Optimization Makes Estimation Much More Worse

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Abstract

Mean-variance optimization as a modern portfolio theory is a major model for theoretical purposes, however, in practice portfolio managers don't have enough interest despite some other ad hoc methods for many reasons such as estimation errors. Recently, the significance of modern portfolio theory has been analyzed that it doesn't beat the simple naïve 1/N rule not only in many real empirical databases but also in a simulation. By this paper, due to inherent weakness of Sharpe ratio we first express more common use and adjusted measurements such as adjusted expected utility of portfolio under ambiguity aversion to analyze their effects on portfolio optimization after this consideration, because using only sample mean and variance (Sharpe ratio) to evaluate performance value for the portfolio models may be subject to considerable bias. Second, we propose a new model based on the new measurement (adjusting ambiguity Sharpe ratio) to improve portfolio optimization problem. Our result states that by using the new measurement mean- variance optimization beats the naïve rule by applying the adjusted measurement and also the novel model outperforms Markowitz in terms of Sharpe ratio while the interesting is that for adjusting Sharpe ratio inverse result exists. Therefore, our study expresses optimization makes estimation almost worse when we try to use a measurement as an optimization target.

Keywords: portfolio selection, optimization, measurement, Sharpe ratio.

1. Introduction

Markowitz (1952) seminal paper as the base of the modern portfolio theory provided widely uses in academic research for constructing portfolios of assets. So far, the idea of portfolio selection and diversification has been exhausting in the explaining and understanding of risk and return for making a decision by maximizing the expected utility function. Markowitz (1952) provided the theory of portfolio selection which it is popularly called as a mean–variance optimization problem (MVO). He provided a suggestion that investors should consider jointly return and risk on the basis of a trade-off between their security returns and risk to determine how to allocate their funds among investment choices. He assumed that utility of terminal wealth is well approximated by a two Taylor expansion. Hence, applying the expected operator (E[X]), one can approximate the expected utility which is derived from an investment in risky assets by the first two moments of portfolio return distribution. As a result, the portfolio selection is a trade-off between expected return and risk.

However, it is well-known that the mean-variance portfolio is not well-diversified (Jorion 1985; GREEN & Hollifield 1992) due to the estimation error in the forecasting process. Moreover, in a recent study by DeMiguel et al. (2009) found that equally weighted portfolios (1/N rule) often outperform mean-variance portfolio and its extensions in terms of Sharp ratio, certainty–equivalent return and turnover surprisingly. Due to estimation error, a considerable effort has been devoted to the solving portfolio problems and estimating the parameter values of them. While these parameter values are estimated from time series sample of past returns, various portfolio weights and then unstable portfolios will be obtained over time. Although a vast literature is played to handle estimation error of moments, minimum-variance portfolio surprisingly outperforms other portfolios and it has a highest sharp ratio (Jorion 1986; Merton 1980; Jagannathan & Ma 2003). They explore the estimation error in the sample mean is so larger than the variance which ignoring the mean improves the portfolio performance. Indeed, Merton (1980) argued that the instability of portfolio weights and sampling errors are due more to estimate the amount of mean and it is difficult to estimate the expected return from time series of realized expected return. "The estimates of variances or covariances from the available time series will be much more accurate than the corresponding expected return estimates." Merton (1980) said.

Measuring and evaluating expected utility of portfolios strategies is central to investing wealth because in the presence of risky assets it represents the significant effect on decision making of any investor and people preferences as a rule of choice. Recent research studies in measuring portfolio performance have applied out-of-sample Sharpe ratio of mean and risk of sample portfolio returns or its certainty-equivalent return ratios like as DeMiguel et al. (2009; 2012), Tu & Zhou (2011), Kirby & Ostdiek (2012) and Jagannathan & Ma (2003). However, Tu & Zhou (2011) measured their portfolio utility in addition to Sharp ratio to compare different strategies. Kirby & Ostdiek (2012) and Tu & Zhou (2011) suggest timing portfolio strategy and the combination of the sophisticated portfolio to solve this problem, respectively. To explore why mean-variance portfolios perform so poorly in their study, we should address this problem by considering two different aspects. First, we ask how Markowitz obtained the idea of portfolio diversification or mean-variance optimization and based on what assumptions. Second, we focus on the measurement tools which they used to evaluate the performance of each strategy. As a result, we should clear that are two concepts consistent to do an evaluation of each strategy? Or applying another measurement could lead to different result.

To address the first aspect, we see that the idea of portfolio selection and diversification has come from explaining and understanding of risk and return for making a decision by maximizing the expected utility function which it is assumed that utility of terminal wealth is well approximated by a two Taylor expansion. The classical portfolio optimization model hypothesizes that the investors are perfectly aware of their preferences by a utility function, therefore, they maximize expected utility function. However, some studies show that this is Incompatible with actual choices. One in particular surveys on decision-making under ambiguity aversion because of poorly performance in actual choices and dissatisfaction of expected theory framework introduced by Von Neumann & Morgenstern (1944) and earlier made by Denial Bernoulli in 1738, which individual welfare can be measured by computing the expected utility. The linearity (affine transformation) of expected utility function with respect to probability and risk preferences implies that the expected theory is neutral with any uncertainty about probability and risk preferences effect. By looking historical data, the investor may become confident about forecasting returns but there is some hidden information that would affect the quality of judgment, therefore, the investors will consider it as ambiguous which makes different ambiguous permia from risk premia. The more recent general model of expected theory under ambiguity introduced by (Epstein & Schneider (2008), Epstein & Schneider (2010), Ju & Miao (2012) and Gilboa & Marinacci (2011)). Under the expected utility theory all idiosyncratic shocks will wash away by well diversification and in this framework investors like expected portfolio returns and dislike variance of portfolio returns which the effect of imposing this model often perform poorly out of sample but in the general and more real preferences model, this no longer holds and well diversified portfolio may collapse (see, Klibanoff et al. (2005), Easley & O'Hara (2009) and Maccheroni et al. (2013)). As a result, ambiguity does matter then this paper wants to capture ambiguity aversion through smoothing the preferences utility function.

