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The Cyclic Decomposition of the Factor Group cf $(D_n \times C_3, Z)/\overline{R}(D_n \times C_3)$ When n is an Odd Number

Kareem Abass Layith AL-Ghurabi
Babylon University College of Education Department of Mathematics
Email:kareemalghurabi@yahoo.com
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Abstract

Let D_n be the dihedral group , C_3 be the cyclic group of order 3 and D_{nh} is the direct product group of D_n and C_3 (i.e. $D_n \times C_3$) . Let cf($D_n \times C_3$,Z) be the abelian group of Z-valued class functions of the group $D_n \times C_3$. The intersection cf($D_n \times C_3$,Z) with the group of all generalized characters of $D_n \times C_3$ which is denoted by R ($D_n \times C_3$) , is a normal subgroup of cf($D_n \times C_3$,Z) denoted by $\overline{R}$ ($D_n \times C_3$) , then factor group cf($D_n \times C_3$,Z)/ $\overline{R}$ ($D_n \times C_3$) is a finite abelian group denoted by K($D_n \times C_3$) .

The problem of determining the cyclic decomposition of the group K($D_n \times C_3$) seem to be untouched .

The aim of this paper is to find the cyclic decomposition of this group.

We find that when n is an odd number such that $n = \prod_{i=1}^{m} p_i^{\alpha_i}$, where all p_i 's are distinct primes , then

$$K(D_n \times C_3) = \bigoplus_{i=1}^{4}\left[\oplus\sum_{i=1}^{m}\left(\oplus\sum K\left(C_{p_i}^{\alpha_i}\right)\right)\left[\prod_{\substack{j\neq i\\ j=1}}^{m}(\alpha_j+1)\right] time\right] \bigoplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1)+2} C_4 \bigoplus_{i=1}^{2(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1)+1} C_3 \oplus C_8.$$

1. Introduction

We can use The general features of the representation theory of finite group G over the complex numbers were discovered by Ferdinand Geory Frobenius in the years before 1900 Later on the modular representation theory of Richard Brauer was developed.

Representation theory is a tool which reduces group theortic problems into problems in linear algebra which is a very well-understood theory . As the representation theory is a very useful method to study abstract groups , it may be hard to deal with it when the order of the finite group is too large Fortunately this is not the case in the character theory which is used to study groups via their representations the theory of invariant factors to obtain the direct sum of the cyclic Z-module of orders the distinct invariant factors of $\equiv^*$(G) to find the cyclic decomposition of K(G).In1982 M.S.Kirdar[4] studied the K(C_n).In 1994 H.H. Abass[2]studied the K(D_n) and found $\equiv^*$(D_n).In 1995 N.R. Mahamood [5] studied the factor group cf(Q_{2m},Z) / $\overline{R}$ (Q_{2m}).In 2005 N.S. Jasim [6] studied the factor group cf(G,Z)/ $\overline{R}(G)$ for the special linear group SL(2,p).

In this paper we study K($D_n \times C_3$) and find $\equiv^*$($D_n \times C_3$) when n is an odd number .

2. Preliminaries

In this section we present some basic concepts of representation theory , character theory, the Dihedral group D_n and the group $D_n \times C_3$ which will be used in later chapters

Definition (1.1):[1]

The set of all $l \times l$ non-singular matrices over the field F forms a group under the operation of matrix multiplication which is called ***the general linear group*** of the dimension n over the field F, denoted by GL (n ,F).

Definition (1.2): [1]

A matrix representation of a group G is a group homomorphism T of G into GL (l ,F) , l is called ***the degree of matrix representation T*** .

Definition (1.3): [1]

The trace of an $l \times l$ matrix is the sum of the main diagonal elements , denoted by tr(A) .

Example (1.4):

Consider the symmetric group S_3 of order 6 ,define $T:S_3 \rightarrow GL(3,C)$ as follows :

$$T((1))=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad T((12))=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\quad T((13))=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

$$T((23))=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\quad T((123))=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\quad T((132))=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

T is a matrix representation of the group S_3 of degree 3 .

