{k+1} + e^{k+1} + i^{k+1} + r^{k+1} = 1,$$

therefor following Anguelov, et al., (2014) the new scheme preserves the conservation law. □

**Proposition 2.** The scheme (8) is boundedness.

**Proof.** Since the proposed scheme is positivity preserving, then all variable are nonnegative real numbers and according to the conservation law, sum of theme variables equal to one, it follows that the scheme is boundedness. □

## 4. Numerical simulations

In order to illustrate the advantages of the proposed scheme, we apply it to the MSEIR model system (2) with $d = 1/(40*365)$, $\beta = .14$, $\delta = 1/180$, $\varepsilon = 1/14$, $\gamma = 1/7$, $q = 0$ and

$$\phi(h) = \frac{1 - e^{-Qh}}{Q}, \qquad Q = \max\{\delta, \varepsilon + d + q, \gamma + d + q\},$$

the parameters used in these simulations have been taken from Anguelov, et al., (2014). In the Figure.1- 3, we see the new scheme preserves the positivity property.

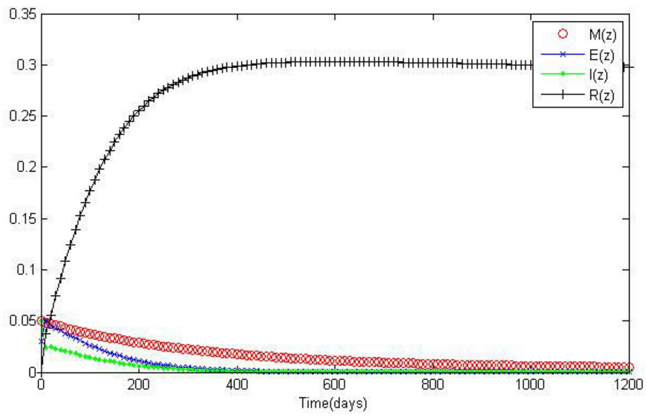


Figure 1. Numerical result of the new scheme with
$m(0) = 0.05, s(0) = 0.87, e(0) = 0.03, i(0) = 0.05, r(0) = 0, h = 10.$

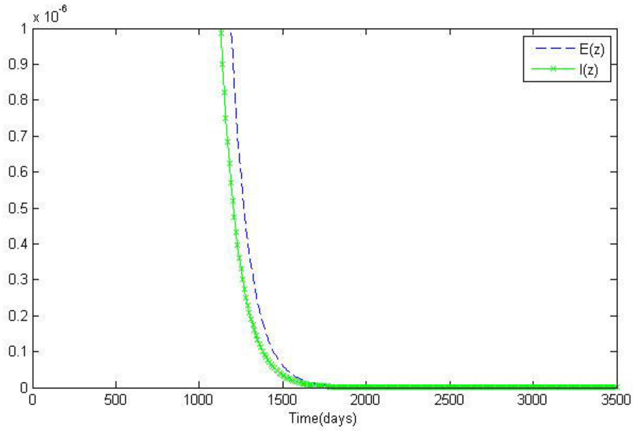


Figure 2. Numerical result of the new scheme with
$m(0) = 0.05, s(0) = 0.87, e(0) = 0.03, i(0) = 0.05, r(0) = 0, h = 10..$

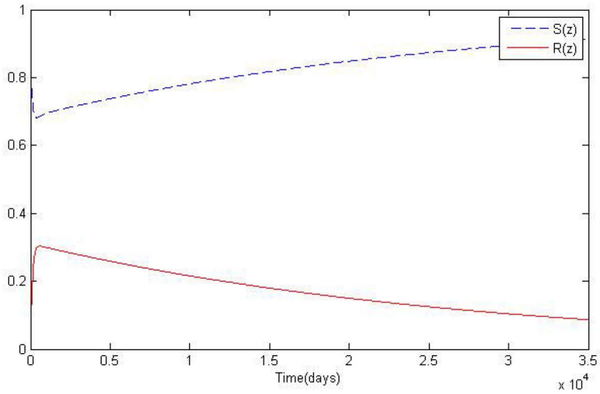


Figure 3. Numerical result of the new scheme with
$m(0) = 0.05, s(0) = 0.87, e(0) = 0.03, i(0) = 0.05, r(0) = 0, h = 10.$

As we can see in Figure.4-5 the Euler method and the second-order Runge-Kutta method (RK2) give negative values when applied to MSEIR model.

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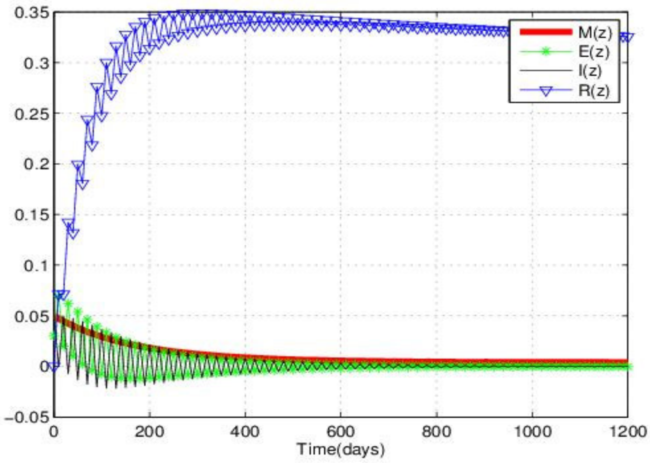


Figure 4. Numerical result for Euler method with
$m(0) = 0.05, s(0) = 0.87, e(0) = 0.03, i(0) = 0.05, r(0) = 0, h = 10.$

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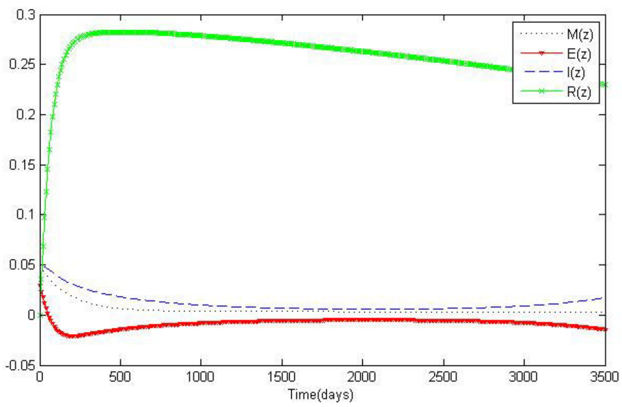


Figure 5. Numerical result for RK2 with
$m(0) = 0.05, s(0) = 0.87, e(0) = 0.03, i(0) = 0.05, r(0) = 0, h = 10.$

 

## 5. Discussions and Conclusion

In this article, we applied the nonstandard finite difference scheme for the MSEIR model by the renormalization of denominator of the discrete derivatives and the nonlocal approximation of nonlinear terms. We show that the proposed scheme is boundedness and positivity preserving. Also this scheme preserves the conservation law. A future work can be investigating of the stability requirement and the necessity condition of positivity property.

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