With respect to the 2nd aspect, a distinguished number of articles in portfolio selection concentrate on Sharp ratio as a main well known measurement tool, while it is incompatible with non-normal distribution of expected return which tends to unsatisfactory opportunities in the case of asymmetric expected return. Thus, we provide a measurement that controls the Sharp ratio of expected utility instead of expected return and ambiguity aversion related to the hidden information mentioned earlier. The empirical evidence shows that mean-variance portfolio in terms of proposed measure which is more consistent with the target of investors outperform naïve diversification. By this finding, though we cannot claim on affectless of estimation error on the weights of the portfolio which there has been notable literature, in turn, this implies that the usefulness of mean-variance framework and in the worst circumstances, mean-variance optimization outperform the simple naïve rule. It is interpreted that it is not necessarily a sign to flaw the mean-variance framework.

While there are extensive surveys to improve portfolio performance by solving the parameter uncertainty due to true parameters are not known for decision makers and applying other policies like as Bayesian portfolio, short-sale constraints and optimal combination methods, by this study we also focuses on how well their measurements assess the out-of-sample performance of portfolios in many different strategies. They provide several models to treat classical framework for a better performance. Using higher-frequency data by Merton (1980) and shrinkage estimators by Ledoit & Wolf (2004) to estimate more accurate covariance, considering Bayesian method by Pástor & Stambaugh (2000) and Pástor (2000) to reduce estimation errors in mean, imposing moment restrictions to portfolio by Jagannathan & Ma (2003) and incorporating combination of sophisticated portfolios by Kan & Zhou (2007) and Tu & Zhou (2011) to show that this improve mean-variance model.

This study extends previous works by applying the corrected measurement to indicate that mean-variance optimization is useful by evaluating adjusted measurements and argues that the possible result of modern portfolio theory outperforms the simple equally weighted portfolio (1/N rule); thus, Sharp ratio of expected return (or excess return) is not a fair consistent measure to assess the performance of portfolio for obtaining the preference of investors. Our current study constructs an appropriate measurement to compare different strategies. In the empirical evidence, we find that the out-of-sample performance of mean-variance portfolio beats naïve rule by incorporating proposed measurement.

The paper is categorized as follows. Section 2 describes the concept of several measurements of portfolio performance and the proposed one. Section 3 illustrates briefly some strategies and suggests a new model under ambiguity aversion for considering the effect of some hidden information to explain Ellsberg (1961)'s paradox. Section 4 explains the methodology and discusses the result of empirical data among these approaches.

2. Measurement Description

In this section, we briefly discuss a set of main performance measurements which applied by authors to evaluate different portfolio strategies discussing about usefulness or huge bias results of mean-variance optimization respect to benchmark model including Sharp ratio, Certainty-equivalent returns, turnover and reward to risk ratio and then present adjusted measurement to address mysterious question raised by some authors specially DeMiguel et al. (2009).

2.1. Current Measurements

When comparing the performance of different models to each other or versus a benchmark model to characterizes how well the utility of investors will be constructed to provide their preferences, some measurements should be derived from expected utility of return rather than the expected return. Several measurements of performance have been proposed by different authors which we briefly introduce them as follow:

A: Sharpe Ratio:

a strategy's performance was suggested first by Roy (1952) as a risk-reward ratio and then the ratio was derived by William Sharp which it was introduced in Sharpe (1966) as a reward-to-volatility ratio. The measure examines how much return you receive for enduring the additional risk. The main essential vulnerability of sharp index is that the ratio consider the volatility of portfolio as the risk of the portfolio and this is a big flaw because if we have asymmetric returns, the volatility will not calculate the appropriate risk of strategies then Sharpe ratio can be problematic significantly. Subsequently, other measures such as alpha and market timing were introduced by Jensen (1969) and Henriksson & Merton (1981) respectively.

By the way, Sharpe ratio is a popular and simple way to compare the return of portfolios obtained by different strategies. It has been used particularly in most articles to address out-of-sample performance of mean-variance portfolio and its extensions to defend or reject the futility of its performance, see Merton (1980), Pástor (2000), Jagannathan & Ma (2003), Kan & Zhou (2007), Kirby & Ostdiek (2012) and DeMiguel et al. (2014). The ratio is calculated by historical data by assuming that these data have productivity ability. It is calculated as follow:

$$SR=\frac{\widehat{R_p}-R_f}{\widehat{\sigma_p}} \tag{1}$$

which, $\widehat{R_p}$ denotes expected portfolio return, $\widehat{\sigma_p}$ is portfolio standard deviation and R_f is defined as the risk-free rate.

B: Certainty-Equivalent return:

Certainty-Equivalent return (CE) is defined as the certain rate of return (zero risk) with utility equals to the utility of expected return with an associated risk for a given wealth. If too many investors are risk averse, portfolio managers would need to offer excess return more than expected return to convince them to consider riskier assets. It depends on the risk tolerance of any investor to be encouraged by the amount of certain equivalent return and the characteristics of portfolios (expected return and the volatility of expected return). Previously, McCulloch & Rossi (1990) use this measure to assess economic significance of Bayesian decision framework from Arbitrage Pricing Theory (APT). Kandel & Stambaugh (1996) incorporate this ratio to investigate the economic significance of predictability of stock return.