Theorem (1.5):[1]

Let $T_1 : G_1 \rightarrow GL(V_1)$ and $T_2: G_2 \rightarrow GL(V_2)$ be two irreducible representations of the groups G_1 and G_2 respectively, then $T_1 \otimes T_2$ is irreducible representations of the group $G_1 \times G_2$.

Definition (1.6): [3]

Let T be a matrix representation of a group G over the field F, ***the character*** χ of a matrix representation T is the mapping χ: $G \rightarrow F$ defined by $\chi(g)=Tr(T(g))$ for all $g\in G$ where $Tr(T(g))$ refers to the trace of the matrix $T(g)$ and $\chi(1)$ is the degree of χ .

Remark (1.7):

(i)A finite group G has a finite number of conjugacy classes and a finite number of distinct irreducible character, the group character of a group representation is constant on a conjugacy class , the values of irreducible characters can be written as a table whose columns are the conjugacy class and rows the value of irreducible characters on each conjugacy class , this table of the group G, denoted by $\equiv(G)$.

(ii) If $G = C_n = < r >$ is the cyclic group of order n generated by r .If $\omega = e^{2\pi i/n}$ is the primitive n-th root of unity , the

$\equiv(C_n)=$

CL_α	1	r	r^2	$\cdots$	r^{n-1}
$\lvert CL_\alpha \rvert$	1	1	1	$\cdots$	1
$\lvert C_G(C_\alpha) \rvert$	n	n	n	$\cdots$	n
χ_1	1	1	1	$\cdots$	1
χ_2	1	ω	ω^2	$\cdots$	ω^{n-1}
χ_3	1	ω^2	ω^4	$\cdots$	ω^{n-2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_n	1	ω^{n-1}	ω^{n-2}	$\cdots$	ω

Definition (1.8):[3]

Let χ and ψ as characters of a group G, then :

1. The sum of characters is defined by:
   $(\chi+\psi)(g) = \chi(g)+\psi(g)$, for all $g\in G$
2. The product of characters is defined by : $(\chi.\psi)(g) = \chi(g).\psi(g)$, for all $g\in G$.

Theorem (1.9):[3]

Let $T_1 : G_1 \rightarrow GL(n,K)$ and $T_2: G_2 \rightarrow GL(m,K)$ are two matrix representations of the groups G_1 and G_2 , χ_1 and χ_2 be two characters of T_1 and T_2 respectively, then the character of $T_1 \otimes T_2$ is $\chi_1\chi_2$.

Definition (1.10):[1]

A rational valued character θ of G is a character whose values are in the set of integers Z, which is $\theta(g) \in Z$, for all $g \in G$.

Proposition (1.11):[4]

The rational valued characters $\theta_i = \sum_{\sigma \in Gal(Q(\chi_i)/Q)} \sigma(\chi_i)$ form basis for $\overline{R}$(G), where χ_i are the irreducible characters of G and their numbers are equal to the number of all distinct Γ- classes of G.

3. The factor group K(G)

In this section , we study the factor K(G) and discuss the cyclic decomposition of the factor groups $K(C_n)$ and $K(D_n)$.

Definition (2.1):[4]

Let M be a matrix with entries in a principal ideal domain R , a k-minor of M is the determinate of k×k sub matrix preserving rows and columns order.

Definition (2.2):[4]

A k-th determinant divisor of M is the greatest common divisor (g.c.d) of all the k-minors of M . This is denoted by D_k (M).

Lemma (2.3):[4]

Let M, P and W be matrices with entries in a principal ideal domain R , if P and W are invertible matrices, then D_k (P M W)= D_k (M) modulo the group of unites of R.

Theorem (2.4):[4]

Let M be an l ×l matrix entries in a principal ideal domain R, then there exists matrices P and W such that:

1- P and W are invertible.
2- P M W = D.
3- D is diagonal matrix.
4- if we denote D_{ii} by d_i then there exists a natural number m ;
$0 \le m \le l$ such that $j > m$ implies $d_j = 0$ and $j \le m$ implies $d_j \ne 0$
and $1 \le j \le m$ implies $d_j \mid d_{j+1}$.

Definition (2.5):[4]

Let M be a matrix with entries in a principal ideal domain R be equivalent to a matrix D=diag $\{d_1, d_2, \ldots, d_m, 0, 0, \ldots, 0\}$ such that $d_j \mid d_{j+1}$ for $1 \le j < m$.

We call D ***the invariant factor matrix of*** M and $d_1, d_2, \ldots, d_m$ the invariant factors of M.

Theorem (2.6):[4]

Let K be a finitely generated module over a principal ideal domain R, then K is the direct sum of a cyclic submodules with an annihilating ideal $\langle d_1\rangle, \langle d_2\rangle, \ldots, \langle d_m\rangle$, $d_j \mid d_{j+1}$ for j = 1, 2, … , K-1.

Proposition(2.7):[4]

Let A and B be two non-singular matrices of the rank n and m respectively, over a principal ideal domain R . Then the invariant factor matrices of A $\otimes$ B equals D(A)$\otimes$D(B) , where D(A)and D(B) are the invariant factor matrices of A and B respectively .

Theorem(2.8):[4]

Let H and L be p_1-group and p_2-group respectively ,where p_1 and p_2 are distinct primes . Then , $\equiv^*(H \times L) = \equiv^*(H) \otimes \equiv^*(L)$.

Remark (2.9):[4]

Suppose cf(G,Z) is of the rank l , the matrix expressing the $\overline{R}$ (G) basis in terms of the cf(G,Z) = Z^l basis is $\equiv^*$(G).

Hence by theorem (2.4),we can find two matrices P and Q with a determinant ± 1 such that P. $\equiv^*$(G).Q =D($\equiv^*$(G))= diag$\{d_1, d_2, \ldots\ldots, d_l\}$,

$$d_i = \pm D_i(\equiv^*(G)) / \pm D_{i-1}(\equiv^*(G)) .$$

this yields a new basis for $\overline{R}$ (G) and Cf(G,Z),$\{v_1, v_2, \ldots\ldots, v_l\}$and $\{u_1, u_2, \ldots\ldots, u_l\}$ respectively with the property $v_j = d_j u_j$.

Hence by theorem (2.6) the Z-module K(G) is the direct sum of cyclic submodules with annihilating ideals $<d_1>, <d_2>, \dots, <d_l>$.

Theorem(2.10) :[4]

Let p be a prime number, then :

$$K(G) = \oplus\sum_{i=1}^{s} C_{d_i} \quad \text{such that } d_i = \pm D_i(\equiv^*(G)) / \pm D_{i-1}(\equiv^*(G)).$$

Theorem (2.11):[4]

$$|K(G)| = \det(\equiv^*(G))$$

Proposition (2.12):[4]

The ***rational valued characters table of the cyclic group*** C_{p^s} of the rank s+1 where p is a prime number which is denoted by ($\equiv^*(C_{p^s})$), is given as follows:

Γ-classes	[1]	$[r^{p^{s-1}}]$	$[r^{p^{s-2}}]$	$[r^{p^{s-3}}]$	…	$[r^{p^2}]$	$[r^p]$	[r]
θ_1	$p^{s-1}(p-1)$	$-p^{s-1}$	0	0	…	0	0	0
θ_2	$p^{s-2}(p-1)$	$p^{s-2}(p-1)$	$-p^{s-2}$	0	…	0	0	0
θ_3	$p^{s-3}(p-1)$	$p^{s-3}(p-1)$	$p^{s-3}(p-1)$	$-p^{s-3}$	…	0	0	0
⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮
θ_{s-1}	p(p-1)	p(p-1)	p(p-1)	p(p-1)	…	p(p-1)	-p	0
θ_s	p-1	p-1	p-1	p-1	…	p-1	p-1	-1
θ_{s+1}	1	1	1	1	…	1	1	1

where its rank s+1 represents the number of all distinct Γ-classes.