Although CE returns are very similar to target utility function of investors, the results for any strategies are related to certain return and it clearly doesn't represent certain utility or utility preference of investor. Lack of consideration of utility function is a potential problem and biased estimate may result, however a vast literature has been focused on this rate to affirm or criticize the men-variance portfolio, for example see Campbell & Viceira (1999), Pástor & Stambaugh (2000), DeMiguel & Uppal (2005) and Hong et al. (2007). By solving the following equation, the CE is obtained:

$$U(W(1+CE))=E(U) \tag{2}$$

Where W denotes the investor's wealth and U is the utility function:

$$U(W)=\begin{cases}\frac{1}{1-\lambda}W^{1-\lambda} & \text{for } \lambda>0 \text{ and } \lambda\neq 1\\ \ln W & \text{for } \lambda=1\end{cases} \tag{3}$$

here λ is risk aversion coefficient of the investor. CE also can be used as the inverse utility function of expected utility see DeMiguel & Uppal (2005).

$$CE=U^{-1}(E(U)) \tag{4}$$

DeMiguel et al. (2009) and Tu & Zhou (2011) explained CE as the risk-free rate of return which individual likes to accept in turn of investing in different risky portfolio strategies. It can approximately interpret as the level of expected utility of investor with a quadratic utility function. It is calculated by

$$CEQ=\hat{r}-\frac{\lambda}{2}\hat{\sigma}^2 \tag{5}$$

which $\hat{r}$ is the realized out-of-sample return of risky strategy and $\hat{\sigma}^2$ is its related variance.

C: Turnover:

In the presence of transaction costs, an investor should consider how frequently assets are traded across total assets by taking total new assets are bought and sold over a particular period. A company with higher turnover ratio will require more transaction cost rather than a firm with a lower rate, thus strategy with high turnover will reduce the total return of any period. It is calculated by the absolute average number of total shares traded in any period for each portfolio strategy.

$$TRO=\frac{1}{T-M}\sum_{t=1}^{T-M}\sum_{j=1}^{N}\left(\left|w_{j,t+1}-w_{j,t}\right|\right) \tag{6}$$

here $w_{j,t}$ denotes the wealth of portfolio obtained by strategy j at period t and N is the total number of available assets for T observing period and M window estimation period.

2.2. Alternative Suggested Measurement

A: Generalized Sharpe Ratio (GSR)

Clearly, comparison based on mean and standard deviation of distribution by Sharpe ratio, certainty equivalent (CE) rate and their descendants measures for evaluating constructed optimal portfolios do not consider possible differences of portfolios and investor's preferences due to first, the assumption of normality in return distribution criticized by many authors such as Jean (1971) developed utility function based on more parameters, Harvey & Siddique (2000) showed that skewness in stock return should be included in portfolio selection, Harvey et al. (2004) discussed how important is incorporating higher order moments in portfolio selection and recently DeMiguel et al. (2012) determined optimal portfolio weight by using option-implied skewness to improve out-of-sample performance of portfolios which leads substantial improvement in Sharpe ratio. Whereas in Sharpe index and its descendants are assumed returns are fully Gaussian distributed. In such situation that we do not take into account other moments, thus return mean and variance of the portfolio may not sufficient to examine the level of investor utility. This disadvantage of Sharpe ratio can be shown by the following simple example of Hodges (1998) in table 1 which provides two probability distribution of excess return over a period.

Table 1:

Distribution I							
Excess return	-20	-10	-5	5	10	20	30
Probability	0.01	0.04	0.25	0.4	0.25	0.04	0.01
	Mean	3.75	Variance	60.18	Sharpe ratio	0.48	
Distribution II							
Excess return	-20	-10	-5	5	10	20	40
Probability	0.01	0.04	0.25	0.4	0.25	0.04	0.01
	Mean	3.85	Variance	66.42	Sharpe ratio	0.47	

As it can be seen distribution type II has more preference for investor rather than type I, because of shifting outcome from 30 to 40 and the others are same, whereas distribution type I has more Sharpe ratio (0.48) rather than type II.

Second, even we assume the normality of return distribution for Sharpe ratio, it cannot differentiate investor preferences which is higher return and lower risk when the excess return has a lower risk free rate then the Sharp ratio is negative. As a result, fund with higher standard deviation and lower return is more worthwhile than the fund with lower standard deviation and higher return especially during bear market. Then, it doesn't measure exactly the quadratic utility of investor which we are going to maximize it in different portfolio strategies especially in mean-variance portfolio theory. Under Sharpe ratio assumption, portfolio optimization could be reward maximization and risk minimization problem. It formally can be written as the following program:

$$\max_{w} \ SR: \frac{\widehat{R_p} - R_f}{\widehat{\sigma_p}}$$
$$s.t: \ w'\iota=1, \ \iota'=[1,1,...,1] \tag{7}$$

where w is portfolio weight. Although this formulation has some parameters as the same as mean-variance problem especially in mean and variance of the portfolio are related, not identical, the result obtained from them is not the same especially when we use it to compare different strategies with the naïve strategies which it doesn't have any volatility in weights of portfolios. Thus, a further correction in measurement is needed particularly in comparing different strategies with benchmark strategy (1/N rule).

Third, some researchers have shown that Sharpe ratio is very prone to manipulation, see for example Leland (1999), Goetzmann et al. (2002) and more recently Goetzmann et al. (2007). As a simple example, it is possible to a successful strategy with high return with low volatility may produce a remarkably low Sharpe ratio. To see this, Goetzmann et al. (2002) show that without adding any value to investor's preferences in option-like strategies can increase Sharpe ratio. Furthermore, Auer (2013) finds that a negative excess return can cause an increase in Sharpe ratio of a fund with a certain poor performance.

Now we should ask this question why the 1/N rule has better performance than mean-variance measuring by Sharpe ratio. When we apply the rolling window to calculate the Sharpe ratio, the mean of Sharpe ratio of naïve diversification will be the sample mean of all assets over the observation which is the consistent estimation of Sharpe ratio. Although for the variance of Sharpe ratio calculated from naïve diversification is not the same as the variance of all assets, yet it is a consistent estimation of the variance. So if we have a market that has high positive sample mean and low variance, this helps the naïve diversification to be optimal or has a better Sharpe ratio performance.