Example (2.13):

Consider the cyclic group C_{49} by using table (2.3), we can find the rational valued characters table of C_{49} as follows:

$\equiv^*(C_{49}) = \equiv^*(C_{7^2}) =$

Γ-classes	[1]	$[r^7]$	$[r]$
θ_1	42	-7	0
θ_2	6	6	-1
θ_3	1	1	1

Remark (2.14):

In general, for $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdot \dots \cdot p_m^{\alpha m}$ where g.c.d $(p_i, p_j) = 1$, if $i \neq j$, p_i's are prime numbers and $\alpha_i \in Z^+$, then we have the following formula :

$$\equiv^*(C_n) = \equiv^*(C_{p_1^{\alpha 1}}) \otimes \equiv^*(C_{p_2^{\alpha 2}}) \otimes \dots \otimes \equiv^*(C_{p_m^{\alpha m}}).$$

Proposition (2.14):[4]

If p is a prime number , then

$$D(\equiv^*(C_p{}^s)) = \text{diag}\{p^s, p^{s-1}, \dots\dots, p, 1\}.$$

Remark (2.15):[4]

For $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdot \dots \cdot p_m^{\alpha m}$ where p_i's are distinct primes and $\alpha_i \in Z^+$, then : $D(\equiv^*(C_n)) = D(\equiv^*(C$

$p_1^{\alpha 1}$)) ⊗D(≡*(C $p_2^{\alpha 2}$)) ⊗ … ⊗ D(≡*(C $P_m^{\alpha m}$)) .

Theorem(2.16) :[4]

Let p be a prime number , then :

$$K(C_p^{\,s}) = \oplus \sum_{i=1}^{s} C_{P^i} \quad .$$

Example(2.17):-

$$K(C_{25}) = K(C_{5^2}) = C_5 \oplus C_{5^2}$$

Proposition(2.18):[4]

Let $n = \prod_{i=1}^{k} P_i^{a_i}$,where p_i's are distinct primes and $a_i \in Z^+$,then :

$$K(C_n) = \oplus \sum_{i=1}^{k} \left(\oplus \sum K\left(C_{P_i^{a_i}}\right)\right) \left[\prod_{\substack{j \neq i \\ j=1}}^{k} \left(a_j + 1\right) \right] \; time \; .$$

Example(2.19) :

To find the cyclic decomposition of group $K(C_{15435})$

$$K(C_{15435}) = K(C_{3^2 \cdot 7^3 \cdot 5}) = \underbrace{K\left(C_{3^2}\right) \oplus \cdots \cdots \oplus K\left(C_{3^2}\right)}_{(3+1).(1+1)\ times}$$

$$\oplus \underbrace{K\left(C_{7^3}\right) \oplus \cdots \cdots \oplus K\left(C_{7^3}\right)}_{(2+1).(1+1)\ times}$$

$$\oplus \underbrace{K\left(C_5\right) \oplus \cdots \cdots \oplus K\left(C_5\right)}_{(2+1).(3+1)\ times}$$

$$= \bigoplus_{i=1}^{8} K(C_{3^2}) \bigoplus_{i=1}^{6} K(C_{7^3}) \bigoplus_{i=1}^{12} K(C_5)$$

$$= \bigoplus_{i=1}^{8} C_{3^2} \bigoplus_{i=1}^{8} C_3 \bigoplus_{i=1}^{6} C_{7^3} \bigoplus_{i=1}^{6} C_{7^2} \bigoplus_{i=1}^{6} C_7 \bigoplus_{i=1}^{12} C_5 \; .$$

Definition (2.20):[3]

For a fixed positive integer n ≥ 3 ,***the dihedral group D_n*** is a certain non- abelian group of the order 2n . In general can write it as:

$$D_n = \{ S^j\ r^k : 0 \leq k \leq n\text{-}1 \ , \ 0 \leq j \leq 1\}$$

which has the following properties :

$$r^n = 1 \ , \ S^2 = 1 \ , \ Sr^k S^{-1} = r^{-k}$$

. *Definition $D_n \times C_3$ (2.21) :[3]*

The group $D_n \times C_3$ is the direct product group $D_n \times C_3$,where C_3 is a cyclic group of the order 3 consisting of elements {1, r^* , r^* }with $(r^*)^2=1$ .It is order 4n .