Therefore, we test the out-of-sample performance of portfolios by applying measures that cover the weakness of Sharpe ratio like Generalized Sharpe Ratio. Generalized Sharpe ratio introduced by Hodges (1998) which measures optimal expected utility of investors with constant absolute risk aversion which is obtained by the following formulation:

$$GSR = \sqrt{\frac{-2}{T-M}\ln(-U^*)} \tag{8}$$

here, $T-M$ denotes the length of observation. There are two methods of estimation of GSR; nonparametric and parametric. The nonparametric method needs to numerical method but the parametric method has a closed-form solution presented by Zakamouline & Koekebakker (2009) as (ASSR) the adjusted for skewness Sharpe ratio and the adjusted for skewness and kurtosis Sharpe ratio (ASKSR). In the mean-variance-skewness framework, ASSR can measure the adjusted skewness Sharpe ratio for investor's preferences which is given by:

$$ASSR = \frac{\mu}{\sigma}\sqrt{1+(\frac{\lambda(\lambda+1)}{6})\frac{s}{3}(\frac{\mu}{\sigma})} \tag{9}$$

also, ASKSR measures first four moments of distribution with the assumption that return distribution follows Normal Inverse Gaussian by parameter vector ($\alpha, \beta, \gamma, \delta$). ASKSR can be obtained by:

$$ASKSR = \sqrt{2(\lambda a^*(\gamma - r_f) - \delta(\theta - \sqrt{\alpha^2 - (\beta - \lambda a^*)^2}))} \tag{10}$$

where

$$\theta = \sqrt{\alpha^2 - \beta^2}$$
$$\alpha = \frac{3\sqrt{3k-4s^2-9}}{\sigma^2(3k-5s^2-9)}, \beta = \frac{3s}{\sigma^2(3k-5s^2-9)}, \gamma = \mu - \frac{3s\sigma}{\sigma^2(3k-5s^2-9)}, \delta = 3\sigma\frac{\sqrt{3k-4s^2-9}}{(3k-5s^2-9)}$$
$$a^* = \frac{1}{\lambda}(\beta + \frac{\alpha(\gamma - r_f)}{\sqrt{\sigma^2 + (\gamma - r_f)^2}}) \tag{11}$$

here, μ, σ, s, k denote mean, standard deviation, skewness and kurtosis of distribution respectively

B: Adjusted for Ambiguity Sharpe Ratio (AASR)

The expected utility theory is defined that a decision maker could choose the higher expected value as the preferred one. Then based on this theory, the classical portfolio optimization model hypothesizes that the investors are perfectly aware of their preferences by utility function, therefore, they maximize expected utility function. However, some studies show that this is Incompatible with actual choices. One, in particular, surveys on decision-making under ambiguity aversion because of poorly performance in actual choices and dissatisfaction of expected theory framework introduced by Von Neumann & Morgenstern (1944) and earlier made by Denial Bernoulli in 1738, which individual welfare can be measured by computing the expected utility. The linearity (affine transformation) of expected utility function with respect to probability and risk preferences implies that the expected theory is neutral with any uncertainty about probability and risk preferences effect. This raises the question of interaction between the property which is referred to as the "independent axiom" and "Ellsberg paradox" the most famous challenge has been proposed by Ellsberg (1961). He proposed the following challenge. An urn contains 90 balls that thirty of them are red balls and the rest are black and white balls which the proportion of them is unknown. There are four games that players are confronted to take at random a ball from an urn. The prize depends upon the color of the ball is taken out, as are expressed in the following table.

Table 2: Ellsberg paradox

Game	Red	Black	Withe
M1	10	0	0
M2	0	10	0
N1	10	0	10
N2	0	10	10

Ellsberg observed that many players who preferred M1 over M2 also prefer N2 over N1. This is incompatible with the "independent axiom" of expected utility. To explain this paradox, Gilboa & Schmeidler (1989) proposed a *decision criterion* that players always react to select worst possible probability distribution because in this way they do not need to know the distribution of hidden information (Maxmin Expected Utility theory). Their welfare is measured by minimizing the various expected utility. It follows that the behavioral properties of ambiguity aversion which can explain Ellsberg paradox.

By looking historical data, the investor may become confident about forecasting returns but there is some hidden information that would affect the quality of judgment, therefore, the investors will consider it as ambiguous which makes different ambiguous permia from risk premia. The more recent general model of expected theory under ambiguity introduced by (Epstein & Schneider (2008), Epstein & Schneider (2010), Ju & Miao (2012) and Gilboa & Marinacci (2011)). Under the expected utility theory all idiosyncratic shocks will wash away by well diversification and in this framework investors like expected portfolio returns and dislike variance of portfolio returns which the effect of imposing this model often perform poorly out of sample but in the general and more real preferences model, this no longer holds and well diversified portfolio may collapse (see, Klibanoff et al. (2005), Easley & O'Hara (2009) and Maccheroni et al. (2013)). As a result, ambiguity does matter then this paper wants to present a measurement which capture ambiguity aversion through smoothing the preferences utility function based on Gilboa & Schmeidler (1989) *decision criterion (or Maxmin theory)* as an adjusted for ambiguity Sharpe ratio. This is also possible based on mathematical intuitions by minimizing the variance of utility and maximizing the expected utility. Economically, the ratio measures utility premium per standard deviation of utility which compensates the investor for the volatility by the utility. It is simply defined as

$$\text{AASR}=\frac{m}{v}$$

$$m=\frac{1}{T-M}\sum_{t=M}^{T-1}(\text{E}_t[\text{U(W)}])$$

$$v^2=\frac{1}{T-M-1}\sum_{t=M}^{T-1}(\text{E}_t[\text{U(W)}]-m)^2$$

$$\text{E}_t[\text{U(W)}]=\text{U(E[W])}+\frac{\text{U}^{(2)}\text{(E[W])}}{2}\mu^{(2)}=\text{w}_t'\mu_t-\frac{\lambda}{2}\text{w}_t'\Sigma_t\text{w}_t$$

3. Models

In this section, we briefly describe some strategies which we apply to evaluate different measurements as we mentioned formerly. We utilize them, first because of employing these models by different authors to investigate the value of their approaches to analyzing the stock market. Second, to find out how well the results will be by suggested measurements. We consider there is N risky asset available to invest in period t.

3.1. Equally weighted diversification portfolio (Naïve Model)

Naïve diversification refers to 1/N rule or the formation of the same importance to each stock in a portfolio. There are not estimation in this approach because of no parameters to estimate. Although it is easy to compute the weight of portfolio, in practice it is difficult to manage the stocks to be equally weighted due to not only constant price changing over the time, but also we have to include all number of stocks in the portfolio.

3.2. Mean-variance portfolio

Markowitz (1952) gives a model that the investor obtains efficient frontier which is the efficient trade-off between return and the risk of diversified portfolios. The investor can reduce only unsystematic risk through diversification, but systematic risk cannot be moderated in this approach because it is unpredictable. In Markowitz's seminal paperwork, he minimizes the amount of risk portfolio for a given portfolio expected return, which is called as the mean-variance framework. The following formulation can express this:

$$\min_{w}\ w'\Sigma w \qquad \text{s.t: } w'\mu \geq \mu_0;\ w'\iota=1,\ \iota'=[1,1,...,1] \tag{12}$$

here, $w=(w_1,w_2,...,w_n)'$ is the weight vector of N risky assets, Σ is an $N \times N$ covariance matrix of returns between N risky assets, $\mu=(\mu_1,\mu_2,...,\mu_N)'$ is the vector of expected returns, μ_0 is the target expected return. For the single-period framework, a rational investor with U a utility function and W_0 initial wealth chooses his portfolio to maximize his expected utility. At the end of period, his wealth becomes:

$$W=W_0(1+w'\mu) \tag{13}$$

Let λ denote an investor risk-aversion coefficient. Under the assumption that an investor's utility function is given by quadratic utility function (that is, asset returns are fully described by mean and variance), the expected utility of terminal wealth can be approximated through a second-order Taylor expansion such that the following equation holds:

$$E[U(W)] \approx U(E[W]) + \frac{U^{(2)}(E[W])}{2}(E[(W\text{-}E(W)^2)]) \tag{14}$$

here, $U^{(i)}(E[W])$ denote ith-order derivative of the utility function, where W is the terminal wealth of investor. Then, by considering CRRA[1] investors, define $\lambda=(-W*U^{(2)}(E[W]))/U^{(1)}(E[W]);\ \Sigma=E[(W\text{-}E(W)^2)]$. Finally, it can be shown that the Markowitz's model can be written as following:

$$\max_{w}\ (w'\mu-\frac{\lambda}{2}w'\Sigma w) \qquad \text{s.t: } w'\iota=1,\ \iota'=[1,1,...,1] \tag{15}$$

for a general utility function, the above problem will no longer be expressed by the Markowitz framework which is the trade-off between risk and return.

3.3. Global Minimum-variance Portfolio

In this model, it only considers variance of historical past return of assets to maximize the expected utility. As we have noted this model ignores the mean of sample return and chooses a set of assets which minimize the variance of returns.

$$\min_{w}\ (w'\Sigma w) \qquad \text{s.t: } w'\iota=1,\ \iota'=[1,1,...,1] \tag{16}$$

[1] Constant Relative Risk Aversion

3.4. Variance-Skewness Portfolio Model

Suppose that the price changes in excess of the risk-free rate are independently and identically distributed with mean vector μ and define $\sum$ as the matrix of covariance of asset returns. We will construct the expected volatility utility of terminal wealth by:

$$Var(U(W))=E[(U(W)-E[U(W)])^2]=E[U^2(W)]-E[U(W)]^2 \quad (17)$$

We then minimize this variance of utility to better diversify efficient portfolios from sample moments. This problem can become even more well-diversified portfolios because the extreme behavior of the weights is more due to the estimation of the sample first moment which obviously disappear from our analysis in the following calculations.

First, for calculating $E[U(W)]^2$ let $\mu^{(i)}$ denote *i*th central moment. The following equation holds if we approximate the expectation of utility wealth by a second-order Taylor expansion at $\mu=E(W)$:

$$E[U(W)] \approx U(E[W])+\frac{U^{(2)}(E[W])}{2}\mu^{(2)} \quad (18)$$

multiply above equation by itself to get:

$$E[U(W)]^2 \approx (U(E[W]))^2+U^{(2)}(E[W])\mu^{(2)}U(E[W])+\frac{(U^{(2)}(E[W]))^2}{4}(\mu^{(2)})^2 \quad (19).$$

Then, similarly we take the first term $E[U^2(W)]$ by implying a second-order Taylor expansion for utility function at $\mu=E(W)$ gives:

$$U(W) \approx \frac{(W-E(W))^0}{0!}U(E(W))+\frac{(W-E(W))^1}{1!}U^{(1)}(E(W))+\frac{(W-E(W))^2}{2!}U^{(2)}(E(W)) \quad (20)$$

multiply above by itself to get

$$U(W)^2 \approx \left[\frac{(W-E(W))^0}{0!}U(E(W))+\frac{(W-E(W))^1}{1!}U^{(1)}(E(W))+\frac{(W-E(W))^2}{2!}U^{(2)}(E(W))\right]^2 \quad (21)$$

applying both sides by expected operation to get

$$\begin{aligned}E[U(W)^2] \approx\ & E[U(\mu)^2+2U(\mu)(W-\mu)U^{(1)}(\mu)+(W-\mu)^2(U^{(1)}(\mu))^2 \\ & +(U(\mu)+(W-\mu)U^{(1)}(\mu))(W-\mu)^2U^{(2)}(\mu)+\frac{1}{4}(W-\mu)^4(U^{(2)}(\mu))^2]\end{aligned} \quad (22).$$