So the group $D_n \times C_3$ is the direct product group $D_n \times C_3$, then the order of $D_n \times C_3$ is 6n .

Lemma (2.22):[2]

The rational valued characters table of D_n when n is an odd number is given as follows:

$\equiv^*(D_n)=$

	Γ- classes of C_n	[S]
θ_1	$\equiv^*(C_n)$	0
⋮		⋮
θ_{l-1}	1 1 1 … 1 1	0
θ_l		1
θ_{l+1}	1 1 1 … 1 1	-1

Where l is the number of Γ- classes of C_n .

Example (2.23):

To find the rational valued characters table of D_{49}, From example (2.13), we obtain $\equiv^*(C_{49})$ and by using lemma (2.22) , we have.

$\equiv^*(D_{49}) = \equiv^*(D_{7^2}) =$

Γ-classes	[1] $[r^7]$ $[r]$	[S]
θ_1	$\equiv^*(C_{7^2})$	0
θ_2	1 1 1	0
θ_3		1
θ_4	1 1 1	-1

Table (2.10)

=

Γ-classes	[1]	$[r^7]$	$[r]$	[S]
θ_1	42	-7	0	0
θ_2	6	6	-1	0
θ_3	1	1	1	1
θ_4	1	1	1	-1

Proposition(2.24):[2]

$$D(\equiv^*(D_n))=\left[\begin{array}{c|c} D\left(\equiv^*(C_n)\right) & 0 \\ \hline 0 & -2 \end{array}\right]$$ Where $D(\equiv^*(D_n))$ and $D(\equiv^*(C_n))$ are the invariant factors matrices of $\equiv^*(D_n)$ and $\equiv^*(C_n)$ respectively .

Theorem(2.25) :-[7]

If n is an odd number ,then :

$$\equiv^*(D_{nh}) = \equiv^*(D_n) \otimes \equiv^*(C_2) .$$

Theorem(2.26) :- [7]

For a fixed positive odd integer n such that $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdots p_m^{\alpha m}$ where $p_1, p_2, \ldots, p_m$ are distinct primes and $\alpha_1, \alpha_2, .., \alpha_m$ are positive integers, then ;

$$K(D_{nh})=\bigoplus_{i=1}^{2} K(D_n) \bigoplus_{i=1}^{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)-2} C_2 \oplus K(C_4) .$$

Example(2.27):-

To find $K(D_{15435h})$

$$K(D_{nh}) = \bigoplus_{i=1}^{2} K(D_n) \bigoplus_{i=1}^{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)-2} C_2 \oplus K(C_4) .$$

$$K(D_{15435h}) = K(D_{3^2.7^3.5h}) = \bigoplus_{i=1}^{2} K(D_{3^2.7^3.5}) \bigoplus_{i=1}^{22} C_2 \oplus K(C_4) .$$

3. The Main Results

In this section we find the general form of the rational valued characters table of the group $\boldsymbol{D_n \times C_3}$ (when n is an odd number) .

Theorem(3.1) :-

If n is an odd number , then :

$$\equiv^*(\boldsymbol{D_n \times C_3}) = \equiv^*(D_n) \quad \equiv^*(C_3)$$

Proof :-

From proposition (2.24) we have

$P(D_{nh}) .M(D_{nh}) .W(D_{nh})$ =diag{2,2,2,…,-2,1,1,1}= $\{d_1, d_2, \ldots\ldots\ldots,$

$$d^{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)-1)}, d^{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))-1}, d^{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))},$$

$$d^{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))+1}\ d^{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))+2}\}.$$

By theorem (2.25) we get

$$AC(D_{nh}) = \bigoplus_{i=1}^{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))-1} C_{di}$$

$$= \bigoplus_{i=1}^{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))-1} C_2$$

From theorem(2.6) we have :

$$AC(D_{nh}) = \bigoplus_{i=1}^{2} AC(D_n) \oplus C_2$$

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