Finally, we can use equation (20) and (23) to calculate equation (18):

$$\begin{aligned}Var(U(W)) &= E[(U(W)-E[U(W)])^2]=E[U(W)^2]-E[U(W)]^2 \\ &= [(U^{(1)}(\mu))^2]\mu^{(2)}+U^{(1)}(\mu)\,U^{(2)}(\mu)\mu^{(3)}+\frac{1}{4}(U^{(2)}(\mu))^2\mu^{(4)}-\frac{1}{4}(U^{(2)}(\mu))^2(\mu^{(2)})^2\end{aligned} \quad (23)$$

if suppose the investors have CRRA preferences with risk aversion parameter λ , for example, let define $U=\frac{W^{1-\lambda}}{\lambda}$ be utility function for CRRA investor. Then higher-order moment tensors can easily parametrize portfolio moments as:

$$\begin{aligned}\mu^{(2)} &= w\Sigma w \\ \mu^{(3)} &= w'M_3(w\otimes w) \\ \mu^{(4)} &= w'M_4(w\otimes w\otimes w)\end{aligned} \quad (24)$$

note that

$$\begin{aligned}M_3 &= E[R-E[R]]\otimes E[R-E[R]]\otimes E[R-E[R]]\ , \\ M_4 &= E[R-E[R]]\otimes E[R-E[R]]\otimes E[R-E[R]]\otimes E[R-E[R]]\end{aligned} \quad (25)$$

by taking the initial wealth as a numeraire, the following explanation can be suggested by our analysis:

$$\begin{aligned}U^{(1)}(W) &= 1 \\ U^{(2)}(W) &= -\lambda \\ U^{(3)}(W) &= \lambda(\lambda+1) \\ U^{(3)}(W) &= -\lambda(\lambda+1)(\lambda+2)\end{aligned} \quad (26)$$

therefore, we can rewrite the investor optimization's problem as a minimizing the following portfolio moments:

$$\mu^{(2)}-\frac{(\lambda)^2}{4}(\mu^{(2)})^2+\lambda\mu^{(3)}+\frac{(\lambda)^2}{4}(\mu^{(4)}) \quad (27)$$

which clearly removes the first-moment impact of sample return in trade-off among moments.

Interestingly, it shows that if only the sample second moment is between zero and $(\lambda)^2/4$, the investor are willing to accept higher variance in exchange of lower skewness and higher kurtosis, otherwise vice versa. As compared to consequences portfolio constructed using mean-variance framework, only when the sample second moment is bounded between zero and $(\lambda)^2/4$, we find a consistent result. This behavior unlikely can explain why optimized portfolios are not optimum.

4. Methodology for Evaluating the Performance of Models

In this section, we express our methodology in two important dimensions. First, we describe how to measure the outperformance of proposed model. This model has some significant advantages. Specifically, it avoids estimation of the first moment of past sample returns, it provides a way to evaluate the investor portfolio selection problem which assumes the sample mean of returns estimation fluctuate a lot when we rebalance the portfolio according to some investigates as the same as Merton (1980) and Jagannathan & Ma (2003). Although there has been considerable effort to improve the estimation of the expected return such as using Bayesian estimation, robust optimization and option-implied information, the estimation of the expected retunes in empirical and simulation-based analysis is poorly behaved and needs very long time series data. Then, we conclude this with a discussion on the relation between expected utility and variance of expected utility objective function. We focus on minimizing the expected variance utility function as it avoids the expected return than maximizing the mean-variance models because the estimation error would result in extreme rebalancing portfolios even run the model by robust estimation. Therefore, we are tremendously interested to discover the behavior of portfolio asset when we bound variance in the posited area of the variance-skewness portfolio with empirical data and compare the result with mean-variance portfolio

Second, we explore the out-of-sample performance of different strategies by current and suggested measurements as an alternative ranking metrics, employing historical market data. To identify the out-of-sample performance of strategies, we have to specify a well efficient measurement which is satisfying information regarding the real data. Sharpe ratio is a widely used measure as a benchmark ratio to gauge the performance of portfolio; however, recently a number of papers have shown that this measure can result in misleading outcome as the estimation errors do. The nature of measure error can be explored by examining the performance of portfolios with two or more ranking metrics. This error may be heavily weighted toward portfolios with low performance. Similarly, measure error may rank the high-performance portfolio as a worse portfolio. Thus, any distortion introduced by measurement should consider during our calculation, regardless of whether a particular metric was most wieldy used for performance evaluation. For comparing the results come from Sharpe ratio with alternative metrics, we use rolling estimation window to compare different policies by Sharpe Ratio, Certainty Equivalent return, Ambiguity Ratio and ASKSR.

4.1. Performance Evaluation of Proposed Model

We consider an economy with the R returns vector of N different risky assets. Let M_i denote ith higher order moment tensor for the assets which is introduced by Jondeau & Rockinger (2003) using Kronecker product as equation (29). The investor's terminal wealth can be defined such that equation (14) and considered the initial wealth as a numeraire. Then, the central moments of portfolio returns can satisfy equation (30).

$$\mathrm{M_i}=E[R-E[R]]^{\otimes i} \;;\; i>1 \quad (28)$$

$$\mu^{(i)}=w'M_i w^{\otimes(i-1)};\; i>1 \quad (29)$$

we can rewrite the investor optimization's problem as a function portfolio of weight vector with two first moment tensors (only trade-off between variance and skewness):

$$\min_w \; w'\Sigma(w)-\frac{(\lambda)^2}{4}(w'\Sigma(w))^2+\lambda w'M_3(w\otimes w) \quad (30)$$

which define the trade-off between variance and skewness. To analyze the out-performance of constructed portfolio, we compare certainty equivalents for an investment in different competing portfolios. Differentiating the above objective function with respect to (w) gives optimum weight values of portfolio:

$$2\Sigma w-(\lambda)^2\Sigma w(w'\Sigma w)+3\lambda M_3(w\otimes w)=0 \quad (31)$$

write the third order moment tensor for n assets:

$$M_3=[S_1|S_2|\cdots|S_i|\cdots|S_n] \quad (32)$$

then the above equation is equivalent to:

$$[2\Sigma\text{-}(\lambda)^2\Sigma(w'\Sigma w)+3\lambda(\sum_{i=1}^{n} w_i S_i)]w=0$$

or (33)

$$[2\Sigma\text{-}(\lambda)^2\Sigma(w'\Sigma w)+3\lambda w'S]w=0$$

so, the explicit solutions can be written as following:

$$(\lambda^2 w'\Sigma w\text{-}2)=\frac{3\lambda M_3(w\otimes w)}{\Sigma w}$$

$$w=\{w|(\lambda^2 var\text{-}2)=\frac{3\lambda\ skew}{var};\ var=w'\Sigma w,\ skew=w'M_3(w\otimes w)\ \} \quad (34)$$

$$w=\frac{(\text{-}3M_3\pm\sqrt{(3M_3)^2+8\Sigma^3})\Sigma^{-2}}{\text{-}2\lambda}\ \text{or}\ w=0$$

Based on our strategy, we can now formally say that variance-skewness portfolio optimization (VSO) is well diversified at skewness, therefore, we can make portfolio diversification based on considering jointly securities skewness and their co-movements. While it is difficult to make an accurate estimation of return due to the direct impact of idiosyncratic volatility on the first moment of individual security rather than other moments, the estimation error of mean significantly moves portfolio weights from optimum one. We conclude from our obtained objective function (equation 31) that if we decrease the variance of the portfolio, in fact, we increase proportionally the variance of expected utility it means that we make worse our utility portfolio.

This portfolio is the Global Minimum Variance of Utility portfolio (GMVU) which can be formulated by the optimization portfolio

$$\min_{w} VarE(U)=w'\Sigma(w)\text{-}\frac{(\lambda)^2}{4}(w'\Sigma(w))^2+\lambda w'M_3(w\otimes w)$$

$$S.T: \quad (35)$$

$$w'i=1;\ i=[1,\cdots,1,1]$$

refer to GMVU we can obtain the efficient frontier of skewness and variance which is totally different with the efficient portfolio of risk-return. Another efficient portfolio that we can introduce is an efficient portfolio of expected variance and return of utility which we call it EVU. We can formulate it as following

$$\min_{w} VarE(U)$$

$$S.T:$$

$$E(U)\geq\mathfrak{M}$$

$$w'i=1;\ i=[1,\cdots,1,1] \quad (36)$$

Our goal is to analysis the performance of our model compared to a benchmark portfolio on the asset allocation of the data set. In order to improve the result of our model, we need to mitigate the impact of estimation error in portfolio optimization which it increases exponentially with the number of risky assets. Following the literature on improved estimation method, the shrinkage estimators are the most effective approach suggested by Ledoit & Wolf (2004) and Martellini & Ziemann (2010) for covariance, skewness and kurtosis respectively, gives better performance than original sample estimator and easy to implement which gives us more motivation to consider these estimators. They define the posterior misspecification function of convex linear combination estimator as:

$$L(\delta)=||\delta\mathbb{F}+(1\text{-}\delta)\mathbb{S}\text{-}\Omega||^2 \quad (37)$$

here, δ is the shrinkage intensity which is between 0 and 1; $\mathbb{F}$ the shrinkage target which we estimate by the sample constant correlation approach, $\mathbb{S}$ the sample estimator and Ω is the true moment tensor matrix. Note that Frobenius norm of a matrix is defined as

$$||s||^2=\sum_i s_i^2 \quad (38)$$

by finding the optimum shrinkage intensity, the expected value of loss will be minimized and asymptotically behave like a constant over time period T . This optimal value can be written

$$\delta^{*}=\frac{1}{T}\frac{\pi-\rho}{\gamma} \quad (39)$$

where, π denotes an asymptotic tensor moment of the sample estimator, ρ represents the asymptotic tensor moment between the sample and structured estimator and γ represents misspecification of the structured estimator. Then, the shrinkage estimators are calculated by:

$$\delta^{*}\mathbb{F}+(1-\delta^{*})\mathbb{S} \quad (40)$$

4.2. Empirical results

In this subsection, we examine the performance of our constructed portfolio using returns from Center for Research in Security Prices (CRSP) monthly returns data. We consider the sample period January 2004 to October 2013. We select top 7 firms from CRSP database, and then we collect monthly returns for these stocks from January 2004 to October 2013. As a result, we obtain valid monthly returns of 7 socks for 118 periods. We first set window estimation M=60 and then to measure the stability of each portfolio we estimate moment and comoments parameter which obviously are not known by using (Martellini & Ziemann 2010; Ledoit & Wolf 2004) shrinkage estimation method. The relative risk aversion coefficient is taken equal to different cases of $\lambda=1,3$.

Table 3 compares the results of Markowitz and naive rules with various measurements. It shows that the result of the Markowitz portfolio is better than the naïve rule with Adjusted for Ambiguity Sharpe Ratio (AASR) and Adjusted for Skewness Sharpe Ratio (ASSR). Clearly, however, the results in terms of Sharpe ratio and Certainty Equivalent return measurements go to inverse results. We view these results as evidence that the adjusting measurements are the source of the usefulness of the Markowitz theory. This is because of two reasons. First, we provide a measurement that considers some possible explanation of hidden information simply through the ambiguity theory. Second, imposing skewness of return distribution to the measurement leads to the expression of non-normality of distribution which is assumed to be a normal distribution in Markowitz theory and its extension.

Table 3: How well is Naïve rather than Markowitz by some new measurements?

PANEL A: N=7	$\lambda=1$		$\lambda=3$	
Measurements\ Models	Naïve	Markowitz	Naïve	Markowitz
Sharp Ratio	0.091706467	0.088835632	0.0917065	0.088835632
Adjusted for Ambiguity Sharpe Ratio	0.001051963	0.206016897*	0.001052	0.177975527*
Certainty Equivalent Return	0.003542018	-0.179559716	0.0004849	-0.662098514
Adjusted for Skewness Sharpe Ratio	0.091631	0.090964793	0.0920721	0.097386299*

Although this was presented the Markowitz model by some new measurements has superior out-of-sample performance than the naïve rule, we developed the Markowitz model to consider ambiguity and hidden information as a novel model as mentioned in subsection 3.4 in more detail. Table 4 describes the summarized results of empirical data for novel model and Markowitz mean-variance theory. We see the novel model almost has better Sharpe ratio, CER and Adjusted for Skewness Sharpe Ratio than mean-variance strategy. For example, the out-of-sample Sharpe ratio for Markowitz is 0.088, while the novel model has 0.091 monthly out-of-sample ratio. Similarly, the Certainty Equivalent Return (CER) for Markowitz is negative, while that for the novel approach is strongly positive in different coefficients of risk aversion. Moreover, both strategies almost have same Adjusted for Skewness Sharpe Ratio around 0.09. The comparison of different measurements typically enhances the improvement results of using the novel model at dealing with estimation error. Thus, considering portfolio under ambiguity as an optimal target is very successful and much more reasonable measuring with current gauges like Sharpe ratio and CER.

Table 4: how well is the novel model rather than Markowitz?

PANEL A: N=7	λ=1		λ=3	
Measurements\ Models	Novel	Markowitz	Novel	Markowitz
Sharp Ratio	0.091599491*	0.088835632	0.091599491*	0.088835632
Certainty Equivalent Return	0.003185329*	-0.179559716	0.001008434*	-0.662098514
Adjusted for Skewness Sharpe Ratio	0.092350955	0.090964793	0.092036462	0.097386299

Our next observation in table 5 is that contrary to the mentioned view in table 4, the Adjusted for Ambiguity Sharpe Ratio is not improved for the novel model which extended from Maxmin Expected Utility theory to explain Ellsberg paradox. For instance, although the novel model has better Sharpe ratio (0.091) than Markowitz (0.088), it has definitely less Adjusted for Ambiguity Sharpe Ratio (0.002) than Markowitz (0.206).

Table 5: how well is the novel model rather than naïve and Markowitz by proposed measurement?

PANEL A: N=7	λ=1		
Measurements\ Models	Naïve	Markowitz	Novel
Sharp Ratio	0.091706467	0.088835632	0.091599491
Adjusted for Ambiguity Sharpe Ratio	0.001051963	0.206016897	0.002367035*
Certainty Equivalent Return	0.003542018	-0.179559716	0.003185329
Adjusted for Skewness Sharpe Ratio	0.091631	0.090964793	0.092350955
PANEL B: N=7	λ=3		
Measurements\ Models	Naïve	Markowitz	Novel
Sharp Ratio	0.091706467	0.088835632	0.091599491
Adjusted for Ambiguity Sharpe Ratio	0.001051972	0.177975527	0.002367753*
Certainty Equivalent Return	0.000484853	-0.662098514	0.001008434
Adjusted for Skewness Sharpe Ratio	0.092072075	0.097386299	0.092036462

This novel model is constructed based on optimization of the portfolio to improve ambiguity for investors who concern about hidden information. Table 5 reports surprising and interesting results where the Adjusted for Ambiguity Sharpe Ratio (AASR) for the novel model is less than Markowitz model and Naïve rule in panel A and B which must be better than others, since the target of optimization for the novel model is maximizing the ambiguity Sharpe ratio. In fact, this finding indicates that it optimizing models based on measuring target by estimation from historical data the worst case will happen no matter it is Markowitz model or any other model.

5. Conclusion

We have introduced some measurements to evaluate different portfolio strategies and then proposed new scale to describe ambiguity aversion. In particular, we find that by some new measurements Markowitz strategy will outperform better than the naïve model. This finding is in contrary to many studies, showing that the naïve rule outperforms better than Markowitz strategy and its extensions. We further find a new model by this new proposed measurement, enhancing that the Sharpe ratio of this model outperforms better than Markowitz strategy and naïve rule. In addition, our study has shown that the new model, however, has higher Sharpe ratio and CER than Markowitz and naïve, by the proposed measurement which is the optimal target of the new model it has a worse result. Consequently, this appears the fact that optimization makes estimation much worse.

In general, our finding has two important insights. First, in this paper we study how mean-variance strategy is misused by ambiguity and the adjusted model takes in an ambiguity premium. Next, when an investor's optimal target incorporates its performance measurement, optimized estimated portfolios generate an inappropriate out-of-sample result in that measurement context, this naturally extends optimization can't help diversification toward right portfolios by estimation data. This concluding take up the question of whether we should stop any academic effort on optimization when there exists estimation. It seems useful to offer that other researchers find other strategies to construct and support portfolio rather than optimization method.

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